wi4243AP: Complex Analysis

week 7, Friday

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Outline

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Bromwich integral

The formula

Assume $|f(t)| \leqslant Me^{\alpha t}$ for some α and let $a > \alpha$. Then

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{a-iR}^{a+iR} e^{st} F(s)\, \mathrm{d}s$$

the integral is along the vertical straight line $\operatorname{Re} s = a$.

So, e.g.,

$$\sin t = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{a-iR}^{a+iR} e^{st} \frac{1}{s^2+1}\, \mathrm{d}s$$

For what functions does this work?

Sufficient condition

If F is analytic on an annulus of outer radius ∞ and $\lim_{s\to\infty} F(s) = 0$ then

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_R} e^{st} F(s)\,\mathrm{d}s = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{a-iR}^{a+iR} e^{st} F(s)\,\mathrm{d}s$$

Write $M_R = \max\bigl\{|F(s)| : |s| \geqslant R\bigr\}$, so $\lim_{R\to\infty} M_R = 0$.

For what functions does this work?

$[a+iR, iR]$ length: a, upper bound for modulus: $e^{at}M_R$

$[-iR, a-iR]$ length: a, upper bound for modulus: $e^{at}M_R$

γ_R use Jordan's Lemma, suitably modified:

$$\left|\int_{\gamma_R} e^{st}F(s)\,\mathrm{d}s\right| \leqslant \frac{\pi M_R}{t}(1-e^{-tR})$$

for details: work problem 7.20

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An example

Find the inverse Laplace transform of $F(s) = \arctan\frac{1}{s}$ (principal branch).

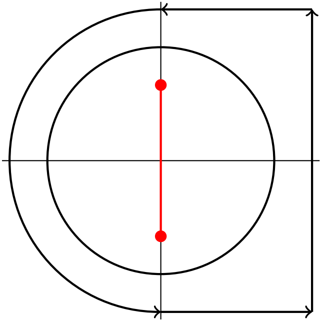
Note: $\lim_{s\to\infty}\arctan\frac{1}{s} = 0$.

So, we know

$$\lim_{R\to\infty}\frac{1}{2\pi i}\oint_{\Gamma_R} e^{st}\arctan\frac{1}{s}\,\mathrm{d}s = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{a-iR}^{a+iR} e^{st}\arctan\frac{1}{s}\,\mathrm{d}s$$

An example

We deform the contour described above.



The branch cut is $[-i, i]$

We shrink Γ_R to a circle C

We calculate

$$\frac{1}{2\pi i}\oint_C e^{st}\arctan\frac{1}{s}\,\mathrm{d}s$$

Cauchy-Goursat: this suffices

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An example

The function $\arctan \frac{1}{s}$ is analytic on the annulus $\{s : |s| > 1\}$ and given by the Laurent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{s}\right)^{2n+1}$$

this series converges uniformly on C, so

$$\frac{1}{2\pi i} \oint_C e^{st} \arctan \frac{1}{s} \, \mathrm{d}s = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{1}{2\pi i} \oint_C \frac{e^{st}}{s^{2n+1}} \, \mathrm{d}s$$

An example

Cauchy's formula (general version):

$$\frac{1}{2\pi i}\oint_C \frac{e^{st}}{s^{2n+1}}\,\mathrm{d}s = \frac{1}{(2n)!}\frac{\partial^{2n}}{\partial s^{2n}}e^{st}\Big|_{s=0} = \frac{t^{2n}}{(2n)!}$$

so we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!} = \frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!} = \frac{\sin t}{t}$$

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One-dimensional Klein-Gordon equation

We were busy solving

$$\frac{\partial^2 g}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2 g}{\partial t^2} + \alpha^2 g\right) = -\delta(x-\xi)\delta(t-\tau)$$

with $-\infty < x, \xi < \infty$ and $t, \tau > 0$ and where c is a positive constant.

Initial conditions: $g(x, 0|\xi, \tau) = g_t(x, 0|\xi, \tau) = 0$

Boundary conditions: $\lim_{|x|\to\infty} \bigl|g(x, t|\xi, \tau)\bigr| < \infty$.

