Delft University of Technology
Faculty EEMCS
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Exam Complex Analysis (wi4243AP)
Thursday 30 October 2014; 09:00 – 12:00.

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This exam consists of six questions.
Marks per question: see margin.
Resources allowed: calculator

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1. Let α be a complex number such that $\operatorname{Im}\alpha \neq 0$ and consider the bilinear transformation given by

$$w = \frac{z-\alpha}{z-\bar{\alpha}}.$$

(2) a. Show that this transformation maps the real line onto the unit circle.

(2) b. What is the image of the upper half plane under this transformation?

(2) c. How does w traverse the unit circle as z traverses the real line in the positive direction?

(3) d. Now let $\alpha = \frac{1}{2}i$. Determine and sketch the image of the (solid) rectangle with corners at -1, 1, $1+i$ and $-1+i$ under the transformation.

(3) 2. a. Is there an analytic function f whose real part is given by $u(x,y) = \exp(\frac{y}{x})$? Justify your answer.

(3) b. Determine all analytic functions on the half plane $\{z : \operatorname{Re} z > 0\}$ that have $v(x,y) = \ln(x^2+y^2) - x^2 + y^2$ as their imaginary part and write these as functions of z.

3. Let f be an analytic function from the unit disc $D = \{z : |z| \leqslant 1\}$ to itself and let α be such that $|\alpha| < 1$. We consider the Taylor series of f at α, given by $\sum_n a_n(z-\alpha)^n$.

(6) a. Use Cauchy's estimate to show that $|a_n| \leqslant \frac{1}{(1-|\alpha|)^n}$ for all n.

(5) b. Improve the estimate in part a by integrating over the unit circle.

(8) 4. Let a be a real number such that $a > 1$; evaluate the following integral

$$\int_0^{2\pi} \frac{1}{(a^2 - 2a\cos\theta + 1)^2}\,d\theta$$

Give all details.

(8) 5. Let a and b be positive real numbers. Evaluate the following integral

$$\int_0^{\infty} \frac{\sin ax}{x(x^2+b^2)}\,dx$$

Give all details. *Hint*: A principal value will be involved.

6. We consider the many-valued function $w = (z^2-1)^{-\frac{1}{2}}$.

(3) a. Suppose we use the branch of $z \mapsto z^{\frac{1}{2}}$ that has the positive real axis as a branch cut and that satisfies $(-1)^{\frac{1}{2}} = i$. Determine the image of the upper half plane, $\{z : \operatorname{Im} z > 0\}$, under this mapping

This problem continues on the next page.

From now on we use the principal branch of $z \mapsto z^{\frac{1}{2}}$ with the negative real axis as a branch cut.

(3) b. Show that

$$f(z) = \frac{1}{z-1}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}$$

defines a branch of our function with branch cut $[-1, 1]$. What is the value of $f(2)$?

(3) c. Determine the first four terms of the Laurent series of this branch in the annulus $\{z : |z| > 1\}$.
Hint: $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{65}{256}x^5 + \cdots$ if x is real and $|x| < 1$.

(3) d. Calculate

$$\oint_S f(z)\,dz$$

where S is the square with vertices at $\pm 5 \pm 5i$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 6}{6}$$

and rounded in the standard way.

THE END