wi4243AP: Complex Analysis

week 1, Friday

K. P. Hart

Faculty EEMCS
TU Delft

Delft, 5 September, 2014

Delft University of Technology

Outline

1. Section 1.1
   - Complex numbers
2. Section 1.2
   - Algebra
   - Algebra and geometry
3. Section 1.3
   - More geometry

Delft University of Technology

Definition

A *complex number* is an ordered pair (x, y) of real numbers.
We write

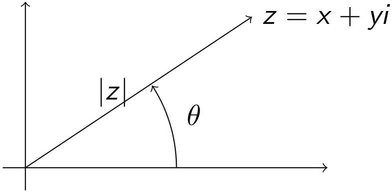
$$z = (x, y) = x(1, 0) + y(0, 1) = x + yi$$

Note: we abbreviate $(1, 0) = 1$ and $(0, 1) = i$ and we shall define multiplication in such a way that $(0, 1)^2 = -(1, 0)$, i.e., $i^2 = -1$.
Notation:

- $\operatorname{Re} z = x$ — real part
- $\operatorname{Im} z = y$ — imaginary part

Representation

Every complex number, $x + yi$, corresponds to a point, (x, y), in the plane.



- $|z| = \sqrt{x^2 + y^2}$; the modulus
- θ; angle, the argument

Representation

We introduce some geometry via polar coordinates.
Properties

- $x = |z| \cos \theta$
- $y = |z| \sin \theta$

There are infinitely many values for θ.
Principal value: $\operatorname{Arg} z$, chosen in interval $(-\pi, \pi]$
Other values: $\arg z = \operatorname{Arg} z + 2k\pi$

Addition and multiplication

Addition: coordinate-wise

$$z + w = x + yi + u + vi = (x + u) + (y + v)i$$

Multiplication (use distributive law and $i^2 = -1$):

$$z \cdot w = (x + yi)(u + vi) = (xu - yv) + (xv + yu)i$$

for example: $(2 + i)(3 + 4i) = (6 - 4) + (8 + 3)i = 2 + 11i$

Delft University of Technology

Division

The formula $(x+iy)(x-iy)=x^2+y^2$ helps with division:

$$\frac{1}{x+iy}=\frac{x-iy}{(x+iy)(x-iy)}=\frac{x-yi}{x^2+y^2}$$

For example

$$\begin{aligned}\frac{16+63i}{3+4i}&=\frac{16+63i}{3+4i}\cdot\frac{3-4i}{3-4i}\\&=\frac{(48+252)+(-64+189)i}{9+16}\\&=\frac{300+125i}{25}=12+5i\end{aligned}$$

TU Delft
Delft University of Technology

Complex conjugate

We write $\overline{x + yi} = x - yi$, the *complex conjugate*.

- Geometrically: reflection in real axis.
- $|z| = |\overline{z}|$ and $z \cdot \overline{z} = |z|^2$
- $\operatorname{Arg} \overline{z} = -\operatorname{Arg} z$ (except when z is real and negative)
- $z + \overline{z} = 2x = 2 \operatorname{Re} z$
- $z - \overline{z} = 2iy = 2i \operatorname{Im} z$
- $\overline{z \pm w} = \overline{z} \pm \overline{w}, \qquad \overline{z \cdot w} = \overline{z} \cdot \overline{w}, \qquad \overline{z/w} = \overline{z}/\overline{w}$

Delft University of Technology

Multiplication

Write $z = r(\cos\varphi + i\sin\varphi)$ and $w = s(\cos\psi + i\sin\psi)$, we get

$$\begin{aligned} zw &= rs(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) \\ &= rs\bigl((\cos\varphi\cos\psi - \sin\varphi\sin\psi) + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi)\bigr) \\ &= rs\bigl(\cos(\varphi+\psi) + i\sin(\varphi+\psi)\bigr). \end{aligned}$$

So *multiply moduli* and *add angles*.

Delft University of Technology

Multiplication: example

Set $z = -1 + i$ en $w = 1 + \sqrt{3}i$.
Then

- $z = \sqrt{2}(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$
- $w = 2(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$
- $zw = (-1 - \sqrt{3}) + (1 - \sqrt{3})i$

And so . . .

