wi4243AP: Complex Analysis

week 5, Friday

K. P. Hart

Faculty EEMCS
TU Delft

Delft, 3 october, 2014

Delft University of Technology

Outline

1. Section 5.4: Laurent series
   - Example: $\arctan z$
   - An old integral
   - An example
2. Section 5.5: Analytic continuations
   - Uniqueness of analytic functions
   - Reflection Principle
3. Section 6.1: Singularities
   - Definition
   - Classification

A reminder

Theorem

Let $f : D \to \mathbb{C}$ be analytic, let $z_0 \in D$ and let R be the distance from z_0 to the complement of D (if $D = \mathbb{C}$ then $R = \infty$). Then on the disc $\{z : |z - z_0| < R\}$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \,\mathrm{d}\zeta = \frac{f^{(n)}(z_0)}{n!}$$

and C is any simple closed contour around z_0 lying inside D.

Delft University of Technology

Main result

Theorem

Let $f : A \to \mathbb{C}$ be analytic, where $A = \{z : R_1 < |z - z_0| < R_2\}$ (an annulus). Then for $z \in A$ we have

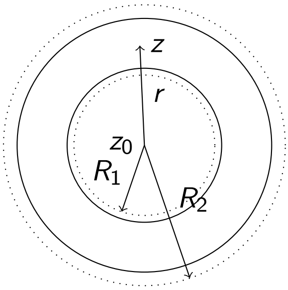
$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$$

where for $k \in \mathbb{Z}$:

$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, \mathrm{d}\zeta$$

where C is any simple closed contour around z_0 lying inside A, oriented anticlockwise.

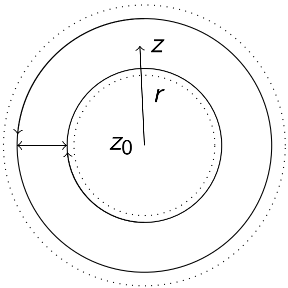
Why?



Take z inside the annulus.

Take r_1 and r_2 with $R_1 < r_1 < |z - z_0| < r_2 < R_2$

Why?



Consider the simple closed curve C consisting of

- the circle C_2 of radius r_2 (anticlockwise),
- the segment γ (inward),
- the circle C_1 of radius r_1 (clockwise),
- the segment γ (outward)

Why?

Apply Cauchy's formula:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \end{aligned}$$

because the integrals along γ cancel.

As in the case of Taylor series:

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta \times (z - z_0)^n$$

Why?

On C_1 we rewrite $1/(\zeta - z) = -1/(z - \zeta)$:

$$\frac{1}{\zeta - z} = -\frac{1}{z - \zeta} = -\frac{1}{z - z_0 + z_0 - \zeta}$$
$$= -\frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}$$
$$= -\frac{1}{z - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^n$$

The quotient has modulus $r_1/r < 1$ on C_1, so we get uniform convergence on C_1

Delft University of Technology

Why?

We get

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = -\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^n\mathrm{d}\zeta$$

$$= -\sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta-z_0)^n\,\mathrm{d}\zeta \times (z-z_0)^{-(n+1)}$$

$$= -\sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-n}}\,\mathrm{d}\zeta \times (z-z_0)^{-(n+1)}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{(\zeta-z_0)^{-n+1}}\,\mathrm{d}\zeta \times (z-z_0)^{-n}$$

Why?

