wi4243AP/wi4244AP: Complex Analysis

week 3, Monday

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Outline

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Changing variables

We have $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$, so f can also be considered as a function of z and $\bar{z}$.

Apply the multi-variable chain-rule:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}$$

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Changing variables

Use real and imaginary parts and the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + iv_x) - \frac{1}{2i}(u_y + iv_y) = \frac{1}{2}(u_x - v_y) - \frac{1}{2i}(v_x + u_y) = 0$$

So, ..., f is (complex) differentiable iff

$$\frac{\partial f}{\partial \bar{z}} = 0$$

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Analytic functions

Definition

A function f is analytic at z_0 if it is differentiable on some neighbourhood $N(z_0, \varepsilon)$ of z_0.

- f is then also analytic at all points of $N(z_0, \varepsilon)$
- the domain of an analytic function is open
- $f(z) = |z|^2$ is differentiable at 0 but not analytic

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Orthogonal curves

Useful fact:
if $f = u + iv$ is analytic then the level curves $u(x, y) = \alpha$ and $v(x, y) = \beta$ are always orthogonal.

Use implicit differentiation

- $\mathrm{d}y/\mathrm{d}x = -u_x/u_y$ on level curves of u
- $\mathrm{d}y/\mathrm{d}x = -v_x/v_y$ on level curves of v

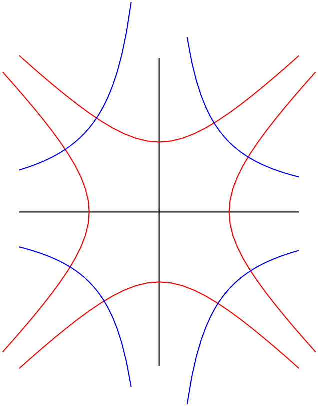
Apply Cauchy-Riemann equations:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_u \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_v = \left(-\frac{u_x}{u_y}\right) \cdot \left(-\frac{v_x}{v_y}\right) = -\frac{u_x}{u_y} \cdot \frac{u_y}{u_x} = -1$$

So there.

Example: z^2

Because $z^2 = x^2 - y^2 + 2xyi$ we have the level curves of



$u = x^2 - y^2$ (red)

$v = 2xy$ (blue) .

Harmonic functions

Definition

A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* on a domain D if it is twice differentiable and $\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$ on all of D

If $f = u + iv$ is analytic then u and v are harmonic.

- $u_{xx} = v_{yx}$ and $v_{xx} = -u_{yx}$
- $u_{yy} = -v_{xy}$ and $v_{yy} = u_{xy}$

Now add.

We will see later that all these derivatives actually exist

Harmonic conjugate

If u and v are harmonic and such that $f = u + iv$ is analytic then v is a *harmonic conjugate* of u.

Equivalently: u and v satisfy $u_x = v_y$ and $u_y = -v_x$ — the Cauchy-Riemann equations.

This is not symmetric: u is not a conjugate of v.

$z^2 = x^2 - y^2 + 2xyi$ is analytic but $i\bar{z}^2 = 2xy + (x^2 - y^2)i$ is not.

Finding harmonic conjugates

If u is harmonic is there a harmonic conjugate?

On simply connected domains: yes.

How to find it?

Force the Cauchy-Riemann equations to hold: v can be written as

$$v(x,y) = \int -u_y \, \mathrm{d}x \text{ and } v(x,y) = \int u_x \, \mathrm{d}y$$

Just try this!

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Example: $\frac{1}{2}\ln(x^2+y^2)$

$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$ is harmonic on the right-hand half plane (check)

We have

$$u_x = \frac{x}{x^2+y^2} \text{ and } u_y = \frac{y}{x^2+y^2}$$

Integrate

$$v(x,y) = -\int \frac{y}{x^2+y^2}\,\mathrm{d}x \text{ and } v(x,y) = \int \frac{x}{x^2+y^2}\,\mathrm{d}y$$

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Example: $\frac{1}{2}\ln(x^2+y^2)$

We get

$$v(x,y) = \int \frac{x}{x^2+y^2}\,\mathrm{d}y = \arctan\frac{y}{x} + h_1(x)$$

and

$$v(x,y) = -\int \frac{y}{x^2+y^2}\,\mathrm{d}x = -\arctan\frac{x}{y} + h_2(y)$$

Remember: if $x>0$ then $\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$ and if $x<0$ then $\arctan x + \arctan\frac{1}{x} = -\frac{\pi}{2}$

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Example: $\frac{1}{2}\ln(x^2+y^2)$

Since $\arctan\frac{y}{x}$ and $-\arctan\frac{x}{y}$ differ by a constant we conclude that

$$v(x,y)=\arctan\frac{y}{x}+c$$

on the right-hand half plane.

