# wi4243AP: Complex Analysis

## week 6, Friday

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# Outline



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# The residue

## Definition

The residue of $f$ at an isolated singularity, $z_0$, is

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z$$

where $C$ is some simple closed contour around $z_0$.

# A formula

Integrate the Laurent series term-by-term and use ‘the heart of Complex Analysis’; what remains (pun intended) is

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \oint_C \frac{b_1}{z - z_0}\,\mathrm{d}z = b_1$$

If $f(z) = \frac{1-\cos x}{x^4}$ then $\operatorname{Res}(f, 0) = 0$ (look at the series).

If $f(z) = \frac{1-\cos x}{x^5}$ then $\operatorname{Res}(f, 0) = -\frac{1}{24}$ (look at the series).

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# Residue Theorem

**Theorem (Cauchy)**

*If $C$ is a simple closed contour and $f$ is analytic on and inside $C$, except at isolated singularities $z_1, z_2, \dots, z_n$. Then*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i\big(\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2) + \cdots + \mathrm{Res}(f, z_n)\big)$$

# Calculating residues

Useful special case: $f$ has a pole of order one at $z_0$: Then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

(Multiply the series by $z - z_0$.)

$$\operatorname{Res}\left(\frac{1}{1+z^2}, i\right) = \lim_{z \to i} \frac{z - i}{(z-i)(z+i)} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i} = -\frac{1}{2} i$$

and

$$\operatorname{Res}\left(\frac{1}{1+z^2}, -i\right) = \lim_{z \to i} \frac{z + i}{(z-i)(z+i)} = \lim_{z \to i} \frac{1}{z-i} = \frac{1}{-2i} = \frac{1}{2} i$$

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# Calculating residues

Consider $\tan z$; the singularities are at $\frac{\pi}{2} + k\pi$ (where $\cos z = 0$).

$$\operatorname{Res}(\tan, \frac{\pi}{2}) = \lim_{z \to \frac{\pi}{2}} \frac{(z - \frac{\pi}{2})}{\cos z} \sin z = \frac{\sin \frac{\pi}{2}}{-\sin \frac{\pi}{2}} = -1$$

This illustrates: *if* $z_0$ is a pole of order 1 of $f/g$ *then*

$$\operatorname{Res}(\frac{f}{g}, z_0) = \lim_{z \to z_0} \frac{z - z_0}{g(z)} f(z) = \frac{f(z_0)}{g'(z_0)}$$

this needs $f(z_0) \neq 0$ and $g(z_0) = 0$, and both analytic at $z_0$.

# Calculating residues

General case: $f$ has a pole of order $k$ at $z_0$: Then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{\bigl((z - z_0)^k f(z)\bigr)^{(k-1)}}{(k-1)!}$$

Multiply the series by $(z - z_0)^k$ and look at the coefficient of $(z - z_0)^{(k-1)}$ in the Taylor series of the new function.

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# Calculating residues

Consider $\tan^3 z$; the singularities are at $\frac{\pi}{2} + k\pi$ (where $\cos z = 0$).
These are poles of order three.
To get the residue: take second derivative of $(z - \frac{\pi}{2})^3 \tan^3 z$.

Let's consult Maple.

# Trigonometric integrals

Calculate $\int_0^{2\pi} \frac{1}{(a-\cos\theta)^2}\,\mathrm{d}\theta$, where $a > 1$.
Substitute $z = e^{i\theta}$, so $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta = iz\,\mathrm{d}\theta$, and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$
We get an integral over the unit circle:

$$\oint_{|z|=1} \frac{1}{\left(a - \frac{1}{2}(z + \frac{1}{z})\right)^2}\frac{1}{iz}\,\mathrm{d}z = \frac{4}{i}\oint_{|z|=1} \frac{z}{(z^2 - 2az + 1)^2}\,\mathrm{d}z$$

The singularities of the integrand, $f(z)$, are the zeros of the denominator:
$\alpha = a - \sqrt{a^2 - 1}$ and $\beta = a + \sqrt{a^2 - 1}$.
Note: $0 < \alpha < 1 < \beta$, so we get…

# Trigonometric integrals

...

$$\oint_{|z|=1} \frac{z}{(z-\alpha)^2(z-\beta)^2}\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f,\alpha)$$

