wi4243AP: Complex Analysis

week 5, Monday

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Outline

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Convergence

Definition

A sequence $\{f_n\}$ of functions converges to a function f (on some domain) if for each individual z in the domain one has

$$\lim_{n\to\infty} f_n(z) = f(z)$$

Definition

$f_n(z) \to f(z)$ *uniformly* if for every $\varepsilon > 0$ there is an $N(\varepsilon)$ such that for all $n \geqslant N(\varepsilon)$ we have

$$\left|f_n(z) - f(z)\right| < \varepsilon$$

for all z in the domain.

Standard example

$D = \{z : |z| < 1\}$ and $f_n(z) = z^n$.

- $f_n(z) \to 0$ for each individual z
- $f_n(z) \to 0$ not uniformly on D
- $f_n(z) \to 0$ uniformly on $D_r = \{z : |z| \leqslant r\}$ if $r < 1$

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Uniform convergence: properties

Theorem

If $f_n \to f$ uniformly on some curve C then

$$\lim_{n\to\infty} \int_C f_n(z)\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z$$

Proof.

By the useful inequality

$$\left|\int_C f_n(z)\,\mathrm{d}z - \int_C f(z)\,\mathrm{d}z\right| \leqslant \int_C |f_n(z) - f(z)|\,\mathrm{d}z \leqslant M_n \cdot L$$

where $M_n = \sup\{|f_n(z) - f(z)| : z \in C\}$ and L is the length of C.
Uniform convergence: $M_n \to 0$. □

Uniform convergence: properties

Theorem

If $f_n' \to g$ uniformly and $f_n \to f$ (in just one point) then $f' = g$.

So

$$\lim_n f_n' = (\lim f_n)'$$

provided $\{f_n'\}$ is known to converge uniformly and $\{f_n\}$ converges somewhere.

Power series

Special form: a fixed number z_0 and a sequence $\{a_n\}$ of numbers are given. Put $f_n(z) = a_n(z - z_0)^n$, we write

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for the resulting series.

Radius of convergence

Theorem

Given a power series $\sum_n a_n(z - z_0)^n$ there is an R such that

- $\sum_n a_n(z - z_0)^n$ *converges if* $|z - z_0| < R$
- $\sum_n a_n(z - z_0)^n$ *diverges if* $|z - z_0| > R$

In addition: if $r < R$ then the series converges uniformly on $\{z : |z - z_0| \leqslant r\}$.

- On the boundary — $|z - z_0| = R$ — anything can happen.
- $R = 0$, $0 < R < \infty$ and $R = \infty$ are all possible.

R is the *radius of convergence* of the series.

Differentiation

Theorem

Let $\sum_n a_n(z-z_0)^n$ be a power series, with radius R, and let $\sum_n na_n(z-z_0)^{n-1}$ be its termwise derivative, with radius R'. Then $R = R'$.

Proof.

If $\sum_n na_n(z-z_0)^{n-1}$ converges absolutely then so does $\sum_n a_n(z-z_0)^n$, by comparison. So $R \geqslant R'$. □

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Differentiation

Proof.

Conversely: if $\sum_n a_n(w-z_0)^n$ converges then so does $\sum_n na_n(z-z_0)^{n-1}$ whenever $|z-z_0|<|w-z_0|$.
Fix N such that $|a_n(w-z_0)^n|\leqslant 1$ for $n\geqslant N$. For those n

$$\left|na_n(z-z_0)^{n-1}\right| = \left|a_n(w-z_0)^n\frac{1}{z-z_0}n\left(\frac{z-z_0}{w-z_0}\right)^n\right| \leqslant \frac{n}{|z-z_0|}\left|\frac{z-z_0}{w-z_0}\right|^n$$

Now use that $\sum_n nz^n$ has radius 1. □

Differentiation/Integration

Theorem

If $R > 0$ *and* $f(z) = \sum_n a_n(z - z_0)^n$ *for* $|z - z_0| < R$ *then*

- $f'(z) = \sum_n na_n(z - z_0)^{n-1}$.
- $\sum_n \frac{a_n}{n+1}(z - z_0)^{n+1}$ *is a primitive function of* f