Example: $\frac{1}{2}\ln(x^2+y^2)$

Note: $\frac{1}{2}\ln(x^2+y^2)$ is harmonic on the whole complex plane, except at $(0,0)$; can we define $v(x,y)$ everywhere too?
Suppose we want $v(1,1)=\frac{\pi}{4}$

We must choose $v(x,y)=\arctan\frac{y}{x}$ on the right-hand half plane

On the upper half plane we (must) take $v(x,y)=-\arctan\frac{x}{y}+\frac{\pi}{2}$

On the lower half plane we must take $v(x,y)=-\arctan\frac{x}{y}-\frac{\pi}{2}$ (because $v(1,-1)=-\frac{\pi}{4}$)

But now we cannot get past the negative real axis.
So: simple connectivity is necessary.

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We look at e^z

Remember

Definition

If $z = x + iy$ then, by definition,

$$e^z = e^x(\cos y + i \sin y)$$

- $\operatorname{Re} e^z = e^x \cos y$ and $\operatorname{Im} e^z = e^x \sin y$
- $|e^z| = e^x$ and $\arg e^z = y$
- $e^z e^w = e^{z+w}$
- $e^{z+2\pi i} = e^z$

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e^z is an entire function

We have seen: e^z is real differentiable everywhere and

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

so, by the C-R equations, it is complex differentiable everywhere.
It is an entire function (analytic on the whole complex plane).

$(e^z)' = e^z$: the matrix on the right represents multiplication by e^z.

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It is the only sensible choice

Theorem

$f(z) = e^z$ *is the only function that satisfies*

1. *it is entire*
2. $f'(z) = f(z)$
3. $f(0) = 1$

See the book for a derivation.

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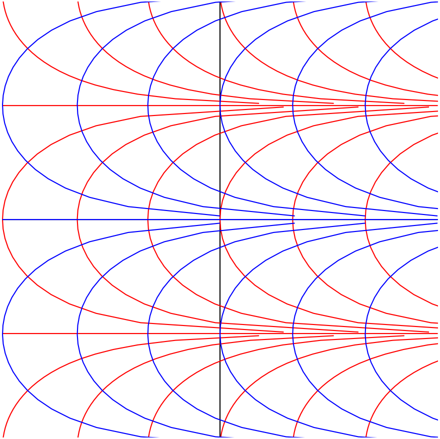
Mapping behaviour

Level curves of $e^x \cos y$ (red) and $e^x \sin y$ (blue)



$\sin z$ and $\cos z$

Definition

We use the Euler formulas to define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Compare with hyperbolic functions:

$$\sin z = \frac{1}{i}\sinh iz \text{ and } \cos z = \cosh iz$$

or

$$\sinh z = \frac{1}{i}\sin iz \text{ and } \cosh z = \cos iz$$

Formulas

Use addition formulas:

$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

and

$$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y$$

After some manipulations:

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y} \quad \text{and} \quad |\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$$

