wi4243AP: Complex Analysis

week 4, Friday

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Outline

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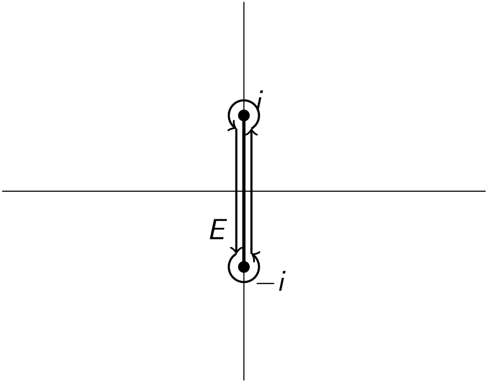
Modulus of the arctangent

We had the branch of $\arctan z$ on $\mathbb{C} \setminus [-i, i]$ that satisfies $\arctan 1 = \frac{\pi}{4}$.
And we were were talking about $\oint_E \arctan z \, \mathrm{d}z$.



Modulus of the arctangent

We wanted to estimate the modulus of the part of the integral along (part of) the circle $\{z : |z - i| = \varepsilon\}$.
For that we needed to know the modulus of $\arctan z$.
Remember

$$\arctan z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i} \ln\left|\frac{1+iz}{1-iz}\right| + \frac{i}{2i} \arg\left(\frac{1+iz}{1-iz}\right)$$

The easy bit: we took the branch of log with $0 < \arg w < 2\pi$, so

$$\left|\frac{1}{2i} \arg\left(\frac{1+iz}{1-iz}\right)\right| \leqslant \pi$$

Modulus of the arctangent

For the other part (assume $|z-i|=\varepsilon$):

$$\ln\left|\frac{1+iz}{1-iz}\right| = \ln|1+iz| - \ln|1-iz| = \ln\varepsilon - \ln|z+i|$$

As ε is about to go to 0 we assume $\varepsilon < \frac{1}{2}$; then $\frac{3}{2} < |z+i| < \frac{5}{2} < e$ and so

$$\left|\ln\left|\frac{1+iz}{1-iz}\right|\right| \leqslant |\ln\varepsilon| + 1$$

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The integral

So, if $|z - i| = \varepsilon$ then

$$|\arctan z| \leqslant \frac{1}{2}|\ln \varepsilon| + \frac{1}{2} + \pi$$

and so the modulus of the integral along the circle is bounded by

$$\pi\varepsilon|\ln \varepsilon| + \pi\varepsilon + \frac{\pi^2}{2}\varepsilon$$

Poisson integral: $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x$

Use $f(z) = e^{-z^2}$, analytic everywhere.

Contour: Γ_a, rectangle with vertices $-a$, a, $a + bi$ and $-a + bi$.

Cauchy-Coursat:

$$\oint_{\Gamma_a} f(z) \, \mathrm{d}z = 0$$

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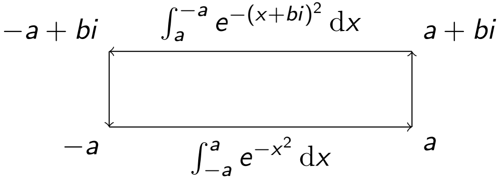
Poisson integral: $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x$

The integral along Γ_a splits into four integrals: along top and bottom:



and along the two sides.

Poisson integral: $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x$

- Along the sides (Γ_2 and Γ_4): $z = \pm a + it$ $(0 \leqslant t \leqslant b)$
- Function value: $e^{-(\pm a+it)^2} = e^{-a^2} \cdot e^{t^2} \cdot e^{\mp 2iat}$
- Modulus: $e^{-a^2} \cdot e^{t^2} \leqslant e^{-a^2} \cdot e^{b^2}$
- Length of side: b

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Poisson integral: $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x$

By last week's useful inequality:

$$\left| \int_{\Gamma_2} f(z) \, \mathrm{d}z \right| \leqslant e^{-a^2} \cdot e^{b^2} \cdot b$$

and so $\lim_{a \to \infty} \int_{\Gamma_2} f(z) \, \mathrm{d}z = 0$ (also for Γ_4).