Multiplication, example

... we find

$$(-1-\sqrt{3}) + i(1-\sqrt{3}) = 2\sqrt{2}\left(\cos\frac{13}{12}\pi + i\sin\frac{13}{12}\pi\right)$$
$$= 2\sqrt{2}\left(\cos -\frac{11}{12}\pi + i\sin -\frac{11}{12}\pi\right)$$

so we write

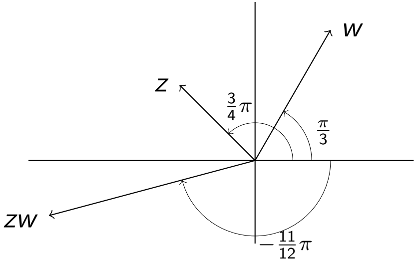
$$\arg zw = \arg z + \arg w$$

but not

$$\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w$$

Delft University of Technology

Multiplication, example



Division

Write $z = r(\cos\varphi + i\sin\varphi)$ and $w = s(\cos\psi + i\sin\psi)$, we get

$$\begin{aligned}
\frac{z}{w} &= \frac{r(\cos\varphi + i\sin\varphi)}{s(\cos\psi + i\sin\psi)} \\
&= \frac{r}{s}\cdot\frac{\cos\varphi + i\sin\varphi}{\cos\psi + i\sin\psi}\cdot\frac{\cos\psi - i\sin\psi}{\cos\psi - i\sin\psi} \\
&= \frac{r}{s}\cdot\frac{(\cos\varphi + i\sin\varphi)(\cos\psi - i\sin\psi)}{\cos^2\psi + \sin^2\psi} \\
&= \frac{r}{s}\bigl(\cos(\varphi - \psi) + i\sin(\phi - \psi)\bigr).
\end{aligned}$$

So *divide moduli* and *subtract angles*.

Delft University of Technology

De Moivre's formula

For any angle θ and any integer n we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Application: easy formulas for $\cos n\theta$ and $\sin n\theta$:

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= \cos^3\theta + 3\cos^2\theta i\sin\theta + 3\cos\theta i^2\sin^2\theta + i^3\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

thanks to the binomial formula

Chebyshev polynomials

Note

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

So $\cos 3\theta = T_3(\cos\theta)$, where $T_3(x) = 4x^3 - 3x$.

T_3 is a *Chebyshev polynomial*; these are used in interpolation and approximation theory.

Chebyshev polynomials

General definition: $\cos n\theta = T_n(\cos\theta)$.

Using De Moivre's identity and the binomial formula we get

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} (1-x^2)^k$$

See Problem 1.18 for an other formula for T_n.

Fractional exponents

What is $(\cos\theta + i\sin\theta)^q$ if q is a (proper) fraction, say $q = \frac{m}{n}$?

Remember: for real positive x, by definition,

$y = x^{\frac{m}{n}}$ is that real positive number with $y^n = x^m$.

In complex numbers: no such choice available, no positive/negative numbers.

Delft University of Technology

Fractional exponents

We solve $z^n = (\cos\theta + i\sin\theta)^m$.

- write $z = r(\cos\varphi + i\sin\varphi)$ ($r = |z|$ and $\varphi = \operatorname{Arg} z$)
- we get $r^n(\cos n\varphi + i\sin n\varphi) = \cos m\theta + i\sin m\theta$
- so $r = 1$ and $\cos n\varphi = \cos m\theta$ and $\sin n\varphi = \sin m\theta$
- we get $n\varphi = m\theta + 2k\pi$ (k an integer)
- as $-\pi < \varphi \leqslant \pi$ there are n suitable values for k
- so: $(\cos\theta + i\sin\theta)^{\frac{m}{n}}$ has n values

Delft University of Technology

Fractional exponents, example

What are the values of $(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i)^{\frac{2}{3}}$?

We have $\theta = \frac{1}{4}\pi$, so we get

$$3\varphi = \frac{1}{2}\pi + 2k\pi \text{ or } \varphi = \frac{1}{6}\pi + \frac{2}{3}k\pi$$

with $k = -1, 0, 1$, to get $-\pi < \varphi \leqslant \pi$.

The values are $-i$ ($\varphi = -\frac{1}{2}\pi$), $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$ ($\varphi = \frac{1}{6}\pi$) and $-\frac{1}{2}\sqrt{3} + \frac{1}{2}i$ ($\varphi = \frac{5}{6}\pi$)

Fractional exponents

Difference between real and complex analysis:

many-valued functions

- $z \mapsto z^{\frac{1}{2}}$ a two-valued function
- $z \mapsto z^{\frac{2}{3}}$ a three-valued function
- $z \mapsto z^{-\frac{2}{5}}$ a five-valued function

Euler's formula (one of the many)

We write (for now as an abbreviation):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

thanks to angle-adding we know

$$e^{i\theta+i\varphi} = e^{i\theta} \cdot e^{i\varphi}$$

(as it should be).