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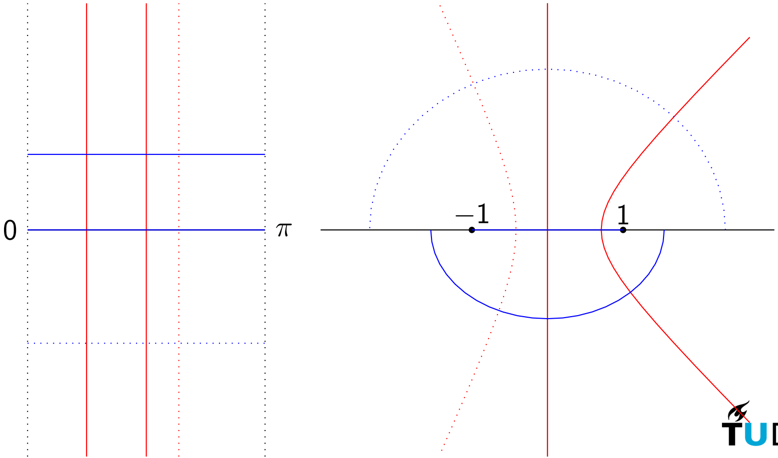
Mapping behaviour of $\cos z$

Consider $w = \cos z$, so $u = \cos x \cosh y$ and $v = -\sin x \sinh y$.

If $x = a$ is fixed then $\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1$ (hyperbola in the w-plane)

If $y = b$ is fixed then $\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$ (ellipse in the w-plane)

Mapping behaviour of $\cos z$



What is $\log z$?

Definition

We write $w = \log z$ to mean that $e^w = z$.

Solve this using modulus, $|e^w| = e^u$, and argument, $\arg e^w = v$.

We get

- $e^u = |z|$, or $u = \ln |z|$ (one value)
- $v = \arg z$ (many values ...)

So, $\log z = \ln |z| + i \operatorname{Arg} z + 2k\pi i$ is many-valued.

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What is Log z?

Definition

The principal branch of the logarithm is Log z, given by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z \qquad (-\pi < \operatorname{Arg} z \leqslant \pi)$$

it is a single-valued function.

Properties

- $z = e^{\operatorname{Log} z}$ is true
- $z = \operatorname{Log} e^z$ is true only if $-\pi < y \leqslant \pi$
- $z = e^{\log z}$ is true, because of periodicity
- $z = \log e^z$ does not make sense (too many right-hand sides)

Derivative

Let's differentiate $\operatorname{Log} z$.

- $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$, so $u_x = \frac{x}{x^2+y^2}$ and $u_y = \frac{y}{x^2+y^2}$
- $v(x,y) = \arctan\frac{y}{x}(\pm\pi)$, so $v_x = -\frac{y}{x^2+y^2}$ and $v_y = \frac{x}{x^2+y^2}$

The matrix

$$\frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

represents multiplication by $\frac{x-iy}{x^2+y^2} = \frac{1}{z}$.

So, yes: $(\log z)' = \frac{1}{z}$.

Heat

Consider the upper half plane and heat up the positive real axis to U Kelvin; keep the negative real axis at absolute zero.

What's the temperature distribution, $T(x, y)$?

$T(x, y)$ is harmonic, $T(x, 0) = U$ $(x > 0)$, $T(x, 0) = 0$ $(x < 0)$.

Take $T(x, y) = \frac{U}{\pi}(\pi - \operatorname{Arg} z)$; it's harmonic and it does the trick.

Later: this is the only solution.

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Heat

Consider the unit disc heat up the top half of the boundary to U Kelvin; keep the other half at absolute zero.

What's the temperature distribution, $T(x, y)$?

Map the disc onto the upper half plane by

$$w = \frac{1}{i}\frac{z-1}{z+1} = \frac{2y - i(x^2 + y^2 - 1)}{(x+1)^2 + y^2}$$

Check that $(1, i, -1) \mapsto (0, 1, \infty)$.

Heat

Now

$$T(x,y) = \frac{U}{\pi}\left(\pi - \operatorname{Arg}\left(\frac{1}{i}\frac{z-1}{z+1}\right)\right)$$

is the solution we seek.

In terms of x and y:

$$T(x,y) = \frac{U}{\pi}\left(\pi - \arctan\left(\frac{x^2+y^2-1}{-2y}\right)\right)$$

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What to do?

From the book: 2.4, 2.5, 2.6, 3.1, 3.2, 3.3.
Suitable exercises: 2.30 - 2.44; 3.1 –3.24
Recommended exercises: 2.30, 2.31, 2.32, 2.34, 2.35; 3.1, 3.4, 3.5, 3.6, 3.10, 3.12, 3.15 (choice), 3.17 (choice), 3.19, 3.23.

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