Poisson integral: $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x$

Along the top:

$$e^{-(x+bi)^2} = e^{-x^2} \cdot e^{b^2} \cdot e^{-2ibx} = e^{-x^2} \cdot e^{b^2}(\cos 2bx - i \sin 2bx)$$

The integral becomes, using $\int_a^{-a} = -\int_{-a}^{a}$:

$$-e^{b^2} \int_{-a}^{a} e^{-x^2} \cos 2bx \, \mathrm{d}x + ie^{b^2} \int_{-a}^{a} e^{-x^2} \sin 2bx \, \mathrm{d}x$$

Imaginary part is zero: the function is odd.

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Poisson integral: $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x$

Put it all together: $0 = \lim_{a\to\infty} \oint_{\Gamma_a} f(z)\,\mathrm{d}z$ gives us

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x - e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,\mathrm{d}x = 0$$

or

$$\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,\mathrm{d}x = e^{-b^2}\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = e^{-b^2}\sqrt{\pi}$$

(as everyone knows ...)

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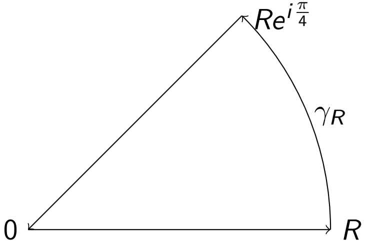
A contour

Let's calculate $\int_0^\infty \cos x^2 \,\mathrm{d}x$ and $\int_0^\infty \sin x^2 \,\mathrm{d}x$.

We use $f(z) = e^{iz^2}$ and the contour Γ_R composed of

The interval $[0, R]$
The arc γ_R from R to $Re^{i\frac{\pi}{4}}$
The line segment from $Re^{i\frac{\pi}{4}}$ to 0

So, $\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 0$



Split the integral

The integral $\oint_{\Gamma_R} f(z)\,\mathrm{d}z$ is the sum of

- $\int_0^R \cos x^2 + i \sin x^2 \,\mathrm{d}x$
- $\int_{\gamma_R} f(z)\,\mathrm{d}z$
- $\int_R^0 e^{i(\alpha t)^2}\,\mathrm{d}\alpha t$, where $\alpha = e^{i\frac{\pi}{4}} = \frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}$

The last integral is equal to $-\alpha \int_0^R e^{-t^2}\,\mathrm{d}t$

The arc, part I

Consider $\int_{\gamma_R} f(z)\,\mathrm{d}z$.

The useful inequality yields

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| \leqslant 1 \cdot R \cdot \frac{\pi}{4}$$

Because $|f(z)|$ attains its maximum, 1, on γ_R at R.

The useful inequality is no panacea.

The arc, part II

Consider $\int_{\gamma_R} f(z)\,\mathrm{d}z$ (again).

Parametrize γ_R: $z(t) = Re^{i\theta}$ (with $0 \leqslant \theta \leqslant \frac{\pi}{4}$).

So, using that $(Re^{i\theta})^2 = R^2(\cos 2\theta + i \sin 2\theta)$,

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| = \left|\int_0^{\frac{\pi}{4}} e^{iR^2\cos 2\theta} \cdot e^{-R^2 \sin 2\theta} iRe^{i\theta}\,\mathrm{d}\theta\right|$$

$$\leqslant R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta}\,\mathrm{d}\theta$$

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An (over)estimate

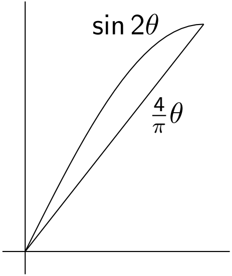
Now work on $\int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} \,\mathrm{d}\theta$.

