Delft University of Technology
Faculty EEMCS
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Exam Complex Analysis (wi4243AP/wi4244AP)
Thursday 31 October 2013; 09:00 – 12:00.

Lecturer: K. P. Hart.
Second reader: H. A. W. M. Kneppers
This exam consists of six questions.
Marks per question: see margin.
Resources allowed: calculator

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1. Let α be a complex number such that $|\alpha| \neq 1$ and consider the bilinear transformation given by

$$w = \frac{z - \alpha}{\overline{\alpha} z - 1}.$$

(2) a. Show that this transformation maps the unit circle onto itself.

(2) b. What is the image of the unit disk under this transformation?

(2) c. How does w traverse the unit circle as z traverses the unit circle in the positive direction?

(3) d. Now let $\alpha = \frac{1}{2}i$. Determine the image of the part of the unit disk that lies in the first quadrant under the transformation.

2. Define $u(x, y) = x \cos x \cosh y + y \sin x \sinh y$

(2) a. Verify that u is harmonic.

(4) b. Determine all analytic functions that have u as their real part and write these as functions of z.

(5) 3. a. Let h be an analytic map from the unit disc $D = \{z : |z| \leqslant 1\}$ to itself such that $h(0) = 0$. Show that $\bigl|h(z)\bigr| \leqslant |z|$ for $z \in D$ and $\bigl|h'(0)\bigr| \leqslant 1$. *Hint*: Consider the function $h(z)/z$.

(6) b. Let f be an analytic map from the unit disc $D = \{z : |z| \leqslant 1\}$ to itself. Show that $\bigl|f'(0)\bigr| \leqslant 1 - |f(0)|^2$. *Hint*: Let $\alpha = f(0)$ and consider the function $g(z) = \frac{f(z)-\alpha}{\overline{\alpha} f(z) - 1}$.

(8) 4. Let a be a real number such that $a > 1$; evaluate the following integral

$$\int_0^{2\pi} \frac{1}{a^2 - 2a \sin\theta + 1} \, \mathrm{d}\theta$$

Give all details.

(8) 5. Let a and b be positive real numbers. Evaluate the following integral

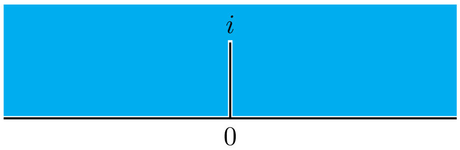
$$\int_0^{\infty} \frac{x \sin ax}{x^2 + b^2} \, \mathrm{d}x.$$

Give all details.

See also the next page.

6. We consider the many-valued function $w = \sqrt{z^2+1}$.

(3) a. Suppose we use the branch of $\sqrt{}$ that has the positive real axis as its branch cut and is such that $\sqrt{-1} = i$. Determine the image of the upper half plane minus the segment $[0, i]$ under this mapping (see below)



From now on we use the principal branch of $\sqrt{}$, that is, the negative real axis is the branch cut and $\sqrt{1} = 1$.

(3) b. Show that

$$f(z) = (z+i)\sqrt{\frac{z-i}{z+i}}$$

defines a branch of our function with branch cut $[-i, i]$. What is the value of $f(1)$?

(3) c. Determine the first four terms of the Laurent series of this branch in the annulus $\{z : |z| > 1\}$.
Hint: $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$ if x is real and $|x| < 1$.

(3) d. Calculate

$$\oint_S f(z)\,\mathrm{d}z$$

where S is the square with vertices at $\pm 5 \pm 5i$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 6}{6}$$

and rounded in the standard way.

THE END