We add the results: $f(z)$ is the sum of

$$-\sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}} \,\mathrm{d}\zeta \times (z - z_0)^{-n}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \,\mathrm{d}\zeta \times (z - z_0)^{n}$$

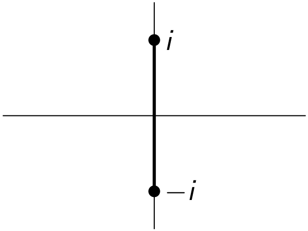
We reverse the orientation of C_1 to get rid of the minus-sign. The integrands $f(\zeta)/(\zeta - z_0)^{n+1}$ are analytic on the whole annulus, so we can replace C_2 and C_1 by one and the same simple closed curve. And this gives us the formula we were looking for.

arctan z once more

We know:

$$\arctan z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$$

The following is another branch cut for arctan z (corresponds to positive real axis):



$\arctan z$ once more

Thus $\arctan z$ has a branch that is analytic on the annulus $\{z : 1 < |z|\}$.
What is the Laurent series?
First for $\frac{1}{1+z^2}$:

$$\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}$$

arctan z once more

Now integrate:

$$\begin{aligned}\arctan z &= \int \frac{1}{1+z^2}\,\mathrm{d}z + c \\ &= \int \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}}\,\mathrm{d}z + c \\ &= \sum_{n=0}^{\infty} \int \frac{(-1)^n}{z^{2n+2}}\,\mathrm{d}z + c \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)z^{2n+1}} + c\end{aligned}$$

arctan z once more

We have, if $|z| > 1$:

$$\arctan z = c + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)z^{2n+1}}$$

Possible values for c?

- $\arctan 1 = \frac{\pi}{4} + k\pi$ (k integer)
- For $z = 1$ the series sums to $-\frac{\pi}{4}$
- So, $c = \frac{\pi}{2} + k\pi$ (k integer)

arctan z once more

Observe that

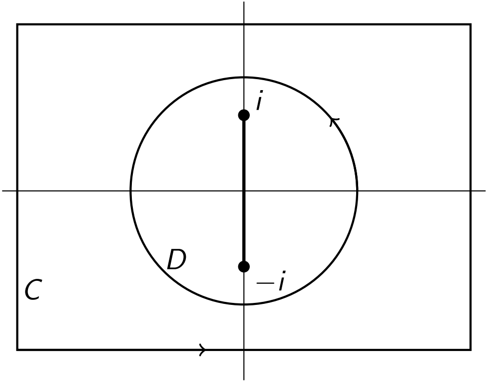
$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)z^{2n+1}}=-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{z}\right)^{2n+1}=-\arctan\frac{1}{z}$$

So, in this case we have $\arctan z = c - \arctan\frac{1}{z}$;
plug in $z = 1$ to get $c = \frac{\pi}{2}$, so for $|z| > 1$:

$$\arctan z=\frac{\pi}{2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)z^{2n+1}}$$

An old integral

Reconsider a problem from week 4 (Monday):
What is $\oint_C \arctan z \, \mathrm{d}z$? Same as $\oint_D \arctan z \, \mathrm{d}z$.



An old integral

Integrate the Laurent series

$$\arctan z = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)z^{2n+1}}$$

because $|\frac{1}{z}| = \frac{1}{r}$, where r is the radius of D and $r > 1$ we have uniform convergence, so

$$\begin{aligned}\oint_D \arctan z \,\mathrm{d}z &= \oint_D \frac{\pi}{2}\,\mathrm{d}z + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)} \oint_D \frac{1}{z^{2n+1}}\,\mathrm{d}z \\ &= 0 - \oint_D \frac{1}{z}\,\mathrm{d}z = -2\pi i\end{aligned}$$

Some Laurent series

Consider $f(z) = \frac{1}{(z-i)(z+2)} = \frac{1}{2+i}\left(\frac{1}{z-i} - \frac{1}{z+2}\right)$.

f is analytic on three annuli around 0:

- $\{z : |z| < 1\}$
- $\{z : 1 < |z| < 2\}$
- $\{z : 2 < |z|\}$

We make the three Laurent series.

Some Laurent series

First annulus: $\{z : |z| < 1\}$. We have

$$\frac{1}{z-i} = -\frac{1}{i}\frac{1}{1+iz} = -\frac{1}{i}\sum_{n=0}^{\infty}(-iz)^n$$

and

$$\frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n$$

Now add.