We have a pole of order 2 at $\alpha$, so we take the derivative of

$$(z-\alpha)^2 f(z) = \frac{z}{(z-\beta)^2}$$

at $\alpha$:

$$\left(\frac{z}{(z-\beta)^2}\right)'\Bigg]_{z=\alpha} = -\frac{z+\beta}{(z-\beta)^3}\Bigg]_{z=\alpha} = -\frac{\alpha+\beta}{(\alpha-\beta)^3} = \frac{2a}{8(a^2-1)^{3/2}}$$

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# Trigonometric integrals

Put it all together: our integral equals

$$\begin{aligned}\frac{4}{i}\oint_{|z|=1}\frac{z}{(z^2-2az+1)^2}\,\mathrm{d}z &= \frac{4}{i}\times 2\pi i\,\mathrm{Res}(f,\alpha)\\ &= 8\pi\frac{2a}{8(a^2-1)^{3/2}}\\ &= \frac{2a\pi}{(a^2-1)^{3/2}}\end{aligned}$$

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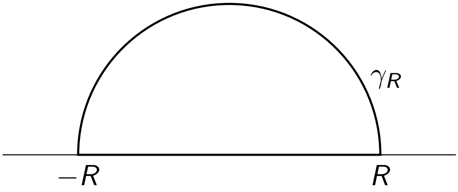
# A rational function

What is the value of $\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x$?

Let $R > 0$ and consider the following contour $\Gamma_R$:




It consists of $[-R, R]$ and the semi-circle $\gamma_R$.

# A rational function

Let $f(z) = \frac{1}{1+z^4}$. We can evaluate $\oint_{\Gamma_R} f(z)\,\mathrm{d}z$ in two ways:

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{1+x^4}\,\mathrm{d}x + \int_{\gamma_R} f(z)\,\mathrm{d}z$$

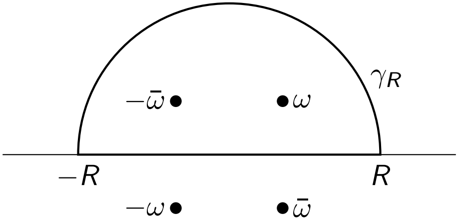
or use the residue theorem.

The singularities of $f$ are the zeros of $z^4+1$, to wit $\omega$, $-\omega$, $\bar{\omega}$ and $-\bar{\omega}$, where

$$\omega = e^{\frac{\pi}{4}i} = \frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}.$$

# A rational function

Here they are:



# A rational function

We find

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\bigl(\mathrm{Res}(f,\omega) + \mathrm{Res}(f,-\bar{\omega})\bigr)$$

The singularities are poles of order 1 so we can calculate the residues using the derivative of the denominator.

# A rational function

We get

$$\operatorname{Res}(f,\omega)=\frac{1}{4\omega^3}=\frac{1}{4i\omega}=\frac{1}{4i}\bar{\omega}$$

and

$$\operatorname{Res}(f,-\bar{\omega})=\frac{1}{4(-\bar{\omega})^3}=\frac{1}{4i\bar{\omega}}=\frac{1}{4i}\omega$$

# A rational function

So, for all $R > 1$ we have

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{2\pi i}{4i}(\bar{\omega} + \omega) = \frac{\pi}{2}(\bar{\omega} + \omega) = \frac{\pi}{2}\sqrt{2}$$

Now on the one hand

$$\lim_{R\to\infty} \oint_{\Gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{2}\sqrt{2}$$

and on the other hand …

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# A rational function

... we have

$$\lim_{R\to\infty}\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty}\frac{1}{1+z^4}\,\mathrm{d}z$$

because

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| \leqslant \frac{\pi R}{R^4-1}$$

so that $\lim_{R\to\infty}\oint_{\gamma_R} f(z)\,\mathrm{d}z = 0$

# The Error integral: $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$

This will be a bit roundabout. We take $a = (1+i)\sqrt{\frac{\pi}{2}} = \sqrt{\pi}e^{\frac{1}{4}\pi i}$. Put

$$g(z) = \frac{e^{-z^2}}{1 + e^{-2az}}$$

Note

- $a^2 = \pi i$
- so $e^{-2a(z+a)} = e^{-2az}$, and
- $g(z) - g(z+a) = \frac{e^{-z^2} - e^{-(z+a)^2}}{1+e^{-2az}} = \frac{e^{-z^2}(1 - e^{-2az}e^{-a^2})}{1+e^{-2az}} = e^{-z^2}$

