wi4243AP/wi4244AP: Complex Analysis

week 3, Friday

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Outline

1. 3.4: Inverse trigonometric/hyperbolic functions
2. 3.5: Exponential and power functions
   - Exponential functions
   - Power functions
3. 3.6: Branch points, branch cuts
   - Branch points
   - Branch cuts
   - An example

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The arctangent function

Wat does the complex arctan look like?
Remember: tan is periodic, with period π.
So if $z = \tan w$, then $z = \tan(w + \pi)$, $z = \tan(w - \pi)$, ...
Hence arctan z has many values: w, $w + \pi$, $w - \pi$, ...

The arctangent function

Let's solve $z = \tan w$ for w:
Start with

$$z = \frac{\sin w}{\cos w} = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}$$

and multiply numerator and denominator by e^{iw}:

$$z = \frac{1}{i}\frac{e^{2iw} - 1}{e^{2iw} + 1} \text{ or } iz = \frac{e^{2iw} - 1}{e^{2iw} + 1}$$

Solve for e^{2iw}:

$$e^{2iw} = \frac{1 + iz}{1 - iz}$$

Thus we find . . .

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The arctangent function

. . . upon taking the logarithm

$$2iw = \log\left(\frac{1+iz}{1-iz}\right)$$

and so

$$w = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}\,\mathrm{Log}\left(\frac{1+iz}{1-iz}\right) + k\pi$$

And yes, the values differ by integer multiples of π

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a^z

We know e^z.
What about 2^z? Or even: 1^z?
In the $\mathbb{R}$eal world: $a^x = e^{x \ln a}$.
In the $\mathbb{C}$omplex world: $a^z = e^{z \log a}$.

Many functions: one for each value of $\log a$.

2^z

So 2^z can be

- $e^{z \ln 2}$
- $e^{z \ln 2 + 2\pi i z}$
- $e^{z \ln 2 - 2\pi i z}$
- ...

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1^z

And 1^z can be

- $e^{z \ln 1} = e^0 = 1$ (constant)
- $e^{2\pi i z}$ (not constant)
- $e^{-2\pi i z}$ (also not constant)
- ...

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z^a, integer a

Integer a: unambiguously defined.
If $a > 0$:

$$z^a = \underbrace{z \times z \times \cdots \times z}_{a \text{ times}}$$

If $a < 0$:

$$z^a = \frac{1}{z^{-a}}$$

and, of course,

$$z^0 = 1$$

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z^a, rational a

Rational a: many-valued (but finitely many):

$$z^{\frac{1}{n}} = |z|^{\frac{1}{n}} \times e^{\frac{\operatorname{Arg} z}{n} i} \times e^{\frac{2k\pi}{n} i}$$

one value for each of $k = 0, 1, \ldots, n-1$.

If $a = \frac{m}{n}$, with $\gcd(m, n) = 1$, then

$$z^a = (z^{\frac{1}{n}})^m$$

again n different values.

z^a, real a

In the real case we define $z^a = e^{a \ln z}$ for *positive* z and arbitrary a.
If a is real and z is complex we use the complex logarithm:

$$z^a = e^{a \log z} = e^{a \ln |z| + ia \arg z} = |z|^a \cdot e^{ia \arg z}$$

Note: $|z^a| = |z|^a$, so "taking powers of moduli" still works.
Similarly: $\arg z^a = a \cdot \arg z$, so "taking multiples of angles" still works.
We actually have $|z|^a \cdot e^{ia(\operatorname{Arg} z + 2k\pi)}$, infinitely many values.

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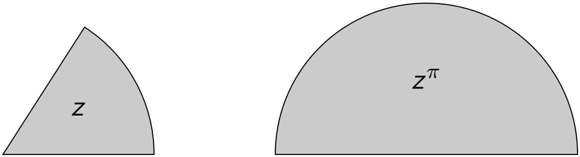
z^a, real a

These functions are useful if you want to smooth out corners in a domain:



The angle on the left is 1 radian.

z^a, arbitrary a

Other $a = \alpha + i\beta$: many-valued (infinitely many):

$$\begin{aligned} z^a = e^{a \log z} &= e^{(\alpha+i\beta)(\ln|z|+i\arg z)} \\ &= e^{\alpha \ln|z|-\beta \arg z} \cdot e^{i(\alpha \arg z+\beta \ln|z|)} \\ &= e^{\alpha \ln|z|-\beta(\operatorname{Arg} z+2k\pi)} \cdot e^{i(\alpha(\operatorname{Arg} z+2k\pi)+\beta \ln|z|)} \end{aligned}$$

one value for each $k \in \mathbb{Z}$.

Note the $\beta \ln|z|$; this will turn a straight line through the origin into a spiral: if $z = re^{i\theta}$, with θ fixed, then $z^a = r^\alpha \cdot e^{-\beta\theta} \cdot e^{i\alpha\theta} \cdot e^{i\beta \ln r}$.