Delft University of Technology

Some Laurent series

Second annulus: $\{z : 1 < |z| < 2\}$. We have

$$\frac{1}{z-i} = \frac{1}{z}\frac{1}{1-\frac{i}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}}$$

and

$$\frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n$$

Now add.

Delft University of Technology

Some Laurent series

Third annulus: $\{z : 2 < |z|\}$. We have

$$\frac{1}{z-i} = \frac{1}{z}\frac{1}{1-\frac{i}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}}$$

and

$$\frac{1}{z+2} = \frac{1}{z}\frac{1}{1+\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(-2)^n}{z^{n+1}}$$

Now add.

Delft University of Technology

An example

Consider these two power series and their sums:

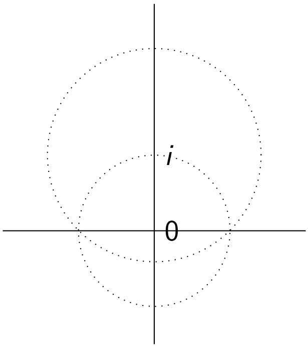
$$f_1(z) = \sum_{n=0}^{\infty} z^n \text{ and } f_2(z) = \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}}(z-i)^n$$

Their circles of convergence are

$$\{z : |z| < 1\} \text{ and } \{z : |z-i| < \sqrt{2}\}$$

respectively.

An example



We have $f_1(z) = \frac{1}{1-z} = f_2(z)$ for z in the intersection of the discs.
These functions are *analytic continuations* of each other.

How unique are analytic functions?

Theorem

Assume f is analytic on a domain D and assume there is a converging sequence $\{z_n\}$ in D with limit $z_0 \in D$ such that $f(z_n) = 0$ for all n. Then $f(z) = 0$ on the whole of D.

Proof.

It will follow that all coefficients in the Taylor series at z_0 are zero.
So $f = 0$ on a disc around z_0.
By analytic continuation this will propagate through all of D. □

$\arctan z$

We know from geometry that $\arctan x = \frac{\pi}{2} - \arctan \frac{1}{x}$ for all positive real x. Hence it holds everywhere.

Therefore we could have found the Laurent series of $\arctan z$, for $|z| > 1$ from the Taylor series for $|z| < 1$:

$$\arctan z = \frac{\pi}{2} - \arctan \frac{1}{z} = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{z}\right)^{2n+1}$$

Delft University of Technology

Reflection Principle

Theorem

Let D be a domain and $f : D \to \mathbb{C}$ analytic. Let $\bar{D} = \{\bar{z} : z \in D\}$ (the reflection of D in the real axis). Then $\bar{f} : \bar{D} \to \mathbb{C}$, defined by $\bar{f}(z) = \overline{f(\bar{z})}$, is analytic on $\bar{D}$.

Proof.

$\bar{f}$ is certainly real differentiable.
$\bar{f}(x, y) = u(x, -y) - iv(x, -y)$; use the Cauchy-Riemann equations. □

Special case

If D is symmetric and intersects the real axis and $f : D \to \mathbb{C}$ is analytic. Then the following are equivalent.

- f is real-valued on the real axis, and
- $f(\bar{z}) = \overline{f(z)}$ for all $z \in D$

Use reflection and uniqueness.

This applies to all familiar functions: e^z, $\sin z$, $\tan z$, $\arctan z$, $\operatorname{Log} z$, $\sqrt{z}$, ...

Definition

A singularity of a function is a point at which it is not analytic.
We are (very much) interested in *isolated* singularities:

Definition

z_0 is an isolated singularity of f if there is an $r > 0$ such that f is analytic on $N(z_0, r) \setminus \{z_0\}$.

Laurent series

Note that $N(z_0, r) \setminus \{z_0\}$ is an annulus, so we have a Laurent series centered at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

The negative powers form the *principal part*.

Delft University of Technology

Removable singularity

Definition

z_0 is a *removable singularity* of f if the principal part is zero.

This means that $f(z_0)$ can be defined so that f becomes analytic on $N(z_0, r)$.

