wi4243AP: Complex Analysis

week 7, Monday

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Outline

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One more example: $\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x$

We take

$$f(z) = \frac{\log z}{1 + z^4}$$

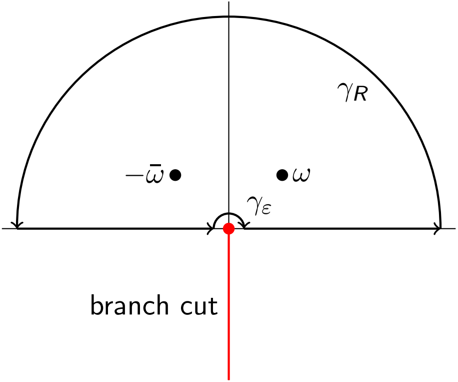
where $\log z$ is the branch of the logarithm defined by

$$\log z = \ln|z| + i \arg z \qquad (-\tfrac{1}{2}\pi < \arg z < \tfrac{3}{2}\pi)$$

so the branch cut is the negative imaginary axis.
Note:

- if $x > 0$ then $\log x = \ln x$;
- if $x < 0$ then $\log x = \ln|x| + i\pi$.

$\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x$: the contour



The contour $\Gamma_{\varepsilon,R}$ consists of

Interval $[\varepsilon, R]$

Semicircle of radius R (anticlockwise)

Interval $[-R, -\varepsilon]$

Semicircle of radius ε (clockwise)

and the singularities

$\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x$: singularities and residues

The singularities of f are the zeros of z^4+1, to wit ω, $-\omega$, $\bar\omega$ and $-\bar\omega$, where

$$\omega = e^{\frac{\pi}{4}i} = \frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}.$$

of these ω and $-\bar\omega$ lie inside $\Gamma_{\varepsilon,R}$ (when $\varepsilon < 1 < R$ of course); the residues are

$$\operatorname{Res}(f,\omega) = \frac{\frac{1}{4}\pi i}{4i\omega} = \frac{\pi}{16}\bar\omega \text{ and } \operatorname{Res}(f,-\bar\omega) = \frac{\frac{3}{4}\pi i}{4i\bar\omega} = \frac{3\pi}{16}\omega$$

so

$$\oint_{\Gamma_{\varepsilon,R}} f(z)\,\mathrm{d}z = 2\pi i \cdot \frac{\pi}{16}(\bar\omega + 3\omega) = -\frac{\pi^2\sqrt{2}}{8} + \frac{\pi^2\sqrt{2}}{4}i$$

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$\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x$: Splitting the integral

The integral splits into

- $\displaystyle\int_\varepsilon^R \frac{\ln x}{1+x^4}\,\mathrm{d}x$
- $\displaystyle\int_{\gamma_R} f(z)\,\mathrm{d}z$
- $\displaystyle\int_{-R}^{-\varepsilon} \frac{\ln|x|+\pi i}{1+x^4}\,\mathrm{d}x = \int_\varepsilon^R \frac{\ln x}{1+x^4}\,\mathrm{d}x + \pi i\int_\varepsilon^R \frac{1}{1+x^4}\,\mathrm{d}x$
- $\displaystyle\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z$

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$\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x$: Estimates

The big semicircle:

$$\left|\oint_{\gamma_R} f(z)\,\mathrm{d}z\right| \leqslant \pi R\frac{\ln R+\pi}{R^4-1}$$

the right-hand side converges to 0 as $R\to\infty$

The small semicircle

$$\left|\oint_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right| \leqslant \pi\varepsilon\frac{-\ln\varepsilon+\pi}{1-\varepsilon^4}$$

the right-hand side converges to 0 as $\varepsilon\to 0$

$\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x$: Putting it together

After we let $R \to \infty$ and $\varepsilon \to 0$ we get

$$2\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x + \pi i \int_0^\infty \frac{1}{1+x^4}\,\mathrm{d}x = -\frac{\pi^2\sqrt{2}}{8} + \frac{\pi^2\sqrt{2}}{4}i$$

so that

$$\int_0^\infty \frac{\ln x}{1+x^4}\,\mathrm{d}x = -\frac{\pi^2\sqrt{2}}{16}$$

and

$$\int_0^\infty \frac{1}{1+x^4}\,\mathrm{d}x = \frac{\pi\sqrt{2}}{4}$$

(which we knew already).

Definition

If $u : \mathbb{R} \to \mathbb{R}$ is such that $\int_{-\infty}^{\infty} \bigl|u(x)\bigr| \,\mathrm{d}x < \infty$ then we put

$$\mathcal{F}\{u(x)\}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} u(x) \,\mathrm{d}x \qquad (\omega \text{ real})$$

the Fourier transform of u
(common abbreviation: $\mathcal{F}\{u(x)\}(\omega) = U(\omega)$).

