wi4243AP: Complex Analysis

week 4, Monday

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Outline

1. 4.1: Definition and properties
   - $f : [a, b] \to \mathbb{C}$
   - $f : \mathbb{C} \to \mathbb{C}$
   - The heart of complex analysis
2. 4.2: Cauchy's integral theorem
   - Cauchy's integral theorem
   - Applications

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Definition

If $f : [a, b] \to \mathbb{C}$ then write $f(t) = u(t) + iv(t)$ and define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t$$

Useful properties

Linear,
real/imaginary part of integral is integral of real/imaginary part,
and

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \int_a^b \bigl|f(t)\bigr|\,\mathrm{d}t$$

Well-known for real-valued functions and
a cute argument for complex-valued functions

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Proof of 4.1.3.d

Let $\theta = \operatorname{Arg}\int_a^b f(t)\,\mathrm{d}t$, so that $\left|\int_a^b f(t)\,\mathrm{d}t\right| = e^{-i\theta}\int_a^b f(t)\,\mathrm{d}t$.

But, then $\int_a^b e^{-i\theta} f(t)\,\mathrm{d}t$ is real, hence equal to (the integral of) its own real part, so

$$\begin{aligned}\left|\int_a^b f(t)\,\mathrm{d}t\right| &= \int_a^b \operatorname{Re}\bigl(e^{-i\theta}f(t)\bigr)\,\mathrm{d}t \\ &\leqslant \int_a^b \left|e^{-i\theta}f(t)\right|\mathrm{d}t \\ &= \int_a^b \left|f(t)\right|\mathrm{d}t\end{aligned}$$

A useful inequality

Along the unit circle we have

$$|e^{2\alpha\pi i} - 1| \leqslant 2\pi|\alpha| \qquad \alpha \text{ real}$$

It follows from Problem 3.9.a, but also thus:

$$\int_0^{2\pi} e^{i\alpha t}\,\mathrm{d}t = \frac{1}{i\alpha}(e^{2\alpha\pi i} - 1) \qquad (\dagger)$$

and

$$\int_0^{2\pi} |e^{i\alpha t}|\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi \qquad (\ddagger)$$

Now combine (†) and (‡) (with moduli).

Definition

Given

- a curve C in an open set O in the plane
- a function $f : O \to \mathbb{C}$

what is $\int_C f(z)\,\mathrm{d}z$?

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Definition

First parametrize C: say $z(t) = x(t) + iy(t)$ $(0 \leqslant t \leqslant 1)$
If z is 'sufficiently nice' (piecewise differentiable) then

$$\int_C f(z)\,\mathrm{d}z = \int_0^1 f\bigl(z(t)\bigr)z'(t)\,\mathrm{d}t$$

See page 122 for a complete expansion of this in the real and imaginary parts of f and z.

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Orientation

Definition depends on the direction of the parametrization. For $w(t) = x(1-t) + iy(1-t)$ $(0 \leqslant t \leqslant 1)$ is also a parametrization and

$$\int_0^1 f\bigl(w(t)\bigr)w'(t)\,\mathrm{d}t = -\int_0^1 f\bigl(z(t)\bigr)z'(t)\,\mathrm{d}t$$

Moral: always mention the orientation.

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A useful inequality

If $M \geqslant \bigl|f(z)\bigr|$ for $z \in C$ and L is the length of C then

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leqslant ML$$

A useful inequality

Proof:

$$\begin{aligned} \left|\int_a^b f\bigl(z(t)\bigr)z'(t)\,\mathrm{d}t\right| &\leqslant \int_a^b \left|f\bigl(z(t)\bigr)z'(t)\right|\mathrm{d}t \\ &\leqslant M\int_a^b \left|z'(t)\right|\mathrm{d}t \\ &= ML \end{aligned}$$

An important special case

Let $z_0 \in \mathbb{C}$ and let C be any circle centered at z_0, oriented anticlockwise, then

$$\oint_C \frac{1}{z - z_0} \, \mathrm{d}z = 2\pi i$$

(we write $\oint_C$ if C is a closed curve).