On the interval $[0, \frac{\pi}{4}]$ we have

$$\sin 2\theta \geqslant \frac{4}{\pi}\theta$$

hence

$$\int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} \,\mathrm{d}\theta \leqslant \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4}{\pi}\theta} \,\mathrm{d}\theta$$



An (over)estimate

We have

$$\int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} \, \mathrm{d}\theta \leqslant \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4}{\pi}\theta} \, \mathrm{d}\theta = \frac{\pi}{4R^2}(1 - e^{-R^2})$$

So that

$$\left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| \leqslant \frac{\pi}{4R}(1 - e^{-R^2})$$

and so

$$\lim_{R \to \infty} \int_{\gamma_R} f(z) \, \mathrm{d}z = 0$$

The answers

We put it all together:

$$0 = \lim_{R\to\infty} \oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_0^\infty \cos x^2 + i \sin x^2 \,\mathrm{d}x - \alpha \int_0^\infty e^{-x^2}\,\mathrm{d}x$$

and so

$$\int_0^\infty \cos x^2\,\mathrm{d}x = \int_0^\infty \sin x^2\,\mathrm{d}x = \frac{1}{2}\sqrt{2}\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{1}{4}\sqrt{2\pi}$$

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The formula

Theorem (Cauchy integral formula)

Let f be analytic in some open set, C a contour and z inside C. Then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

Note: the integrand is analytic everywhere in the open set, except at z. So, integral independent of contour (around z).

Why it is true

Use only circles around z and subtract and add $f(z)$ in the integral:

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta) - f(z) + f(z)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\oint_C \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta + f(z)\end{aligned}$$

By 'the heart of complex analysis'.

Losing the first integral

Remember: because $\bigl(f(\zeta)-f(z)\bigr)/(\zeta-z)$ is analytic (except at z)

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta$$

is independent of the radius of C.

If the radius r is small enough then
$\bigl|\bigl(f(\zeta)-f(z)\bigr)/(\zeta-z)-f'(z)\bigr|<1$ for all ζ on C.
By our useful inequality:

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta\right| \leqslant \frac{1}{2\pi}\cdot\bigl(\bigl|f'(z)\bigr|+1\bigr)\cdot 2\pi r = \bigl(\bigl|f'(z)\bigr|+1\bigr)\cdot r$$

The integral must be zero.

Leibniz rules

Differentiate both sides of Cauchy's formula:

$$\begin{aligned} f'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial z}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta \end{aligned}$$

Yes, this is legitimate, see page 138.

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Leibniz rules

Given a curve γ and a continuous $f : \gamma \to \mathbb{C}$ one can define

$$h(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

The argument in the book actually establishes that h is analytic off γ and

$$h'(z) = \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta$$

This is an important method of generating analytic functions.

The General Formula

Keep differentiating:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta$$

This only uses

$$\frac{\partial^n}{\partial z^n} \frac{1}{\zeta - z} = \frac{n!}{(\zeta - z)^{n+1}}$$

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Some consequences

Theorem

If f is differentiable on an open set then it is, at once, infinitely often differentiable.

Theorem (Cauchy's Estimate)

If $M(r) = \max\{|f(z)| : |z - z_0| = r\}$ *then*

$$\frac{|f^{(k)}(z_0)|}{k!} \leqslant \frac{M(r)}{r^k}$$

for all k.

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Example 4.3.2 done right

Given $|f(z)| \leqslant 1/(1-|z|)$ give an estimate for $|f^{(n)}(0)|$.

Use the circle with radius $n/(n+1)$; on the circle we have $1-|\zeta| = 1/(n+1)$, hence $|f(\zeta)| \leqslant n+1$.

Apply Cauchy's estimate:

$$\frac{|f^{(n)}(0)|}{n!} \leqslant \frac{n+1}{\left(\frac{n}{n+1}\right)^n} = (n+1)\left(\frac{n+1}{n}\right)^n$$

Done!