Properties

linearity $\mathcal{F}\{\alpha u + \beta v\} = \alpha\mathcal{F}\{u\} + \beta\mathcal{F}\{v\}$

shifting $\mathcal{F}\{u(x-a)\} = e^{ia\omega}\mathcal{F}\{u(x)\}$

shifting $\mathcal{F}\{e^{-i\omega_0 x}u(x)\} = U(\omega - \omega_0)$

scaling $\mathcal{F}\{u(bx)\} = \frac{1}{b}U(\frac{1}{b}\omega)$

derivatives $\mathcal{F}\{u'(x)\} = -i\omega\mathcal{F}\{u(x)\}$

derivatives $\frac{\mathrm{d}}{\mathrm{d}\omega}\mathcal{F}\{u(x)\} = i\omega\mathcal{F}\{u(x)\}$

convolution $\mathcal{F}\{u * v\} = \mathcal{F}\{u\} \cdot \mathcal{F}\{v\}$

Convolution product: $(u * v)(x) = \int_{-\infty}^{\infty} u(\xi)v(x-\xi)\,\mathrm{d}\xi$

Inversion and Parseval's identity

The Fourier tranform is invertible:

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} U(\omega)\, \mathrm{d}\omega$$

It preserves norms:

$$\int_{-\infty}^{\infty} \left|u(x)\right|^2 \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|U(\omega)\right|^2 \mathrm{d}\omega$$

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Warning

Four possible conventions:

- $\displaystyle\int_{-\infty}^{\infty} e^{i\omega x} u(x)\,\mathrm{d}x$ (the present text)
- $\displaystyle\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x} u(x)\,\mathrm{d}x$ (norm-preserving)
- $\displaystyle\int_{-\infty}^{\infty} u(x) e^{-i\omega x}\,\mathrm{d}x$ (looks like complex inner product)
- $\displaystyle\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x) e^{-i\omega x}\,\mathrm{d}x$ (mathematician's favourite)

Just remember to check the convention of the book that you are reading.

$\mathcal{F}$ as an excuse to calculate residues

We can calculate

$$\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,\mathrm{d}x$$

if f extends to a 'tame' complex function.

- a finite number of isolated singularities
- $\lim_{z\to\infty} f(z) = 0$
- no singularities on the real axis

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A useful lemma

Lemma (Jordan)

If $\lim_{z\to\infty} f(z) = 0$ *and if* $\omega > 0$ *then*

$$\lim_{R\to\infty} \int_{C_R} e^{i\omega z} f(z)\,\mathrm{d}z = 0$$

where C_R *is the semi-circle* $\{z : |z| = r, \operatorname{Im} z \geqslant 0\}$.

Write $M_R = \max\{|f(z)| : z \in C_R\}$; by assumption

$$\lim_{R\to\infty} M_R = 0$$

Jordan's Lemma: proof

Parametrize the semi-circle: $z = Re^{it}$ $(0 \leqslant t \leqslant \pi)$ and rewrite the integral:

$$\int_0^\pi f(Re^{it})e^{i\omega R\cos t}e^{-\omega R\sin t}Re^{it}\,\mathrm{d}t$$

From this we get:

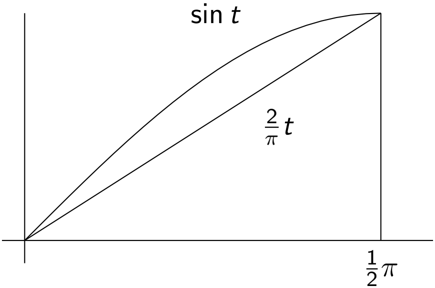
$$\begin{aligned}\left|\int_{C_R} f(z)\,\mathrm{d}z\right| &\leqslant M_R\cdot R\cdot\int_0^\pi e^{-\omega R\sin t}\,\mathrm{d}t\\ &\overset{*}{\leqslant} 2M_R\cdot R\cdot\int_0^{\frac{1}{2}\pi} e^{-\omega R\frac{2}{\pi}t}\,\mathrm{d}t\\ &= 2M_R\cdot R\cdot\frac{\pi}{2\omega R}(1-e^{-\omega R})\end{aligned}$$

Jordan's Lemma: proof

Clearly:

$$\lim_{R\to\infty} \frac{M_R\pi}{\omega}(1 - e^{-\omega R}) = 0$$

And inequality $*$ follows from $\sin t \geqslant \frac{2}{\pi}t$ $(0 \leqslant t \leqslant \frac{1}{2}\pi)$