Later we'll see further justification for this formula.

Euler's formula, use

If $|z| = r$ and $\operatorname{Arg} z = \theta$ we write $z = re^{i\theta}$.

We write the values of $z^{\frac{m}{n}}$ as

$$r^{\frac{m}{n}} e^{i(\frac{m}{n}\theta + 2\frac{km}{n}\pi)} \qquad (k = 0, \ldots, n-1)$$

Note: $r \geqslant 0$ so $r^{\frac{m}{n}}$ is unambiguous.
Also note: the values form a regular n-gon with $r^{\frac{m}{n}} e^{i\frac{m}{n}\theta}$ as its 'first' vertex.

Square roots

Using Euler's formula (and because $e^{i\pi} = -1$) the values of $\sqrt{z}$ are

$$\sqrt{r}e^{i\frac{1}{2}\theta} \text{ and } -\sqrt{r}e^{i\frac{1}{2}\theta}$$

Direct calculation: look at $(u+iv)^2 = x + iy$. We get

- $u^2 - v^2 = x$ (real parts)
- $2uv = y$ (imaginary parts)
- $u^2 + v^2 = \sqrt{x^2 + y^2}$ (moduli)

Now solve ...

Square roots

... the first and third to get

- $u^2 = \frac{1}{2}(\sqrt{x^2+y^2}+x)$ or $u = \pm\sqrt{\frac{1}{2}(\sqrt{x^2+y^2}+x)}$
- $v^2 = \frac{1}{2}(\sqrt{x^2+y^2}-x)$ or $v = \pm\sqrt{\frac{1}{2}(\sqrt{x^2+y^2}-x)}$

Of the four combinations only two survive because of $2uv = y$.

Example: $(u+iv)^2 = 5+12i$ gives $u = \pm 3$ and $v = \pm 2$;
we get $3+2i$ and $-3-2i$ because $2uv = 12 > 0$.

Don't remember the formula but do remember the method.

Delft University of Technology

Modulus again

Useful formulas: with $z = x + iy$ we have

- $|z|^2 = x^2 + y^2 = z \cdot \bar{z}$
- $|x|, |y| \leqslant \sqrt{x^2 + y^2} = |z|$ or
- $|\operatorname{Re} z|, |\operatorname{Im} z| \leqslant |z|$

Delft University of Technology

Triangle inequality

Theorem

Given two complex numbers z and w we have

$$\big||z| - |w|\big| \leqslant |z - w| \leqslant |z| + |w|$$

Proof.

- $|z - w|^2 = (z - w)(\bar{z} - \bar{w}) = |z|^2 + |w|^2 - (z\bar{w} + \bar{z}w)$
- $\big(|z| + |w|\big)^2 = |z|^2 + |w|^2 + 2|zw|$
- $-(z\bar{w} + \bar{z}w) = -2\,\mathrm{Re}\, z\bar{w} \leqslant 2|z\bar{w}| = 2|zw|$

This gives the second $\leqslant$. □

TU Delft
Delft University of Technology

Triangle inequality

Theorem

Given two complex numbers z and w we have

$$\bigl||z| - |w|\bigr| \leqslant |z - w| \leqslant |z| + |w|$$

Proof.

The first $\leqslant$ follows from the second:

- $|z| = |z - w + w| \leqslant |z - w| + |w|$ so $|z| - |w| \leqslant |z - w|$
- $|w| = |w - z + z| \leqslant |w - z| + |z|$ so $|w| - |z| \leqslant |w - z|$

Now combine. □

TU Delft
Delft University of Technology

Triangle inequality, use

Useful in over- and underestimating moduli.
Example: we must often estimate the modulus of something like

$$\frac{1}{Re^{it} - z}$$

We know $|Re^{it} - z| \geqslant \big||Re^{it}| - |z|\big| = \big|R - |z|\big|$.
So, if R is large enough (as it usually is):

$$\left|\frac{1}{Re^{it} - z}\right| \leqslant \frac{1}{\big|R - |z|\big|} = \frac{1}{R - |z|}$$

What to do?

From the book: Sections 1.1, 1.2 and 1.3
Suitable exercises: 1.1 – 1.40
Recommended exercises: 1.1, 1.2, 1.6, 1.9, 1.12, 1.13, 1.23, 1.24, 1.25, 1.30, 1.40