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i^i

Famous example (Euler).
Note

$$\log i = \frac{1}{2}\pi i + 2k\pi i$$

so

$$i^i = e^{i \log i} = e^{i(\frac{1}{2}\pi i + 2k\pi i)} = e^{-\frac{1}{2}\pi - 2k\pi}$$

one (real) value for each integer k

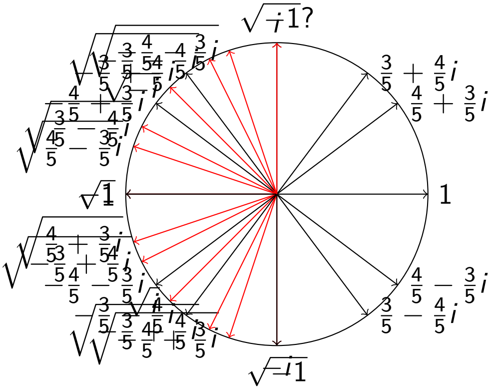
$\sqrt{z}$

The function $\sqrt{z}$ is two-valued:

$$\sqrt{|z|}e^{\frac{1}{2}i\operatorname{Arg} z} \text{ and } \sqrt{|z|}e^{\frac{1}{2}i\operatorname{Arg} z+i\pi} = -\sqrt{|z|}e^{\frac{1}{2}i\operatorname{Arg} z}$$

Choose a value for $\sqrt{-1}$, say $-\sqrt{1}e^{\frac{1}{2}\pi i} = -i$ (the second choice). Walk clockwise along the circle given by $|z| = 1$, retaining this choice.

$\sqrt{z}$


$\frac{3}{5} + \frac{4}{5}i$
$\frac{4}{5} + \frac{3}{5}i$
1
$\frac{4}{5} - \frac{3}{5}i$
$\frac{3}{5} - \frac{4}{5}i$

$\sqrt{z}$

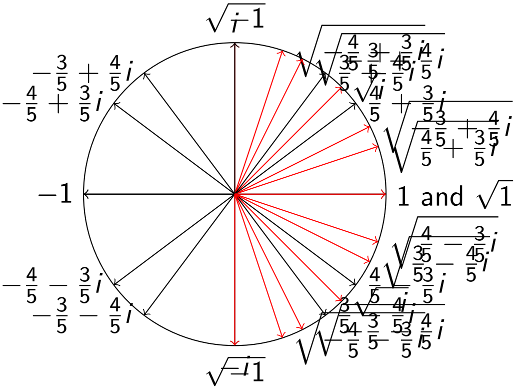
After going round the circle once we end up at *the other square root of* -1.

We have moved to an other branch of $\sqrt{z}$.

We say 0 is a branch point of $\sqrt{z}$.

Also ∞ is a branch point of $\sqrt{z}$
(every circle around 0 is also a circle around ∞).

$\sqrt{z}$, continue the walk



$\sqrt{z}$, continue the walk

After going round the circle once more we end up at *our original square root of* -1.

We are back on our original branch of $\sqrt{z}$.

This makes 0 is a branch point of order 1 of $\sqrt{z}$.

Order: the number of times to go round a branch point to get back to the original value minus one.

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$\sqrt{z}$

One way of splitting a many-valued function into usable single-valued branches is by *cutting* the plane along a suitable curve.

For $\sqrt{z}$ a popular choice is the negative real axis, it connects the branch points 0 and ∞.

On the complement both branches are single-valued analytic functions.

$\log z$

$\log z$ has the same branch points, 0 and ∞, as $\sqrt{z}$: after going round a circle around 0 the argument of z has changed by 2π (positive or negative) hence $\log z$ has changed by $2\pi i$.

If you keep going (in the same direction) you will never get back to the original value; this branch point has order ∞.

Any line from 0 to ∞ can serve as a branch cut of $\log z$; usually one takes the negative real axis.

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The arctangent function

$\arctan z$ is a composition

$$z \mapsto \frac{1+iz}{1-iz} \mapsto \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$$

The bilinear map maps i to 0 and $-i$ to ∞, which are branch points of $\log z$.

So i and $-i$ are branch points of $\arctan z$.

The imaginary axis is mapped onto the real line.
The triple $(i, \infty, -i)$ is mapped to $(0, -1, \infty)$.

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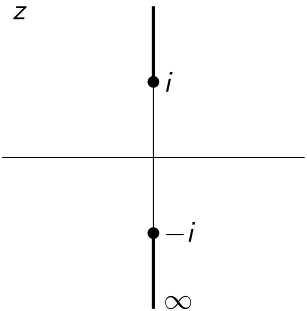
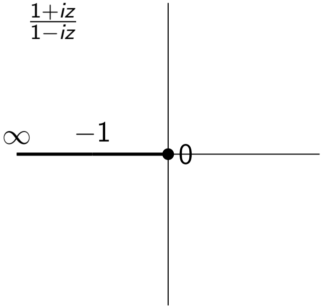
The arctangent function

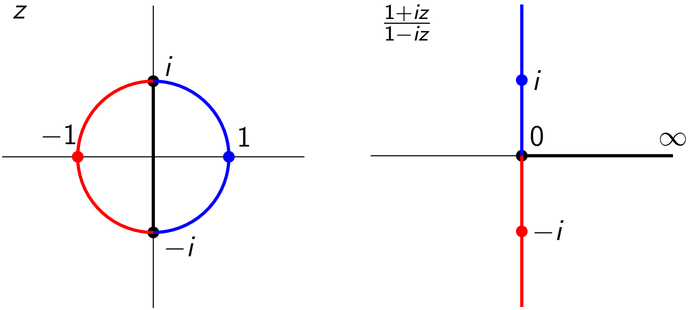
The branch points and a branch cut for $\arctan z$





The arctangent function

The branch points and a few more branch cuts for arctan z



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What to do?

From the book: 3.4, 3.5, 3.6.

Suitable exercises: 3.15 - 3.37

Recommended exercises: 3.19, 3.21, 3.22, 3.23, 3.27, 3.36, 3.37.

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