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An important special case

- Parametrization of C: $z(t) = z_0 + re^{it}$ $(0 \leqslant t \leqslant 2\pi)$
- derivative: $z'(t) = ire^{it}$
- $1/(z - z_0) = e^{-it}/r$
- So,

$$\oint_C \frac{1}{z - z_0}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{r} e^{-it} ire^{it}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i$$

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An almost as important special case

Let $z_0 \in \mathbb{C}$ and let C be any circle centered at z_0, oriented anticlockwise and $n \in \mathbb{Z}$ $(n \neq 1)$, then

$$\oint_C \frac{1}{(z - z_0)^n} \, \mathrm{d}z = 0$$

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An almost as important special case

- Parametrization of C: $z(t) = z_0 + re^{it}$ $(0 \leqslant t \leqslant 2\pi)$
- derivative: $z'(t) = ire^{it}$
- $1/(z - z_0)^n = e^{-int}/r^n$
- So,

$$\oint_C \frac{1}{(z-z_0)^n}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{r^n} e^{-int} ire^{it}\,\mathrm{d}t$$
$$= \int_0^{2\pi} \frac{i}{r^{n-1}} e^{(1-n)ti}\,\mathrm{d}t$$
$$= \frac{i}{r^{n-1}} \left[\frac{e^{(1-n)ti}}{(1-n)i}\right]_0^{2\pi} = 0$$

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The theorem

Theorem (Cauchy)

If f is analytic on and inside a simple closed contour C and if $f'(z)$ is continuous on and inside C then

$$\oint_C f(z)\,\mathrm{d}z = 0$$

It follows from Green's Theorem from Vector Calculus and the Cauchy-Riemann equations.

Goursat's version

Actually

Theorem (Cauchy-Goursat)

If f is analytic on and inside a simple closed contour C then

$$\oint_C f(z)\,\mathrm{d}z = 0$$

In fact: it suffices to prove this for rectangles.

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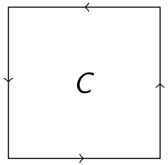
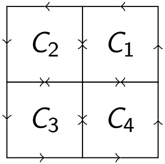
The proof

Let C be a square and assume $\alpha = \oint_C f(z)\,\mathrm{d}z \neq 0$.



Divide the square into four subsquares.

The proof



Then the integrals in the interior cancel, so

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \oint_{C_2} f(z)\,\mathrm{d}z + \oint_{C_3} f(z)\,\mathrm{d}z + \oint_{C_4} f(z)\,\mathrm{d}z$$

The proof

By the triangle inequality
$|\alpha| \leqslant \left|\oint_{C_1} f(z)\,\mathrm{d}z\right| + \left|\oint_{C_2} f(z)\,\mathrm{d}z\right| + \left|\oint_{C_3} f(z)\,\mathrm{d}z\right| + \left|\oint_{C_4} f(z)\,\mathrm{d}z\right|$

Therefore, for one of the squares, call it $C^{(1)}$, we have

$$\left|\oint_{C^{(1)}} f(z)\,\mathrm{d}z\right| \geqslant \frac{1}{4}|\alpha|$$

The proof

Inside $C^{(1)}$ we, likewise, find a square, $C^{(2)}$, such that

$$\left|\oint_{C^{(2)}} f(z)\,\mathrm{d}z\right| \geqslant \frac{1}{16}|\alpha|$$

Inside $C^{(2)}$ we, likewise, find a square, $C^{(3)}$, such that

$$\left|\oint_{C^{(3)}} f(z)\,\mathrm{d}z\right| \geqslant \frac{1}{4^3}|\alpha|$$

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The proof

Inside $C^{(n-1)}$ we, likewise, find a square, $C^{(n)}$, such that

$$\left|\oint_{C^{(n)}} f(z)\,\mathrm{d}z\right| \geqslant \frac{1}{4^n}|\alpha|$$

This sequence of squares shrinks down to one point, say z_0.

The proof

Near z_0 we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)$$

Where $R(z)$ satisfies

$$\lim_{z \to z_0} \frac{R(z)}{z - z_0} = 0 \qquad (*)$$

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The proof

Exercise

Let R be a rectangle. Prove: $\oint_R a\,\mathrm{d}z = 0$ *and* $\oint_R z\,\mathrm{d}z = 0$.

