wi4243AP/wi4244AP: Complex Analysis

week 2, Friday

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Delft, 12 September, 2014

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Outline

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Monday's Example

$(i, 0, \infty) \mapsto (1 + i, 0, 1 - i)$.

First: $(i, 0, \infty) \mapsto (1, 0, \infty)$, via $z \mapsto z/i$.

Second: $(1 + i, 0, 1 - i) \mapsto (1, 0, \infty)$, via

$$w \mapsto \frac{(w - 0)(1 + i - (1 - i))}{(w - (1 - i))(1 + i - 0)} = \frac{2iw}{(1 + i)w - 2}$$

and take its inverse: $z \mapsto \dfrac{-2z}{-(1 + i)z + 2i}$.

Finally take the composition: $z \mapsto \dfrac{2iz}{(i - 1)z + 2i}$

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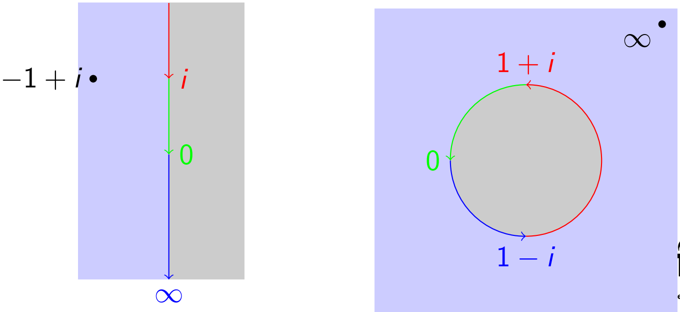
Monday's Example

There was an error on the board: the point that is mapped to ∞ is actually

$$\frac{-2i}{-1+i} = -1+i$$

Here is the correct picture



And so ...

You should now really be able to do Problem 1 on each of last year's exams (they're on Blackboard).

Notation

- $N(z_0, \varepsilon)$, the ε-neighbourhood of z_0; it is $\{z : |z - z_0| < \varepsilon\}$
- $N'(z_0, \varepsilon)$, the reduced/deleted ε-neighbourhood of z_0; it is $N(z_0, \varepsilon) \setminus \{z_0\} = \{z : 0 < |z - z_0| < \varepsilon\}$

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Kinds of points

Let S be a set in $\mathbb{C}$ and z a point in $\mathbb{C}$; we say z is

- an interior point of S if there is $\varepsilon > 0$ such that $N(z, \varepsilon) \subseteq S$
- a boundary point of S if for *every* $\varepsilon > 0$ the set $N(z, \varepsilon)$ contains points of S and its complement
- a limit or accumulation point of S if for *every* $\varepsilon > 0$ the set $N'(z, \varepsilon)$ contains points of S

For example: consider the set $\{z : |z| \leqslant 1\}$. The point $\frac{1}{3} + \frac{1}{3}i$ is an interior point; $\frac{3}{5} + \frac{4}{5}i$ is a boundary point and also an accumulation point; $1 + i$ is none of the above, it is an exterior point.

Open and closed sets

Let S be a set in $\mathbb{C}$, we say S is

- open if every point of S is also an interior point of S
- closed if its complement $\mathbb{C} \setminus S$ is open

Rule-of-thumb: sets defined by strict inequalities are open; sets defined using $\leqslant$ are closed; sets defined using both are neither open nor closed.

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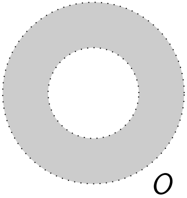
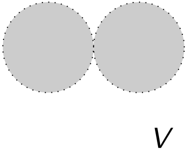
Examples

- $S = \{z : |z| < 1\}$ is open:
  if $z \in S$ put $\varepsilon = 1 - |z|$, then $N(z, \varepsilon) \subseteq S$
- $D = \{z : |z| \leqslant 1\}$ is closed:
  if $z \notin D$ put $\varepsilon = |z| - 1$, then $N(z, \varepsilon) \subseteq \mathbb{C} \setminus D$
- $L = \{z : |z| < 1, \operatorname{Re} z \geqslant 0\}$ is neither open nor closed:
  - not open: $\frac{1}{2}i \in L$ is not an interior point
  - not closed: $\frac{4}{5} + \frac{3}{5}i \notin L$ but it is not an interior point of $\mathbb{C} \setminus L$

Connectedness

- An open set U is connected if any two points can be connected by an arc within U.
- The open set $O = \{z : 1 < |z| < 2\}$ is connected.
- The open set $V = \{z : |z - 1| < 1 \text{ or } |z + 1| < 1\}$ is not: no arc from -1 to 1 stays inside V.

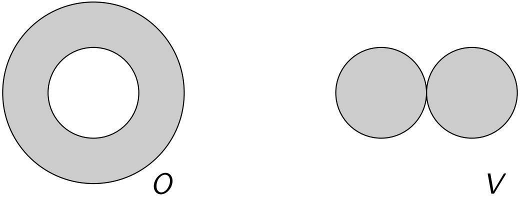




Domains

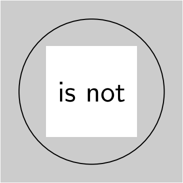
- A connected open set is often called an open domain
- Adding all limit points yields its closure; a closed domain

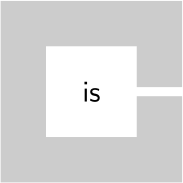
The closure of O is a closed domain, that of V is not.