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What we did thus far, forward

First Laplace ($t \to s$):

$$\frac{\partial^2 G}{\partial x^2} - \left(\frac{s^2 + a^2}{c^2}\right) G = -\delta(x - \xi) e^{-s\tau}$$

then Fourier ($x \to \omega$):

$$\left(-\omega^2 - \frac{s^2 + a^2}{c^2}\right) \mathcal{F}\{G\} = -e^{-s\tau} e^{i\omega\xi}$$

or

$$\mathcal{F}\{G\} = \frac{e^{-s\tau} e^{i\omega\xi}}{\omega^2 + \frac{1}{c^2}(s^2 + a^2)}$$

What we did thus far, inversion

We already found:

$$G(x, s|\xi, \tau) = \frac{ce^{-s\tau}}{2} \frac{\exp\left(-\frac{|x-\xi|}{c}\sqrt{s^2 + a^2}\right)}{\sqrt{s^2 + a^2}}$$

The factor $e^{-s\tau}$ brings us to a shifting formula and $\frac{c}{2}$ is just a constant, so we leave these out.

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Inversion

Once we calculate we calculate the inverse transform

$$g(x,t|\xi,\tau) = \frac{1}{2\pi i}\int_B e^{st}\cdot\frac{\exp\left(-\frac{|x-\xi|}{c}\sqrt{s^2+a^2}\right)}{\sqrt{s^2+a^2}}\,\mathrm{d}s$$

(where B is some Bromwich line) the Green's function will be

$$\frac{c}{2}H(t-\tau)\cdot g(x,(t-\tau)|\xi,\tau).$$

$\sqrt{s^2 + a^2}$

We want a branch of $\sqrt{s^2 + a^2}$ that takes on its normal values on the real axis.
We define a branch of

$$\left(\frac{s - ia}{s + ia}\right)^{\frac{1}{2}}$$

and multiply this by $s + ia$; having a bilinear transformation makes it easier to determine values near branch cuts.

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$\sqrt{s^2 + a^2}$

We take the positive real axis as a branch cut for $\sqrt{z}$ and we define $z^{\frac{1}{2}} = -\sqrt{|z|}e^{\frac{1}{2}\arg z}$ $(0 \leqslant \arg z < 2\pi i)$.
Thus, for us: $(-1)^{\frac{1}{2}} = -i$.

If z approaches 1 from above then $z^{\frac{1}{2}}$ approaches -1.
If z approaches 1 from below then $z^{\frac{1}{2}}$ approaches 1.

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$\sqrt{s^2 + a^2}$

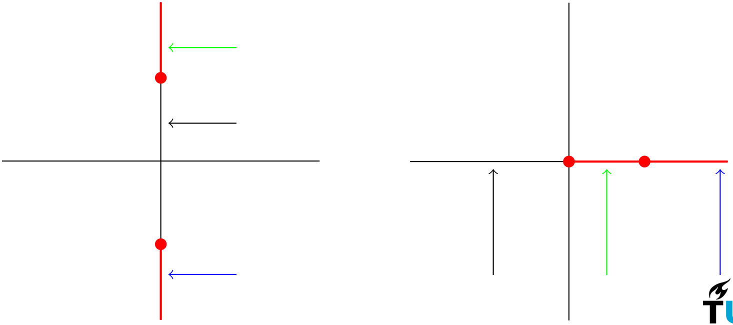
So our definition of $\sqrt{s^2 + a^2}$ is

$$(s + ia)\left(\frac{s - ia}{s + ia}\right)^{\frac{1}{2}}$$

Then at $s = 0$ the value comes out $-i \cdot ai = a$.
This suffices to have $\sqrt{x^2 + a^2}$ positive for all real x.

$\sqrt{s^2 + a^2}$

We shall let the Bromwich line approach the imaginary axis. So we need to know what happens to $\sqrt{s^2 + a^2}$ when s approaches $i\omega$ from the right.



$\sqrt{s^2 + a^2}$

On the imaginary axis, say with $s = \omega i$ we have

$$\frac{s - ia}{s + ia} = \frac{\omega - a}{\omega + a}$$

which is negative if $-a < \omega < a$ and positive elsewhere.

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$\sqrt{s^2 + a^2}$

Black arrow: if s approaches $i\omega$ then $\frac{s-ia}{s+ia}$ approaches (the negative number) $\frac{\omega-a}{\omega+a}$.
Then our $\sqrt{s^2+a^2}$ approaches

$$(i\omega + ia) \cdot -i \left(\frac{a-\omega}{a+\omega}\right)^{\frac{1}{2}}$$

which comes out as the real square root of $a^2 - \omega^2$.
Note: $\omega + a > 0$, hence $\omega + a = \sqrt{(\omega+a)^2}$.