It follows that, for every n,

$$\begin{aligned}\oint_{C^{(n)}} f(z)\,\mathrm{d}z &= \oint_{C^{(n)}} f(z_0)\,\mathrm{d}z + \oint_{C^{(n)}} f'(z_0)(z-z_0)\,\mathrm{d}z + \oint_{C^{(n)}} R(z)\,\mathrm{d}z \\ &= \oint_{C^{(n)}} R(z)\,\mathrm{d}z\end{aligned}$$

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The proof

Let L denote the length of a side of C; the length of a side of $C^{(n)}$ then is $\frac{1}{2^n}L$ and its diameter is $\frac{1}{2^n}L\sqrt{2}$.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|\frac{R(z)}{z-z_0}| < \varepsilon$ whenever $|z - z_0| < \delta$.

Now let n be such that $C^{(n)}$ lies within δ of z_0.

Then

$$\left|\oint_{C^{(n)}} R(z)\,\mathrm{d}z\right| \leqslant \varepsilon\frac{1}{2^n}L\sqrt{2}\cdot\frac{4L}{2^n} = \frac{\varepsilon L^2 4\sqrt{2}}{4^n}$$

The proof

Now choose ε so small that $\varepsilon L^2 4\sqrt{2} < |\alpha|$ and take n as above. Then

$$\frac{1}{4^n}|\alpha| \leqslant \left| \oint_{C^{(n)}} f(z)\,\mathrm{d}z \right| = \left| \oint_{C^{(n)}} R(z)\,\mathrm{d}z \right| < \frac{1}{4^n}|\alpha|$$

Contradiction.

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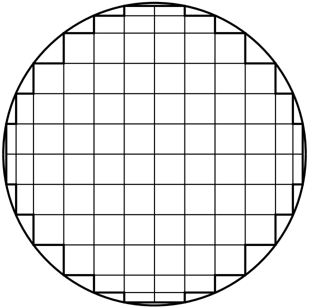
General case

The general case follows by polygonal approximation and continuity.



The integral over the polygon is zero because the integral along each rectangle is.

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Independence of path

If f is analytic on a simply connected domain D and $\alpha, \beta \in D$ then for any two paths Γ_1 and Γ_2 from α to β we have

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z$$

Γ_1 and the reverse of Γ_2 form a closed curve C so

$$0 = \oint_C f(z)\,\mathrm{d}z = \int_{\Gamma_1} f(z)\,\mathrm{d}z - \int_{\Gamma_2} f(z)\,\mathrm{d}z$$

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Primitive functions

Theorem

Let $f : D \to \mathbb{C}$ be analytic, where D is simply connected. Then there is $F : D \to \mathbb{C}$ such that $F' = f$.

Proof.

Fix $z_0 \in D$ and define $F(z) = \int_C f(w)\,\mathrm{d}w$, where C is a (piecewise linear) curve from z_0 to z (and everywhere parallel to the real or imaginary axis).
This is independent of C, by the above.
Proof that $F' = f$ is much like in the real case. □

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Using a primitive of an analytic f

Integrals along curves inside a simply connected domain:

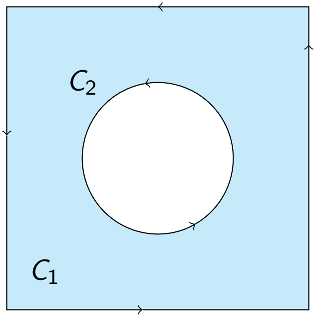
$$\int_\gamma f(z)\,\mathrm{d}z = F(\beta) - F(\alpha)$$

α: initial point; β: end point.
See also Example 4.2.3.

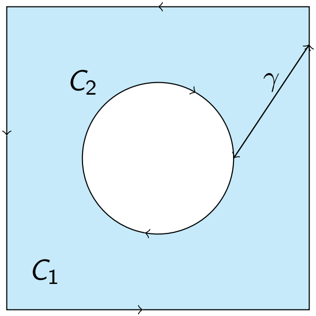
Two closed curves, same integral

Theorem

If C_1 and C_2 are two simple closed curves and f is analytic between C_1 and C_2 then $\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z$, provided the orientations are the same.