Example: $f(z) = \frac{1-\cos z}{z^2}$; the Laurent series is

$$\frac{1}{2} - \frac{1}{4!}z^2 + \frac{1}{6!}z^4 + \cdots + \frac{(-1)^n}{(2n+2)!}z^{2n} + \cdots$$

Set $f(0) = \frac{1}{2}$ to remove the singularity.

TU Delft
Delft University of Technology

Pole

Definition

z_0 is a pole of f if the principal part is finite, its *order* is the *largest* k such that $b_k \neq 0$.

Example: $f(z) = \frac{1-\cos z}{z^4}$; the Laurent series is

$$\frac{1}{2}z^{-2} - \frac{1}{4!} + \frac{1}{6!}z^2 + \cdots + \frac{(-1)^n}{(2n+2)!}z^{2n-2} + \cdots$$

This is a pole of order 2.

Pole

Note

z_0 is a pole of f of order k iff k is such that $\lim_{z\to z_0}(z-z_0)^k f(z)$ exists and is non-zero. (It's the coefficient of $(z-z_0)^{-k}$.)

Example: $f(z) = \frac{1}{\sin z}$; it has a pole of order 1 at 0, because $\lim_{z\to 0}\frac{z}{\sin z} = 1$.

If $k > 1$ then $\lim_{z\to 0}\frac{z^k}{\sin z} = 0$;

If $k < 1$ then $\lim_{z\to 0}\frac{z^k}{\sin z} = \infty$.

The Laurent series of $(\sin z)^{-1}$

We calculate (part of) the Laurent series of $\frac{1}{\sin z}$.
We know:

$$\sin z = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots$$

and

$$\frac{1}{\sin z} = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

Now multiply:

$$1 = \sin z \cdot \frac{1}{\sin z}$$

so . . .

The Laurent series of $(\sin z)^{-1}$

$$1 = (z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots)(\frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots)$$
$$= a_{-1} + a_0 z + (a_1 - \frac{1}{6}a_{-1})z^2 + (a_2 - \frac{1}{6}a_0)z^3$$
$$+ (a_3 - \frac{1}{6}a_1 + \frac{1}{120}a_{-1})z^4 + \cdots$$

And so: $a_{-1} = 1$, $a_0 = 0$, $a_1 = \frac{1}{6}a_{-1} = \frac{1}{6}$, $a_2 = 0$,
$a_3 = \frac{1}{6}a_1 - \frac{1}{120}a_{-1} = \frac{7}{360}$, ...

The Laurent series of $(\sin z)^{-1}$

We find

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots$$

Apparently $\frac{1}{\sin z} - \frac{1}{z}$ has a removable singularity at 0.

The radius of convergence of the resulting power series will be π.

Delft University of Technology

Essential singularity

Definition

z_0 is an essential singularity if the principal part is infinite.

Example: $f(z) = e^{\frac{1}{z}}$; the Laurent series is

$$1 + \sum_{n=1}^{\infty} \frac{1}{n! z^n}$$

Notice: for every $r > 0$ the annulus $\{z : 0 < |z| < r\}$ is mapped onto $\mathbb{C} \setminus \{0\}$.

Structural property

Theorem (Casorati-Weierstraß)

If z_0 is an essential singularity of f then for every $\lambda \in \mathbb{C}$, every $\varepsilon > 0$ and every $r > 0$ there is a z such that $|z - z_0| < r$ and $|f(z) - \lambda| < \varepsilon$.

Even better

Theorem (Picard)

If z_0 is an essential singularity of f and $r > 0$ then f assumes every complex value on $\{z : 0 < |z - z_0| < r\}$ with one possible exception.

What to do?

From the book: 5.4, 5.5, 6.1
Suitable problems: 5.28 – 5.40; 6.1–6.5
Recommended problems: 5.28, 5.29, 5.32, 5.36, 5.37, 5.38, 5.40; 6.1, 6.3, 6.5