wi4243AP: Complex Analysis

week 2, Monday

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Outline

1. 1.3: Geometric properties
   - Roots of unity
   - Cross ratio
   - Symmetry with respect to a circle
2. 1.5: Infinity and the Riemann sphere
3. 8.2:Bilinear transformations
   - Bilinear transforms
   - Preservation of circles
   - Preservation of symmetry

Roots of unity, notation

As we have seen the n values of $1^{\frac{1}{n}}$ are

$$e^{\frac{2k\pi}{n}i} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \qquad (k = 0, 1, \ldots, n-1)$$

We often write $\omega_n = e^{\frac{2\pi}{n}i}$, so that the roots are

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}.$$

- $\omega_2 = e^{\pi i} = -1$
- $\omega_3 = e^{\frac{2}{3}\pi i} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$
- $\omega_4 = e^{\frac{1}{2}\pi i} = i$

Roots of unity, properties

These are the solutions to $z^n - 1 = 0$ or

$$(z-1)(z^{n-1} + \cdots + z + 1) = 0$$

so, . . . ,

$$\omega_n^{n-1} + \cdots + \omega_n + 1 = 0$$

and likewise for ω_n^2, . . . , ω_n^{n-1}.

Fifth roots of unity

We calculate $\omega_5 = \cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$.

Key observations

- $z^4 + z^3 + z^2 + z + 1 = (z - \omega_5)(z - \omega_5^2)(z - \omega_5^3)(z - \omega_5^4)$
- $\omega_5^4 = \overline{\omega_5}$ and $\omega_5^3 = \overline{\omega_5^2}$
- $(z - \omega_5)(z - \omega_5^4) = z^2 - 2\cos\frac{2}{5}\pi z + 1$
- $(z - \omega_5^2)(z - \omega_5^3) = z^2 - 2\cos\frac{4}{5}\pi z + 1$

Fifth roots of unity, continued

Set $a = \cos\frac{2}{5}\pi$ and $b = \cos\frac{4}{5}\pi$ and note that

$$(z^2 - 2az + 1)(z^2 - 2bz + 1) = z^4 + z^3 + z^2 + z + 1$$

and also

$$(z^2-2az+1)(z^2-2bz+1) = z^4-2(a+b)z^3+(2+4ab)z^2-2(a+b)z+1$$

So that $-2(a + b) = 1$ and $2 + 4ab = 1$.

Solution: $\cos\frac{2}{5}\pi = -\frac{1}{4} + \frac{1}{4}\sqrt{5}$ and $\cos\frac{4}{5}\pi = -\frac{1}{4} - \frac{1}{4}\sqrt{5}$.

So ω_5 is ...

Now use $\sin\frac{2}{5}\pi = \sqrt{1-\cos^2\frac{2}{5}\pi}$ to get $\sin\frac{2}{5}\pi = \frac{1}{4}\sqrt{10+2\sqrt{5}}$ and so

$$\omega_5 = -\frac{1}{4} + \frac{1}{4}\sqrt{5} + \frac{i}{4}\sqrt{10+2\sqrt{5}}$$

Bonus

We also get:

$$\cos\frac{1}{5}\pi = -\cos\frac{4}{5}\pi = \frac{1}{4} + \frac{1}{4}\sqrt{5}$$

and

$$\sin\frac{1}{5}\pi = \sin\frac{4}{5}\pi = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$$

Cross ratio

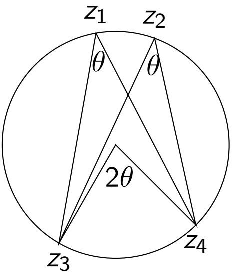
Given four distinct complex numbers z_1, z_2 z_3 and z_4; their *cross ratio* is

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

If the points lie on a circle then their cross ratio is real, and conversely.

Cross ratio

The argument is in this picture



$$\operatorname{Arg} \frac{z_1 - z_3}{z_1 - z_4} = -\theta$$

$$\operatorname{Arg} \frac{z_2 - z_3}{z_2 - z_4} = -\theta$$

$$\operatorname{Arg} \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3} = 0$$

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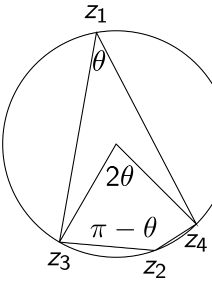
Cross ratio

and in this picture



$$\operatorname{Arg} \frac{z_1 - z_3}{z_1 - z_4} = -\theta$$

$$\operatorname{Arg} \frac{z_2 - z_3}{z_2 - z_4} = \pi - \theta$$

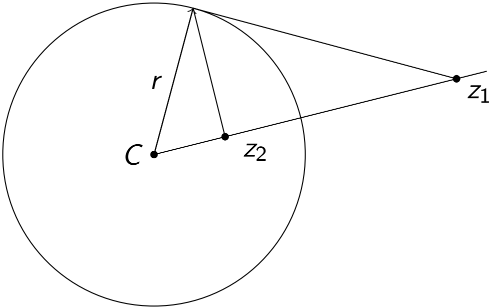
$$\operatorname{Arg} \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3} = -\pi$$

and two more variations.