Consequences

If $\omega > 0$ then

$$\mathcal{F}\{f(x)\} = 2\pi i[\text{ sum of residues of } e^{i\omega z} f(z) \text{ above real axis}]$$

If $\omega < 0$ then we can take the semi-circle below the real axis to obtain

$$\mathcal{F}\{f(x)\} = -2\pi i[\text{ sum of residues of } e^{i\omega z} f(z) \text{ below real axis}]$$

Example: $\mathcal{F}\{\frac{1}{x^4+1}\}$

Take $\omega > 0$ and rename the zeros of $z^4 + 1$ as α, $\bar{\alpha}$, $-\alpha$ and $-\bar{\alpha}$ (still: $\alpha = \frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$).

From earlier examples we know what we will get:

$$\operatorname{Res}(e^{i\omega z}f(z), \alpha) = \frac{1}{4i}\bar{\alpha}e^{i\omega\alpha} \text{ and } \operatorname{Res}(e^{i\omega z}f(z), -\bar{\alpha}) = \frac{1}{4i}\alpha e^{-i\omega\bar{\alpha}}$$

After some 'hard' work:

$$\mathcal{F}\{\frac{1}{x^4+1}\} = \frac{2\pi i}{4i}(\bar{\alpha}e^{i\omega\alpha} + \alpha e^{-i\omega\bar{\alpha}})$$
$$= \frac{\pi}{2}\sqrt{2}e^{-\frac{\omega}{2}\sqrt{2}}(\cos\frac{\omega}{2}\sqrt{2} + \sin\frac{\omega}{2}\sqrt{2})$$

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One-dimensional Klein-Gordon equation

To find its Green's function solve

$$\frac{\partial^2 g}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2 g}{\partial t^2} + \alpha^2 g\right) = -\delta(x-\xi)\delta(t-\tau)$$

with $-\infty < x, \xi < \infty$ and $t, \tau > 0$ and where c is a positive constant.

Initial conditions: $g(x, 0|\xi, \tau) = g_t(x, 0|\xi, \tau) = 0$

Boundary conditions: $\lim_{|x|\to\infty} \bigl|g(x, t|\xi, \tau)\bigr| < \infty$.

Transform this

First Laplace ($t \to s$):

$$\frac{\partial^2 G}{\partial x^2} - \left(\frac{s^2 + a^2}{c^2}\right) G = -\delta(x - \xi)e^{-s\tau}$$

then Fourier ($x \to \omega$):

$$\left(-\omega^2 - \frac{s^2 + a^2}{c^2}\right) \mathcal{F}\{G\} = -e^{-s\tau} e^{i\omega\xi}$$

or

$$\mathcal{F}\{G\} = \frac{e^{-s\tau} e^{i\omega\xi}}{\omega^2 + \frac{1}{c^2}(s^2 + a^2)}$$

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Inversion

So now:

$$G(x, s|\xi, \tau) = \frac{1}{2\pi} e^{-s\tau} \int_{-\infty}^{\infty} e^{-i\omega x} \frac{e^{i\omega\xi}}{\omega^2 + \frac{1}{c^2}(s^2 + a^2)} \, \mathrm{d}\omega$$
$$= \frac{1}{2\pi} e^{-s\tau} \int_{-\infty}^{\infty} \frac{e^{-i(x-\xi)\omega}}{\omega^2 + \frac{1}{c^2}(s^2 + a^2)} \, \mathrm{d}\omega$$

Singularities and residues

The singularities are at $\omega^+ = \frac{i}{c}\sqrt{s^2+a^2}$ and $\omega^- = -\frac{i}{c}\sqrt{s^2+a^2}$, with residues

$$\operatorname{Res}(\cdot,\omega^+) = \frac{e^{\frac{x-\xi}{c}\sqrt{s^2+a^2}}}{2\frac{i}{c}\sqrt{s^2+a^2}}$$

and

$$\operatorname{Res}(\cdot,\omega^-) = \frac{e^{-\frac{x-\xi}{c}\sqrt{s^2+a^2}}}{-2\frac{i}{c}\sqrt{s^2+a^2}}$$

The integrals

From the formula remember that we take the residues above the real axis if $-(x-\xi) > 0$ (i.e., $x < \xi$) and those below the real axis if $-(x-\xi) > 0$ (i.e., $x > \xi$).