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$\sqrt{s^2 + a^2}$

Green and blue arrows: if s approaches $i\omega$ then $\frac{s-ia}{s+ia}$ approaches (the positive number) $\frac{\omega-a}{\omega+a}$ from below, so that $\left(\frac{s-ai}{s+ai}\right)^{\frac{1}{2}}$ approaches

$$-\sqrt{\frac{\omega-a}{\omega+a}}e^{\pi i} = \sqrt{\frac{\omega-a}{\omega+a}}.$$

And so $\sqrt{s^2+a^2}$ approaches

$$i(\omega+a)\left(\frac{\omega-a}{\omega+a}\right)^{\frac{1}{2}} = \begin{cases} i\sqrt{\omega^2-a^2} & \text{if } \omega > a \\ -i\sqrt{\omega^2-a^2} & \text{if } \omega < -a \end{cases}$$

Note: if $\omega < -a$ then $\omega + a = -\sqrt{(\omega+a)^2}$

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The inverse transform

As the Bromwich line approaches the imaginary axis the integral approaches the sum of three integrals, the first of which is the bit between $-ai$ and ai (parametrized by $\omega \mapsto i\omega$):

$$\frac{1}{2\pi i}\int_{-a}^{a} e^{i\omega t}\cdot\frac{\exp\left(-\frac{|x-\xi|}{c}\sqrt{a^2-\omega^2}\right)}{\sqrt{a^2-\omega^2}}\,\mathrm{d}i\omega$$

which equals

$$\frac{1}{\pi}\int_{0}^{a} \cos\omega t\cdot\frac{\exp\left(-\frac{|x-\xi|}{c}\sqrt{a^2-\omega^2}\right)}{\sqrt{a^2-\omega^2}}\,\mathrm{d}\omega$$

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The inverse transform

The other two integrals are

$$\frac{1}{2\pi i}\int_a^\infty e^{i\omega t}\cdot\frac{\exp\left(-i\frac{|x-\xi|}{c}\sqrt{\omega^2-a^2}\right)}{i\sqrt{\omega^2-a^2}}\,\mathrm{d}i\omega$$

and

$$\frac{1}{2\pi i}\int_{-\infty}^{-a} e^{i\omega t}\cdot\frac{\exp\left(i\frac{|x-\xi|}{c}\sqrt{\omega^2-a^2}\right)}{-i\sqrt{\omega^2-a^2}}\,\mathrm{d}i\omega$$

respectively.

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The inverse transform

These simplify to

$$\frac{1}{2\pi i}\int_a^\infty \frac{\exp\left(i\left(\omega t - \frac{|x-\xi|}{c}\sqrt{\omega^2 - a^2}\right)\right)}{\sqrt{\omega^2 - a^2}}\,\mathrm{d}\omega$$

and

$$-\frac{1}{2\pi i}\int_a^\infty \frac{\exp\left(-i\left(\omega t - \frac{|x-\xi|}{c}\sqrt{\omega^2 - a^2}\right)\right)}{\sqrt{\omega^2 - a^2}}\,\mathrm{d}\omega$$

respectively.

The inverse transform

Finally, the sum of these two is

$$\frac{1}{\pi}\int_a^\infty \frac{\sin\left(\omega t - \frac{|x-\xi|}{c}\sqrt{\omega^2 - a^2}\right)}{\sqrt{\omega^2 - a^2}}\,\mathrm{d}\omega$$

This is far from an explicit expression but such an integral representation is better than nothing.

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Back to Fourier

From an old exam

Calculate

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + 25} \, \mathrm{d}x$$

for $\omega \in \mathbb{R}$.

Bad method

Let $f(z) = \frac{\cos \omega z}{z^2+25}$ and integrate along $[-R, R]$ and the semicircle γ_R $(= \{z : |z| = R, \operatorname{Im} z \geqslant 0\})$.
This is bad because . . .

Good method

The integral is the real part of the Fourier transform of $\frac{1}{x^2+25}$; also called its Fourier cosine transform.
So, we use $f(z) = \frac{e^{i\omega z}}{z^2+25}$ and the curve mentioned on the previous slide.

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Finish

The integral along the curve is equal to $2\pi i \operatorname{Res}(f, 5i)$, which is equal to $\frac{\pi}{5} e^{-5\omega}$, provided $\omega > 0$.