Definition

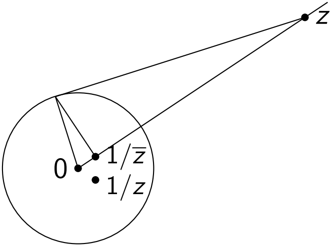
Two points, z_1 and z_2, are *symmetric with repect to the circle with center C and radius r* if

- they are on the same half-line emanating from C
- $|z_1 - C| \cdot |z_2 - C| = r^2$



z and $1/z$

If $z \neq 0$ then z and $1/\overline{z}$ are symmetric with respect to the *unit circle* — the circle with center 0 and radius 1.



So $1/z$ is the complex conjugate of the symmetry point of z with respect to the unit circle.

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Infinity

We add an extra point to the complex plane, that we call ∞.
It will make lots of formulas and formulations nicer.
Arithmetic for the new point:

- $z + \infty = \infty + z = \infty$ (all complex z)
- $z \times \infty = \infty \times z = \infty$ (all complex $z \neq 0$)
- $z/0 = \infty$ (all complex $z \neq 0$)
- $z/\infty = 0$ (all complex z)
- $\infty/z = \infty$ (all complex $z \neq 0$)

Still undefined: $\infty - \infty$, $0 \times \infty$, $0/0$, ∞/∞, ...

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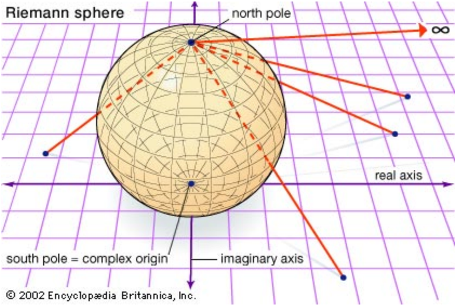
The Riemann Sphere

Here is a picture of how ∞ is attached to $\mathbb{C}$:



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The Riemann Sphere

The equation of the sphere is as in the book:

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}$$

see page 31.

The basic properties are:

Circles and straight lines all become circles on the Riemann sphere; straight lines become circles that pass through ∞ (the North Pole).

This will become important in a few moments when we discuss bilinear transformations.

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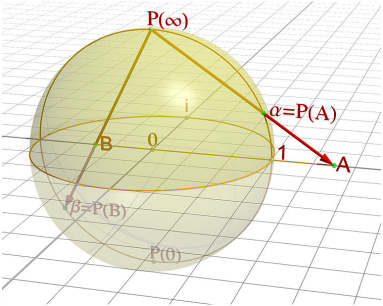
The Riemann Sphere

Here is another way of attaching ∞ to $\mathbb{C}$:



The formules change a bit but the remarks above concerning lines and circles remain valid.

Definition

A bilinear transformation or Möbius transformation is a map $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{az + b}{cz + d}$$

with a, b, c and d complex numbers such that $ad - bc \neq 0$.

Basic examples: $z \mapsto \frac{1}{z}$, $z \mapsto az$ and $z \mapsto z + b$.

Every bilinear transformation is a composition of these (page 376).

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Connection with matrices

Every invertible 2×2-matrix determines a bilinear transformation.
If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

determine $f(z)$ and $g(z)$ respectively then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

determines $f\bigl(g(z)\bigr)$ (write out the formulas).

Inverse

Easy consequence: each bilinear transformation is bijective as a map from the Riemann sphere to itself.
The inverse of

$$z \mapsto \frac{az+b}{cz+d}$$

is given by

$$z \mapsto \frac{dz-b}{-cz+a}$$

Note: the factor $1/(ad-bc)$ cancels.

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Cross ratio

The cross ratio determines a bilinear transform:

$$z \mapsto \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

It maps z_2 to 1, z_3 to 0, and z_4 to ∞
(and the circle through z_2, z_3 and z_4 onto the real axis).

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Other cross ratios

If one of z_2, z_3 or z_4 equals ∞ we can make suitable bilinear transforms too.

If $z_2 = \infty$ use

$$z \mapsto \frac{z - z_3}{z - z_4}$$

If $z_3 = \infty$ use

$$z \mapsto \frac{z_2 - z_4}{z - z_4}$$

If $z_4 = \infty$ use

$$z \mapsto \frac{z - z_3}{z_2 - z_3}$$

Common abbreviation: (z, z_2, z_3, z_4).

Mapping triples to triples

Given two triples, (z_2, z_3, z_4) and (w_2, w_3, w_4) of points on the Riemann sphere there is one bilinear transform M that maps z_i to w_i for all i.

To find it first take the transform that maps (z_2, z_3, z_4) to $(1, 0, \infty)$ and the inverse of the transform that maps (w_2, w_3, w_4) to $(1, 0, \infty)$.