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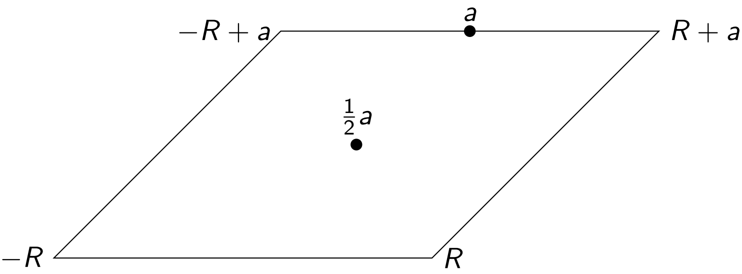
# A contour and a pole




- Let $\Gamma_R$ be the trapezoid shown above.
- $g$ has one singularity inside $\Gamma_R$: at $\frac{1}{2}a$

# Residue and integral

The residue at $\frac{1}{2}a$ of $g$ is

$$\lim_{z\to\frac{1}{2}a}(z-\frac{1}{2}a)g(z) = e^{-\frac{1}{4}a^2}\lim_{z\to\frac{1}{2}a}\frac{z-\frac{1}{2}a}{1+e^{-2az}}$$

$$= \frac{e^{-\frac{1}{4}a^2}}{-2ae^{-a^2}} = \frac{e^{-\frac{1}{4}\pi i}}{-2\sqrt{\pi}e^{\frac{1}{4}\pi i}e^{-\pi i}}$$

$$= -\frac{i}{2\sqrt{\pi}}$$

So

$$\oint_{\Gamma_R} g(z)\,\mathrm{d}z = \frac{2\pi i\cdot -i}{2\sqrt{\pi}} = \sqrt{\pi}$$

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# Split the integral

- Bottom line: $\int_{-R}^{R} g(x)\,\mathrm{d}x$
- Top line: $\int_{R}^{-R} g(x+a)\,\mathrm{d}x$
- Together: $\int_{-R}^{R} g(x) - g(x+a)\,\mathrm{d}x = \int_{-R}^{R} e^{-x^2}\,\mathrm{d}x$

So, when $R \to \infty$ these give us $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$

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# The sides

On the right-hand side $z = R + at$ $(0 \leqslant t \leqslant 1)$, so

- $z^2 = R^2 + \sqrt{2\pi}Rt + (\pi t^2 + \sqrt{2\pi}Rt)i$, hence

$$|e^{-z^2}| = e^{-R^2 - \sqrt{2\pi}Rt} \leqslant e^{-R^2}$$

- $2az = 2aR + 2a^2t = \sqrt{2\pi}R + (\sqrt{2\pi}R + 2\pi t)i$, hence

$$|e^{-2az}| = e^{-\sqrt{2\pi}R}$$

So that

$$\left|\frac{e^{-z^2}}{1 + e^{-2az}}\right| \leqslant \frac{e^{-R^2}}{1 - e^{-\sqrt{2\pi}R}}$$

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# The sides

On the left-hand side $z = -R + at$ $(0 \leqslant t \leqslant 1)$, so

- $z^2 = R^2 - \sqrt{2\pi}Rt + (\pi t^2 - \sqrt{2\pi}Rt)i$, hence

$$\left|e^{-z^2}\right| = e^{-R^2+\sqrt{2\pi}Rt} \leqslant e^{-R^2+\sqrt{2\pi}R}$$

- $2az = -2aR + 2a^2t = -\sqrt{2\pi}R + (-\sqrt{2\pi}R + 2\pi t)i$, hence

$$\left|e^{-2az}\right| = e^{\sqrt{2\pi}R}$$

So that

$$\left|\frac{e^{-z^2}}{1+e^{-2az}}\right| \leqslant \frac{e^{-R^2+\sqrt{2\pi}R}}{e^{\sqrt{2\pi}R}-1} = \frac{e^{-R^2}}{1-e^{-\sqrt{2\pi}R}}$$

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# The sides

For either side, call it $\gamma_R$, we have

$$\left|\int_{\gamma_R} g(z)\,\mathrm{d}z\right| \leqslant \frac{e^{-R^2}}{1 - e^{-\sqrt{2\pi}R}} \cdot \sqrt{\pi}$$

So that $\lim_{R\to\infty} \int_{\gamma_R} g(z)\,\mathrm{d}z = 0$.

Done!

# Integrating round a branch point

We evaluate $\int_0^\infty \frac{x^\alpha}{1+x^2}\,\mathrm{d}x$ ($0 < |\alpha| < 1$, $\alpha$ real).