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Differentiation/Integration

- We have $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$.
- So $\sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}$.
- Also, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ is the derivative of $\sum_{n=1}^{\infty} \frac{1}{n} z^n$.
- So, $\sum_{n=1}^{\infty} \frac{1}{n} z^n = -\operatorname{Log}(1-z) + c$ for some c;
- put in $z = 0$: we get $c = 0$, and so

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n = -\operatorname{Log}(1-z) \qquad (|z| < 1)$$

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Derivatives

If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, with $R > 0$, then

- $f(z_0) = a_0$
- $f'(z_0) = 1 \cdot a_1$
- $f''(z_0) = 2 \cdot 1 \cdot a_2$
- $f^{(k)}(z_0) = k! a_k$

and we get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Main result

Theorem

Let $f : D \to \mathbb{C}$ be analytic, let $z_0 \in D$ and let R be the distance from z_0 to the complement of D (if $D = \mathbb{C}$ then $R = \infty$). Then on the disc $\{z : |z - z_0| < R\}$ we have

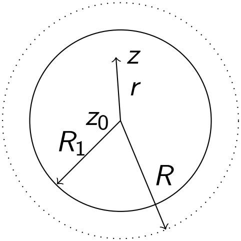
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta = \frac{f^{(n)}(z_0)}{n!}$$

and C is any simple closed contour around z_0 lying inside D.

Why?



Take z inside the circle $\{w : |w - z_0| = R\}$ and take R_1 such that $|z - z_0| = r < R_1 < R$. Work on the circle C_1 of radius R_1 around z_0.

Why?

Apply Cauchy's formula: $f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$.
Transform $1/(\zeta - z)$:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

The modulus, r/R_1, of the quotient is less than 1 on C_1, so this series converges *uniformly* on C_1.
We may interchange sum and integral.

Why?

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \mathrm{d}\zeta \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\,\mathrm{d}\zeta \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,\mathrm{d}\zeta \times (z-z_0)^n
\end{aligned}
$$

Done, by Cauchy's general formula.

What is the radius?

The radius of convergence of the series is the largest R such that f is analytic on $\{z : |z - z_0| < R\}$, possibly $R = \infty$.

For example: the Taylor series of $\arctan z$ centered at 0 has radius 1, because i and $-i$ are branch points:

$\arctan z$ is analytic on $\{z : |z| < 1\}$ but on no larger disc centered at 0.

$\arctan z$

Remember:

$$\arctan z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$$

Also

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

We stick in iz and $-iz$ and subtract the results.

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arctan z

$$\operatorname{Log}(1+iz)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(iz)^n$$

$$\operatorname{Log}(1-iz)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-iz)^n$$

the even-numbered terms drop out;
if n is odd, say $n=2k+1$, the nth terms give

$$\frac{(-1)^{2k+2}}{2k+1}(iz)^{2k+1}-\frac{(-1)^{2k+2}}{2k+1}(-iz)^{2k+1}=\frac{2i^{2k+1}}{2k+1}z^{2k+1}$$

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arctan z

Now note that $i^{2k+1} = (-1)^k i$, so we get

$$\frac{2i(-1)^k}{2k+1} z^{2k+1}$$

and so

$$\arctan z = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{2i(-1)^k}{2k+1} z^{2k+1} + m\pi$$

for some m. If we want $\arctan 0 = 0$ then $m = 0$ and

$$\arctan z = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} z^{2k+1}$$

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$\arctan z$

The series also can be obtained by integrating $\frac{1}{1+z^2}$ term-by-term:

$$\begin{aligned}\arctan z &= \int \frac{1}{1+z^2}\,\mathrm{d}z \\ &= \int \sum_{n=0}^{\infty}(-1)^n z^{2n}\,\mathrm{d}z \\ &= \sum_{n=0}^{\infty}\int(-1)^n z^{2n}\,\mathrm{d}z \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}z^{2n+1}\end{aligned}$$

The integration constant is zero, because $\arctan 0 = 0$.

What to do?

From the book: 5.2, 5.3
Suitable problems: 5.1 - 5.27
Recommended problems: 5.3, 5.7, 5.8, 5.11, 5.12, 5.17.

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