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Mapping triples to triples

Why only one?

Count the number of fixed points:

$$z = \frac{az+b}{cz+d} \text{ iff } cz^2 + (d-a)z - b = 0$$

So

- two fixed points if $c \neq 0$
- one fixed point if $c = 0$ and $d \neq a$
- no fixed point if $c = 0$, $a = d$ and $b \neq 0$
- only fixed points $c = 0$, $a = d$ and $b = 0$ (and so $Mz = z$).

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Mapping triples to triples

So, the possibilities are: 0, 1, 2, all.
Hence if M has three or more fixed points then $Mz = z$.

If S and T both map (z_2, z_3, z_4) to (w_2, w_3, w_4) then $T^{-1}S$ has three fixed points, hence $T^{-1}Sz = z$ or $Sz = Tz$ for all z.

Example

$(i, 0, \infty) \mapsto (1+i, 0, 1-i)$.

First $(i, 0, \infty) \mapsto (1, 0, \infty)$, via $z \mapsto z/i$.

Second $(1+i, 0, 1-i) \mapsto (1, 0, \infty)$, via

$$w \mapsto \frac{(w-0)(1+i-(1-i))}{(w-(1-i))(1+i-0)} = \frac{2iw}{(1+i)w-2}$$

and take its inverse: $z \mapsto \dfrac{-2z}{-(1+i)z+2i}$.

Finally take the composition: $z \mapsto \dfrac{2iz}{(i-1)z+2i}$

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Circles

Given a bilinear transform M and a circle or a line C, the image of C is a circle or a line.

Argument: take three points z_2, z_3 and z_4 on C then M is the composition of the transform that takes (z_2, z_3, z_4) to $(1, 0, \infty)$ and the one that takes $(1, 0, \infty)$ to (Mz_2, Mz_3, Mz_4).

This composition takes C first to the real axis and then to the circle or line determined by Mz_2, Mz_3 and Mz_4.

Complex conjugates

Assume z_2, z_3 and z_4 lie on unit circle; so $z \mapsto (z, z_2, z_3, z_4)$ maps the unit circle onto the real axis.

What can we say about w and z if $(w, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$?

Do the calculations (and use that $\overline{z}_i = 1/z_i$):

$$\begin{aligned}
\frac{(\overline{z} - \overline{z}_3)(\overline{z}_2 - \overline{z}_4)}{(\overline{z} - \overline{z}_4)(\overline{z}_2 - \overline{z}_3)} &= \frac{(\overline{z} - 1/z_3)(1/z_2 - 1/z_4)}{(\overline{z} - 1/z_4)(1/z_2 - 1/z_3)} \times \frac{z_3 z_2 z_4}{z_4 z_2 z_3} \\
&= \frac{(z_3\overline{z} - 1)(z_4 - z_2)}{(z_4\overline{z} - 1)(z_3 - z_2)} \times \frac{\overline{z}}{\overline{z}} \\
&= \frac{(1/\overline{z} - z_3)(z_2 - z_4)}{(1/\overline{z} - z_4)(z_2 - z_3)}
\end{aligned}$$

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Symmetry with respect to a circle

So, because bilinear transforms are bijective:

$$(w, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \text{ iff } w = \frac{1}{\bar{z}}$$

that is, $(w, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$ iff w and z are *symmetric with respect to the unit circle*.
We deduce: bilinear transformations preserve symmetry with respect to circles and lines.

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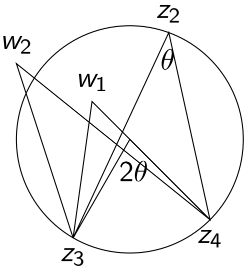
Orientation

Consider this picture again



$$\operatorname{Arg}\frac{w_1-z_3}{w_1-z_4} > \theta$$

$$\operatorname{Arg}(w_1, z_2, z_3, z_4) > 0$$

$$\operatorname{Arg}\frac{w_2-z_3}{w_2-z_4} < \theta$$

$$\operatorname{Arg}(w_2, z_2, z_3, z_4) < 0$$

Orientation

Remember: the cross-ration (z, z_2, z_3, z_4) sends (z_2, z_3, z_4) to $(1, 0, \infty)$ and the circle through them to the real line.

The inside, to the right of (z_2, z_3, z_4), goes to the upper half plane, to the right of $(1, 0, \infty)$.

Thus, Moebius transforms preserve orientation.
They also preserve perpendicularity.

What to do?

From the book: Sections 1.3, 1.5 and 8.2
Suitable exercises: 1.29 – 1.40, 1.47 – 1.49; 8.21 – 8.34
Recommended exercises: 1.30, 1.31, 1.36, 1.37, 1.40, 1.47, 1.48; 8.23, 8.24, 8.25

You should now be able to do Problem 1 on each of last year's exams (they're on Blackboard).

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