If $\omega < 0$ we go through the lower half plane and we get $-2\pi i \operatorname{Res}(f, -5i)$, which is equal to $\frac{\pi}{5} e^{5\omega}$.

The net result is

$$\frac{\pi}{5} e^{-5|\omega|}$$

A Problem and a solution

$\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$ should have the value 0, but it does not because its value does not exist.

A solution by Cauchy: define

$$\mathrm{P}\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} \frac{1}{x}\,\mathrm{d}x + \int_{\varepsilon}^{1} \frac{1}{x}\,\mathrm{d}x\right)$$

the Principal Value of the integral.

General definition

In general, if f has a singularity at c in the interval $[a, b]$ then

$$\mathrm{P}\int_a^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^b f(x)\,\mathrm{d}x\right)$$

so, first calculate $\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^b f(x)\,\mathrm{d}x$ and then let $\varepsilon \to 0$.

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A useful fact

Theorem (See 6.2.4)

Let f have a pole of order 1 at z_0 and let γ_ε denote the arc parametrized by $t \mapsto z_0 + \varepsilon e^{it}$ ($\alpha \leqslant t \leqslant \beta$).
Then

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = (\beta - \alpha) i \operatorname{Res}(f, z_0)$$

The idea is to integrate the Laurent series of f at z_0.
The power $(z - z_0)^{-1}$ gives

$$b_1 \int_{\gamma_\varepsilon} \frac{1}{(z - z_0)}\,\mathrm{d}z = b_1 \int_\alpha^\beta i\,\mathrm{d}t = b_1(\beta - \alpha) i$$

A useful fact

The 'rest' of f, by which we mean $g(z) = f(z) - \frac{b_1}{z-z_0}$, is analytic on a neighbourhood of z_0. We have

$$\left|\int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z\right| \leqslant M_\varepsilon(\beta-\alpha)\varepsilon$$

where M_ε is the maximum of $|g(z)|$ on γ_ε (and $(\beta-\alpha)\varepsilon$ is the length of γ_ε).

As $\varepsilon \to 0$ we have $M_\varepsilon \to |g(z_0)|$, so that

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z = 0.$$

$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$

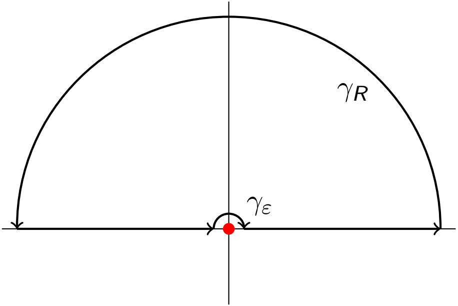
Integrating $\frac{\sin z}{z}$ will not help because that function is analytic everywhere: all contour integrals are zero.

Try $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,\mathrm{d}x$; its imaginary part is twice what we want (as $\frac{\sin x}{x}$ is even).

As the integral does not exist; we use its principal value.

We take $f(z) = \frac{e^{iz}}{z}$ and a familiar contour.

$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,\mathrm{d}x$: the contour



The contour $\Gamma_{\varepsilon,R}$ consists of

Interval $[\varepsilon, R]$

Semicircle of radius R (anticlockwise)

Interval $[-R, -\varepsilon]$

Semicircle of radius ε (clockwise)

$\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,\mathrm{d}x$: continued

- $\oint_{\Gamma_{\varepsilon,R}} f(z)\,\mathrm{d}z = 0$ (0 is outside the contour)
- $\lim_{R\to\infty} \int_{\gamma_R} f(z)\,\mathrm{d}z = 0$ (Jordan's Lemma)
- $\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = -\pi i$ (useful fact)
- $\int_{-R}^{-\varepsilon} f(x)\,\mathrm{d}x + \int_{\varepsilon}^{R} f(x)\,\mathrm{d}x = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,\mathrm{d}x$ ($\frac{\cos x}{x}$ is odd)

We find

$$0 = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \oint_{\Gamma_{\varepsilon,R}} f(z)\,\mathrm{d}z = -\pi i + 2i\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$$

so that

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

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What to do?

From the book: 7.2, 6.5
Suitable problems: 7.16 - 7.25; 6.28 - 6.42.
Recommended problems: 7.22, 7.23, 6.30, 6.32, 6.33, 6.39

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