So if $x < \xi$ we get

$$G(x, s|\xi, \tau) = \frac{1}{2\pi} e^{-s\tau} \cdot 2\pi i \operatorname{Res}(\cdot, \omega^{+}) = \frac{ce^{-s\tau}}{2} \frac{\exp\left(\frac{x-\xi}{c}\sqrt{s^2+a^2}\right)}{\sqrt{s^2+a^2}}$$

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The integrals

And if $x > \xi$ we get

$$\begin{aligned} G(x,s|\xi,\tau) &= \frac{1}{2\pi} e^{-s\tau} \cdot -2\pi i \operatorname{Res}(\cdot, \omega^-) \\ &= \frac{ce^{-s\tau}}{2} \frac{\exp\left(-\frac{x-\xi}{c}\sqrt{s^2+a^2}\right)}{\sqrt{s^2+a^2}} \end{aligned}$$

Note: the minus sign in front of the $2\pi i$ is because we integrate clockwise around the singularity in the lower half plane.

A single formula

We can combine both answers into one:

$$G(x,s|\xi,\tau) = \frac{ce^{-s\tau}}{2}\frac{\exp\left(-\frac{|x-\xi|}{c}\sqrt{s^2+a^2}\right)}{\sqrt{s^2+a^2}}$$

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Definition

The formula

For $f : [0, \infty) \to \mathbb{C}$ its Laplace transform is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

Here $s = \sigma + i\tau$ is complex.

If there are M and a such that $|f(t)| \leqslant Me^{at}$ for $t \geqslant 0$ then $\mathcal{L}\{f(t)\}$ exists whenever $\operatorname{Re} s > a$.

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Properties

The map $\mathcal{L}$ is linear and injective, i.e.,

- $\mathcal{L}\{\alpha f + \beta g\} = \alpha\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}$ (where defined)
- if $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ for σ large enough then $f = g$

Also, Leibniz' rule applies:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty -te^{-st} f(t)\,\mathrm{d}t$$

so, $F'(s) = \mathcal{L}\{-t \cdot f(t)\}$ and $F(s)$ is analytic.

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Example: $\sin t$

We calculate $\mathcal{L}\{\sin t\}$:

$$\frac{1}{2i}\left(\mathcal{L}\{e^{it}\} - \mathcal{L}\{e^{-it}\}\right) = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{s^2+1}$$

Note

- $\int_0^\infty e^{-st} \sin t \,\mathrm{d}t$ exists whenever $\operatorname{Re} s > 0$
- the resulting function has an analytic extension to all of $\mathbb{C}$ (except at i and $-i$).

Luckily this happens a lot, see the formulas on pp 327–9.

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Bromwich integral

The formula

Assume $|f(t)| \leqslant Me^{\alpha t}$ for some α and let $a > \alpha$. Then

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{a-iR}^{a+iR} e^{st} F(s)\,\mathrm{d}s$$

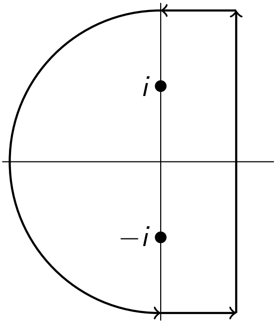
the integral is along the vertical straight line $\operatorname{Re} s = a$.

So, e.g.,

$$\sin t = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{a-iR}^{a+iR} e^{st} \frac{1}{s^2+1}\,\mathrm{d}s$$

Example: $\sin t$

Let's calculate the integral for $\sin t$.
We use the following contour Γ_R with two singularities inside



The segment $[a - iR, a + iR]$

The segment $[a + iR, iR]$

The semi-circle γ_R

The segment $[-iR, a - iR]$

The contour integral

Of course

$$\frac{1}{2\pi i}\oint_{\Gamma_R}\frac{e^{st}}{s^2+1}\,\mathrm{d}s=\operatorname{Res}(f,i)+\operatorname{Res}(f,-i)=\frac{e^{it}}{2i}+\frac{e^{-it}}{-2i}=\sin t$$

So, all we need to do is show that

$$\lim_{R\to\infty}\frac{1}{2\pi i}\oint_{\Gamma_R}\frac{e^{st}}{s^2+1}\,\mathrm{d}s=\lim_{R\to\infty}\int_{a-iR}^{a+iR}\frac{e^{st}}{s^2+1}\,\mathrm{d}s$$

The other bits

$[a+iR, iR]$ length: a, upper bound for modulus: $e^{at}/(R^2-1)$

$[-iR, a-iR]$ length: a, upper bound for modulus: $e^{at}/(R^2-1)$

γ_R length: πR, upper bound for modulus: $1/(R^2-1)$

In all three cases: limit of integral is zero.

Done.

What to do?

From the book: 6.4, 7.2, transforms resulting from PDE
Suitable problems: 6.23 - 6.28; 7.16 - 7.25.
Recommended problems: 6.23, 6.27, 7.22, 7.23

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