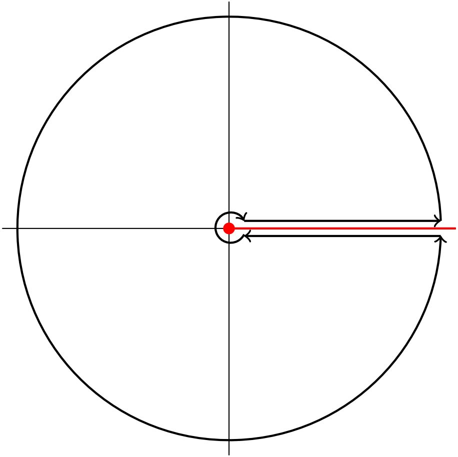
We should (probably) use the complex fuction $f(z) = \frac{z^\alpha}{1+z^2}$ but which branch?

We choose the branch cut along the positive real axis and define a branch of $z^\alpha$ by $e^{\alpha \log z}$, where $\log z = \ln|z| + i \arg z$ with $0 \leqslant \arg z < 2\pi$.

So $z^\alpha = |z|^\alpha \cdot e^{i\alpha \arg z}$.

And what about the contour? Something with the positive real axis . . .

# The Contour



The contour $\Gamma_{\varepsilon,R}$ consists of

Interval $[\varepsilon, R]$

Circle of radius $R$ (anticlockwise)

Interval $[\varepsilon, R]$ (in reverse)

Circle of radius $\varepsilon$ (clockwise)

# Singularities and residues

The function $f(z)$ has two singularities inside $\Gamma_{\varepsilon,R}$:
$i$ and $-i$ (poles of order 1), when $\varepsilon < 1 < R$.

Residues:

$$\operatorname{Res}(f,i) = \frac{1}{2i} i^\alpha = \frac{1}{2i} e^{\frac{1}{2}\alpha\pi i}$$

and

$$\operatorname{Res}(f,-i) = \frac{1}{-2i}(-i)^\alpha = -\frac{1}{2i} e^{\frac{3}{2}\alpha\pi i}$$

So

$$\oint_{\Gamma_{\varepsilon,R}} f(z)\,\mathrm{d}z = \pi\left(e^{\frac{1}{2}\alpha\pi i} - e^{\frac{3}{2}\alpha\pi i}\right)$$

# Splitting the integral

The integral splits into

- $\int_{\varepsilon}^{R} \frac{x^{\alpha}}{1+x^2}\,\mathrm{d}x$
- $\oint_{|z|=R} f(z)\,\mathrm{d}z$
- $\int_{R}^{\varepsilon} \frac{x^{\alpha} e^{2\alpha\pi i}}{1+x^2}\,\mathrm{d}x = -e^{2\alpha\pi i}\int_{\varepsilon}^{R} \frac{x^{\alpha}}{1+x^2}\,\mathrm{d}x$
- $\oint_{|z|=\varepsilon} f(z)\,\mathrm{d}z$

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# Estimates

The big circle:

$$\left|\oint_{|z|=R} f(z)\,\mathrm{d}z\right| \leqslant 2\pi R\frac{R^\alpha}{R^2-1}$$

as $1+\alpha<2$ the right-hand side converges to 0 as $R\to\infty$

The small circle

$$\left|\oint_{|z|=\varepsilon} f(z)\,\mathrm{d}z\right| \leqslant 2\pi\varepsilon\frac{\varepsilon^\alpha}{1-\varepsilon^2}$$

as $1+\alpha>0$ the right-hand side converges to 0 as $\varepsilon\to 0$

# Putting it together

After we let $R \to \infty$ and $\varepsilon \to 0$ we get

$$(1 - e^{2\alpha\pi i}) \int_0^\infty \frac{x^\alpha}{1 + x^2}\,\mathrm{d}x = \pi(e^{\frac{1}{2}\alpha\pi i} - e^{\frac{3}{2}\alpha\pi i})$$

so that

$$\int_0^\infty \frac{x^\alpha}{1 + x^2}\,\mathrm{d}x = \pi \frac{e^{\frac{1}{2}\alpha\pi i}(1 - e^{\alpha\pi i})}{1 - e^{2\alpha\pi i}} = \frac{\pi}{2\cos\frac{1}{2}\alpha\pi}$$

(work it out on paper . . . )

# What to do?

From the book: 6.2, 6.3.

Suitable problems: 6.7 - 6.22; 4.13

Recommended problems: 6.7, 6.8, 6.16, 6.19, 6.20, 6.21; 4.13.

In 6.19–21 don't try all of them right away, save some for later . . .

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