# wi4243AP: Complex Analysis

## week 8, Monday

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# Outline

# The question

http://www.wisfaq.nl/show3archive.asp?id=60954&j=2009
I have following integral

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)}$$

Sing points in the upper half plane are $x = i$ (pole order 1) and $x = 0$ (pole order 1/2).
I can rewrite the integral:

$$\int_+ f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, i) = \int_{-R}^{-\varepsilon} \ldots + \int_{\gamma_\varepsilon} \ldots + \int_\varepsilon^R \ldots + \int_{\gamma_R}$$

# The question

The second and fourth integral (on the right-hand side) tend 0 (can be proved via an estimate, I did that WAS OK.
If I want to calculate the given integral then I have

$$2\pi i \operatorname{Res}(f,i) = \frac{\pi}{\sqrt{i}} = \int_{-\infty}^{0} \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} + \int_{0}^{\infty} \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)}$$

My problem now is: How do I get the desired integral $\int_0^\infty \mathrm{d}x/\sqrt{x}(x^2+1)$

Well ...?

# Answer

- $\sqrt{i} = \frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$ and so $\pi/\sqrt{i} = \frac{\pi}{2}\sqrt{2} - i\frac{\pi}{2}\sqrt{2}$
- For $x < 0$ we have $\sqrt{x} = i\sqrt{|x|}$ so $\int_{-\infty}^{0}$ becomes $-i\int_{0}^{\infty}$

We get

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi\sqrt{2}}{2}$$

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# One of Euler's best

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

# A useful function and contour

We use

$$f(z) = \frac{\pi}{z^2}\frac{\cos \pi z}{\sin \pi z}$$

and large squares for contours.
$\Gamma_N$ is the square with vertices $(\pm 1 \pm i)(N + \frac{1}{2})$.

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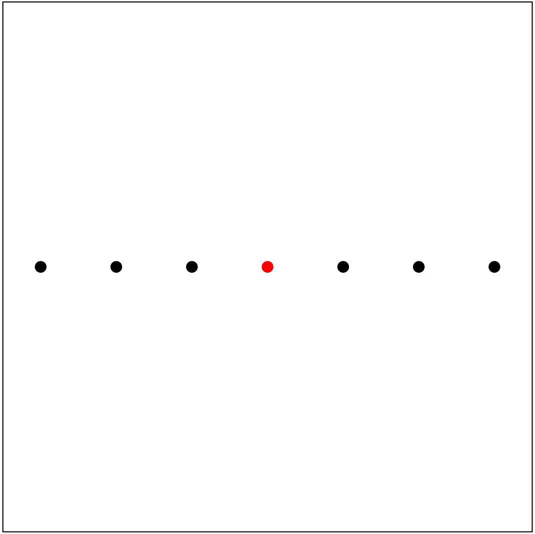
# Here's $\Gamma_3$



The contour $\Gamma_3$

Pole of order 3 at 0

Poles of order 1 at all nonzero integers

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# Residues

The residues at nozero $n$ are easy:

$$\operatorname{Res}(f,n) = \frac{\pi}{n^2}\frac{\cos\pi n}{\pi\cos\pi n}$$

Pole of order 1, so we substitute $n$ in

$$\frac{\pi}{z^2}\frac{\cos\pi z}{(\sin\pi z)'}$$

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# Residues

We calculate a few terms of the Laurent series of $\frac{\cos z}{\sin z}$:

$$\left(\frac{a_{-1}}{z} + a_0 + a_1 z + \cdots\right)\left(z - \frac{1}{6}z^3 + \cdots\right) = 1 - \frac{1}{2}z^2 + \cdots$$

or

$$a_{-1} + a_0 z + \left(a_1 - \frac{1}{6}a_{-1}\right)z^2 + \cdots = 1 - \frac{1}{2}z^2 + \cdots$$

so: $a_{-1} = 1$, $a_0 = 0$, $a_1 = -\frac{1}{2} + \frac{1}{6}a_{-1} = -\frac{1}{3}$.

# Residues

We find

$$f(z) = \frac{\pi}{z^3} - \frac{\pi^2}{3}\frac{1}{z} + \cdots$$

and so

$$\operatorname{Res}(f, 0) = -\frac{\pi^2}{3}$$

# The integrals

We find that, for every $N$,

$$2\pi i \oint_{\Gamma_N} f(z)\,\mathrm{d}z = -\frac{1}{3}\pi^2 + 2\sum_{n=1}^{N}\frac{1}{n^2}$$

Now we show

$$\lim_{N\to\infty}\oint_{\Gamma_N} f(z)\,\mathrm{d}z = 0$$

# Estimates

Length of $\Gamma_N$: $4 \times (2N+1) = 8N+4$.

On $\Gamma_N$ we have $\frac{1}{z^2} \leqslant \frac{1}{N^2}$.

Remember:
$|\cos z| = \sqrt{\sinh^2 y + \cos^2 x}$
and
$|\sin z| = \sqrt{\sinh^2 y + \sin^2 x}$

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# Estimates

If $x = \pm(N + \frac{1}{2})$ then

$$|\operatorname{cotan} \pi z| = \frac{\sqrt{\sinh^2 \pi y}}{\sqrt{\sinh^2 \pi y + 1}} \leqslant 1$$

# Estimates

If $y = \pm(N + \frac{1}{2})$ then

$$
\begin{aligned}
|\operatorname{cotan} \pi z| &= \frac{\sqrt{\sinh^2 \pi(N + \frac{1}{2}) + \cos^2 \pi x}}{\sqrt{\sinh^2 \pi(N + \frac{1}{2}) + \sin^2 \pi x}} \\
&\leqslant \frac{\sqrt{\sinh^2 \pi(N + \frac{1}{2}) + 1}}{\sqrt{\sinh^2 \pi(N + \frac{1}{2})}} \\
&\leqslant 2
\end{aligned}
$$

(when $N \geqslant 1$).