Simply connected

An open domain is simply connected if every closed curve can be shrunk to a point, inside the domain.





How to draw pictures

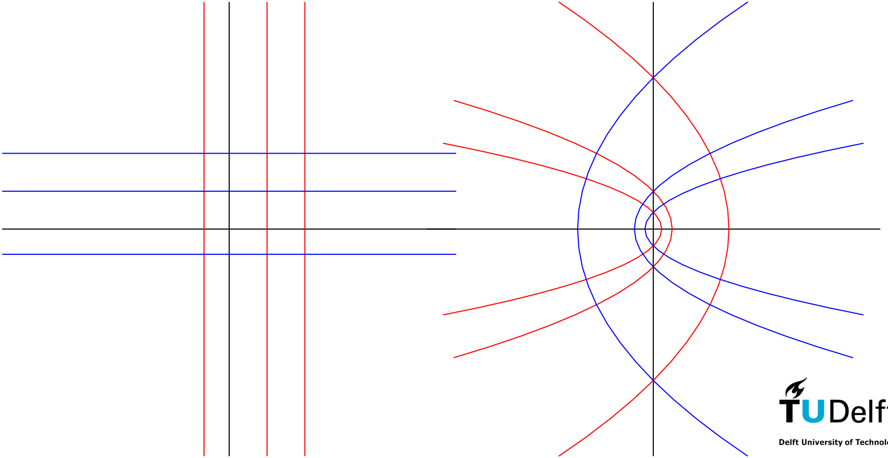
Graphs of complex functions are part of four-dimensional space.

This makes sketching them somewhat difficult.

Usually we draw pictures of images of figures to see how a complex functions works.

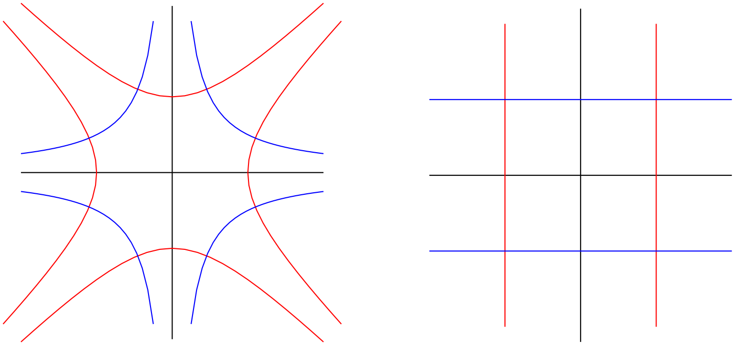
Example: $w = z^2$

One thing one can do: sketch images of horizontal and vertical lines.



Example: $w = z^2$

Or: draw preimages of horizontal and vertical lines

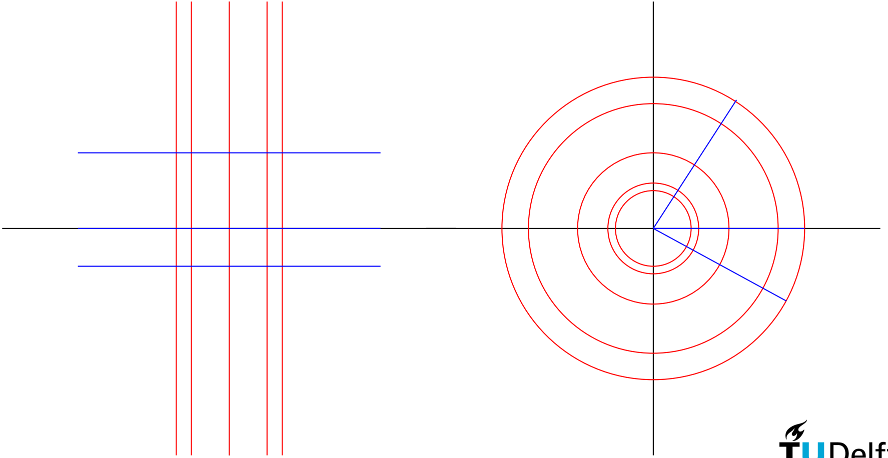


Exponential function

We define

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

Mapping behaviour



$\sin z$ and $\cos z$

Use Euler's formulas:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \text{ and } \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

We use

$$e^{iz} = (\cos x + i \sin x)e^{-y}$$

and

$$e^{-iz} = (\cos x - i \sin x)e^{y}$$

to get ...

sin z and cos z

… the following identities

$$\begin{aligned}\cos z &= \frac{1}{2}(\cos x\, e^{-y} + \cos x\, e^{y}) + \frac{i}{2}(\sin x\, e^{-y} - \sin x\, e^{y}) \\ &= \cos x \cosh y - i \sin x \sinh y\end{aligned}$$

and

$$\begin{aligned}\sin z &= \frac{1}{2i}(\cos x\, e^{-y} - \cos x\, e^{y}) + \frac{i}{2i}(\sin x\, e^{-y} + \sin x\, e^{y}) \\ &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

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Definition

As in the case of real functions: f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and we denote that limit by $f'(z_0)$ of course.