Morera's theorem

Theorem

If f is continuous on a simply connected domain D and

$$\oint_C f(z)\,\mathrm{d}z = 0$$

for all closed curves in D then f is analytic.

It has a differentiable primitive, hence f itself is (infinitely often) differentiable.

Gauss' mean value theorem

Theorem

If f is analytic on and inside the circle $|z - z_0| = r$ then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, \mathrm{d}\theta$$

Simply rewrite Cauchy's formula by substituting $z = z_0 + re^{i\theta}$.
It also works for the real and imaginary parts, i.e., for harmonic functions.

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Liouville's theorem

Theorem

If f is an entire function (analytic on all of $\mathbb{C}$) and bounded (there is B such that $|f(z)| \leqslant B$ for all z), then f is constant.

Apply the useful inequality to

$$f'(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{(\zeta - z)^2} \,\mathrm{d}\zeta$$

(C_R is the circle given by $|\zeta - z| = R$).
$M \leqslant BR^{-2}$ and $L = 2\pi R$, so $|f'(z)| \leqslant B/R$ for all R;
now let $R \to \infty$.

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Extra: Fundamental Theorem of Algebra

Theorem

Let p be a non-constant complex polynomial.
Then $p(z) = 0$ has a solution in $\mathbb{C}$.

It follows that a polynomial of degree n has n zeros, counting multiplicities; e.g., $z^{10} = 0$ has ten solutions (all 0).

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Extra: Fundamental Theorem of Algebra

We assume $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ (with $n \geqslant 1$) and that $p(z) = 0$ has no solutions.

- Observe that $\lim_{z\to\infty} p(z)z^{-n} = 1$
- fix R so large that $|p(z)z^{-n}| \geqslant \frac{1}{2}$ whenever $|z| \geqslant R$ and $\frac{1}{2}R^n \geqslant |p(0)| + 1$.
- So, if $|z| \geqslant R$ then $|p(z)| \geqslant |p(0)| + 1$.
- Pick z_0 such that $|p(z_0)|$ is the minimum of $|p(z)|$ on the closed and bounded set $\{z : |z| \leqslant R\}$.

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Extra: Fundamental Theorem of Algebra

We have R and z_0 such that

- If $|z| \leqslant R$ then $|p(z_0)| \leqslant |p(z)|$
- If $|z| \geqslant R$ then $|p(z_0)| \leqslant |p(0)| < |p(0)| + 1 \leqslant |p(z)|$
- So, $|p(z_0)|$ is the global minimum of $|p(z)|$ on $\mathbb{C}$
- By assumption $p(z_0) \neq 0$
- For all z we have $|\frac{1}{p(z)}| \leqslant |\frac{1}{p(z_0)}|$

We see that $\frac{1}{p(z)}$ is entire (because p is) and bounded, but not constant; a contradiction.

Maximum Modulus theorem

Theorem

If f is analytic on a domain D then the maximum value of $|f(z)|$, if any, occurs on the boundary of D. (Unless f is constant.)

By the mean value theorem: if $|f(z_0)|$ is a (local) maximum in the interior then, as soon as $N(z_0; r) \subseteq D$

$$|f(z_0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})| \, \mathrm{d}\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| \, \mathrm{d}\theta = |f(z_0)|$$

This implies $|f(z_0)| = |f(z_0 + re^{i\theta})|$ for all θ and all such r.
Apply Exercise 2.21.b: f is constant on a disc around z_0.

Maximum Modulus theorem

Theorem

If u is harmonic on a domain D then the maximum value of $|u(x, y)|$, if any, occurs on the boundary of D. (Unless u is constant.)

Find a complex conjugate v and consider $f = e^{u+iv}$.
Then $|f(z)| = e^{u(x,y)}$ ...

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What to do?

From the book: 4.2, 4.3
Suitable problems: 4.11-4.34
Recommended problems: 4.11 (choice), 4.14, 4.16, 4.17, 4.21, 4.22, 4.25, 4.30

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