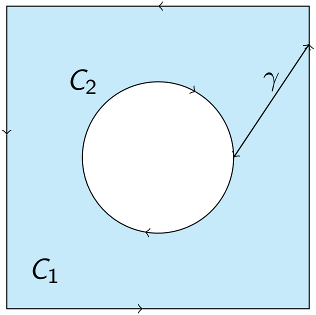


Two closed curves, same integral



C consists of: C_1 (anticlockwise), then γ (inward), then C_2 (clockwise), then γ (outward). f analytic inside C so $\oint_C f(z)\,\mathrm{d}z = 0$

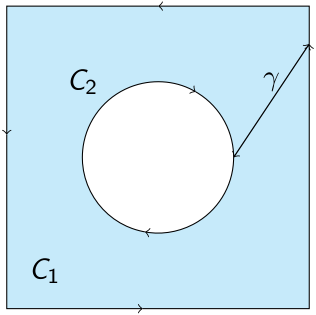
Two closed curves, same integral



We get

$$0 = \oint_{C_1} f(z)\,\mathrm{d}z + \int_\gamma f(z)\,\mathrm{d}z - \oint_{C_2} f(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z$$

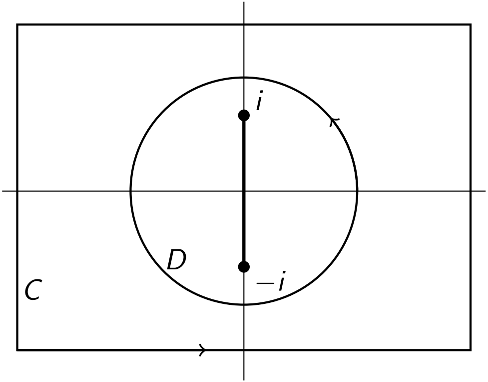
Two closed curves, same integral



And so $\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z$.

The arctangent function

Take the branch of $\arctan z$ on $\mathbb{C} \setminus [-i, i]$ that satisfies $\arctan 1 = \frac{\pi}{4}$.
What is $\oint_C \arctan z \, \mathrm{d}z$? Same as $\oint_D \arctan z \, \mathrm{d}z$.



The arctangent function

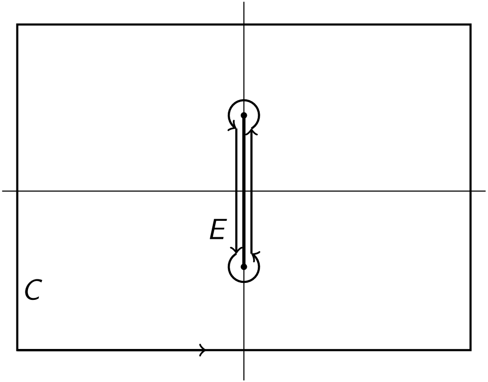
Parametrize D: $z(t) = re^{it}$ $(0 \leqslant t \leqslant 2\pi)$; we get

$$\int_0^{2\pi} \frac{1}{2i} \log\left(\frac{1 + ire^{it}}{1 - ire^{it}}\right) ire^{it}\,\mathrm{d}t$$

Too difficult, let's try something else ...

The arctangent function

Take the branch of $\arctan z$ on $\mathbb{C} \setminus [-i, i]$ that satisfies $\arctan 1 = \frac{\pi}{4}$.
What is $\oint_C \arctan z \, \mathrm{d}z$? Same as $\oint_E \arctan z \, \mathrm{d}z$.



The arctangent function

Let E shrink to the branch cut. The integral stays the same, but

- integrals along circular arcs converge to 0
- integral along right-hand line converges to

$$\int_{-1}^{1} \frac{1}{2i} \ln\left(\frac{1-t}{1+t}\right) i \, \mathrm{d}t$$

- integral along left-hand line converges to

$$\int_{1}^{-1} \frac{1}{2i} \left(\ln\left(\frac{1-t}{1+t}\right) + 2\pi i\right) i \, \mathrm{d}t$$

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The arctangent function

So $\int_C \arctan z \, \mathrm{d}z$ is (apparently) equal to

$$\int_{-1}^{1} \frac{1}{2i} \ln\left(\frac{1-t}{1+t}\right) i \, \mathrm{d}t + \int_{1}^{-1} \frac{1}{2i} \left(\ln\left(\frac{1-t}{1+t}\right) + 2\pi i\right) i \, \mathrm{d}t = -2\pi i$$

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What to do?

From the book: 4.1, 4.2
Suitable problems: 4.1 – 4.12
Recommended problems: 4.1, 4.2, 4.3, 4.6, 4.7, 4.9, 4.11 (choice).

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