Examples

$\bar{z}$ is nowhere differentiable

$$\lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} = \lim_{(x,y) \to (x_0,y_0)} \frac{(x - x_0) - i(y - y_0)}{(x - x_0) + i(y - y_0)}$$

does not exist:
1 along the horizontal line $y = y_0$;
-1 along the vertical line $x = x_0$.

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Examples

$|z|^2$ is differentiable at 0 (the limit is zero) but nowhere else:

$$\begin{aligned}\frac{|z|^2 - |z_0|^2}{z - z_0} &= \frac{z\overline{z} - z_0\overline{z} + z_0\overline{z} - z_0\overline{z}_0}{z - z_0} \\ &= \frac{\overline{z}(z - z_0) + z_0(\overline{z} - \overline{z}_0)}{z - z_0} \\ &= \bar{z} - z_0\frac{\bar{z} - \bar{z}_0}{z - z_0}\end{aligned}$$

now use the same argument as for $z \mapsto \overline{z}$: now the limits are $\overline{z}_0 + z_0$ (horizontally) and $\overline{z}_0 - z_0$ (vertically)

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e^z

If all is as it should be we should have

$$\lim_{z\to z_0} \frac{e^z - e^{z_0}}{z - z_0} = e^{z_0}$$

for all z_0.
Note

$$\lim_{z\to z_0} \frac{e^z - e^{z_0}}{z - z_0} = e^{z_0} \lim_{z\to z_0} \frac{e^{z-z_0} - 1}{z - z_0}$$

so we ask

$$\lim_{z\to 0} \frac{e^z - 1}{z} \stackrel{?}{=} 1$$

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e^z, the hard way

Well, ..., if we write $z = x + iy$ we get, after some rewriting:

$$\begin{aligned}\frac{e^z - 1}{z} &= \frac{(e^x \cos y - 1) + ie^x \sin y}{x + iy} \\ &= \frac{((e^x \cos y - 1) + ie^x \sin y)(x - iy)}{x^2 + y^2} \\ &= \frac{x(e^x \cos y - 1) + ye^x \sin y}{x^2 + y^2} \\ &\qquad + i\frac{xe^x \sin y - y(e^x \cos y - 1)}{x^2 + y^2}\end{aligned}$$

and now let $x, y \to 0$

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e^z, the hard way

How to tackle such a limit?

Taylor to the rescue:

- $e^x \approx 1 + x + \frac{1}{2}x^2$
- $\cos y \approx 1 - \frac{1}{2}y^2$
- $\sin y \approx y - \frac{1}{6}y^3$

Stick this into the fractions and, …, done.

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Real differentiability

Consider a complex function f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$; then it is real differentiable at (x_0, y_0) if there is a 2×2-matrix A such that

$$f(x,y) - f(x_0,y_0) = A\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + R(x,y)$$

where $\lim_{z\to z_0} \frac{\|R(x,y)\|}{\|z-z_0\|} = 0$
Remember

$$A = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix}$$

where $f = u + iv$.

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Complex differentiability

Complex differentiability can be restated in a similar fashion

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + R(z)$$

where $\lim_{z \to z_0} \frac{R(z)}{z - z_0} = 0$.

If $f'(z_0) = a + ib$ then $f'(z_0)(z - z_0)$ can be expanded as

$$\bigl(a(x - x_0) - b(y - y_0)\bigr) + i\bigl(b(x - x_0) + a(y - y_0)\bigr)$$

we can recast this as a matrix multiplication.

Complex versus Real

Complex:

$$f(z) - f(z_0) \approx \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

Real:

$$f(z) - f(z_0) \approx \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

The crucial point is the shape of the matrix in the complex case.
This leads to . . .

Cauchy-Riemann equations

f is complex differentiable at z_0 if it is real differentiable there and

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } v_x(x_0, y_0) = -u_y(x_0, y_0)$$

these are the Cauchy-Riemann equations.

The function $\bar{z}$ is real differentiable with

$$u_x = 1,\ u_y = 0,\ v_x = 0 \text{ and } v_y = -1$$

and so nowhere complex differentiable.

e^z, the easy way

With $f(z) = e^z$ we have $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Clearly f is real differentiable and the matrix of derivatives at z_0 is

$$\begin{pmatrix} e^{x_0} \cos y_0 & -e^{x_0} \sin y_0 \\ e^{x_0} \sin y_0 & e^{x_0} \cos y_0 \end{pmatrix}$$

This is of the right form and it represents multiplication by e^{z_0}.

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What to do?

From the book: Sections 1.4; 2.1, 2.3 and 2.4
Suitable exercises: 1.42 – 1.46; 2.1 – 2.4, 2.12 – 2.22
Recommended exercises: 1.43, 1.44, 1.45; 2.2, 2.3, 2.15, 2.19, 2.22