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# Finishing up

Everything combined:

$$\left| \oint_{\Gamma_N} f(z)\,\mathrm{d}z \right| \leqslant \frac{(8N+4) \times \pi \times 2}{N^2}$$

which suffices to have the limit be equal to zero.

# A differential equation

A string or belt runs on two wheels one unit apart at a speed of $v$ units.
The function $w(x,t)$ describes the lateral displacement of the string at position $x$ $(0 < x < 1)$ and time $t$ $(t > 0)$.
It satisfies the following (partial) differential equation

$$\frac{\partial^2 w}{\partial t^2} + 2v\frac{\partial^2 w}{\partial x \partial t} + (v^2 - 1)\frac{\partial^2 w}{\partial x^2} = 0$$

If $v = 0$ then we have the wave equation.

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# Boundary conditions

We impose boundary conditions:

$$w(0,t) = 0 \text{ and } w(1,t) = f(t)$$

One wheel is stable, the other one wiggles a bit.

We start at rest:

$$w(x,0) = 0 \text{ and } w_t(x,0) = 0 \text{ and } w_x(x,0) = 0$$

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# Apply Laplace transform

Transform the equation:

$$s^2 W(x,s) - s \cdot w(x,0) - w_t(x,0) + 2v\big(sW_x(x,s) - w_x(x,0)\big) + (v^2-1)W_{xx}(x,s) = 0$$

this can be rewritten as

$$W_{xx} + \frac{2vs}{v^2-1} W_x + \frac{s^2}{v^2-1} W = 0.$$

Note: linear with constant coefficients

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# Solve the differential equation

Characteristic equation:

$$\lambda^2 + \frac{2vs}{v^2-1}\lambda + \frac{s^2}{v^2-1} = 0$$

with solutions

$$\lambda_1 = \frac{-s}{v+1} \text{ and } \lambda_2 = \frac{-s}{v-1}$$

Hence

$$W(x,s) = C_1(s)e^{\lambda_1 x} + C_2(s)e^{\lambda_2 x}$$

The 'constants' $C_1$ and $C_2$ depend on $s$.

# $C_1$ and $C_2$

$W(0,s)=0$, hence $C_1(s)+C_2(s)=0$.
$W(1,s)=F(s)$ (transform of $f$), hence $C_1(s)(e^{\lambda_1}-e^{\lambda_2})=F(s)$
and so $C_1(s)=\frac{F(s)}{e^{\lambda_1}-e^{\lambda_2}}$.
We get

$$W(x,s)=F(s)\cdot\underbrace{\frac{\exp(\frac{-sx}{v+1})-\exp(\frac{-sx}{v-1})}{\exp(\frac{-s}{v+1})-\exp(\frac{-s}{v-1})}}_{H(x,s)}$$

If we can find the inverse transform, $h(x,t)$, of $H(x,s)$ then we are done:

$$w(x,t)=f(t)*h(x,t)=\int_0^t f(\tau)h(x,t-\tau)\,\mathrm{d}\tau$$

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# Inversion

Unfortunately $H(x, s)$ has many singularities: the solutions to

$$\exp\left(\frac{-s}{v+1}\right) = \exp\left(\frac{-s}{v-1}\right)$$

or

$$\exp\left(\frac{2s}{v^2-1}\right) = 1$$

these are $s_n = (v^2 - 1)n\pi i$, with $n \in \mathbb{Z}$.

$s_0 = 0$ is a removable singularity, all others are poles of order 1.

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# Inversion formula

We know that

$$h(x,t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{1-iR}^{1+iR} H(x,s)e^{st}\,\mathrm{d}s$$

As we have infinitely many singularities we have no annulus; we can try a trick like with my favorite formula.

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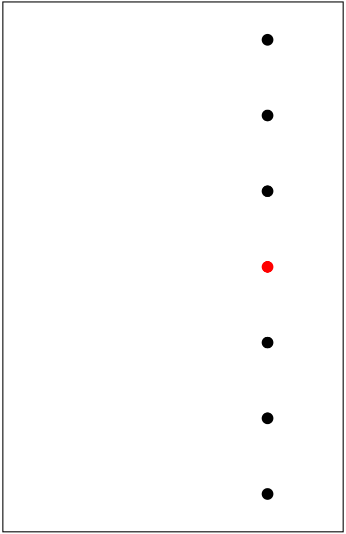
# Inversion formula



The contour $\Gamma_3$

Removable singularity at 0

Poles of order 1 at $(v^2 - 1)n\pi i$

# Slight problem

The integrals along the three other sides do not converge to 0.
What is true is that $H(x, s)$ is bounded on those rectangles, provided the horizontals are at $\pm(n + \frac{1}{2})(v^2 - 1)\pi i$:
So, we divide by $s^2$:

$$W(x, s) = \bigl(s^2 F(s) - sf(0) - f'(0)\bigr)\frac{H(x, s)}{s^2} + \bigl(sf(0) + f'(0)\bigr)\frac{H(x, s)}{s^2}$$

This makes inversion possible, at the cost of a more complicated convolution.
Note that 0 is now a pole of order 2.

# Residues

We have

$$H(x,s) = \frac{\exp(\frac{-sx}{v+1}) - \exp(\frac{-sx}{v-1})}{\exp(\frac{-s}{v+1}) - \exp(\frac{-s}{v-1})}$$

At $s_n = (v^2 - 1)n\pi i$ we have $\frac{s_n}{v+1} = (v-1)n\pi i$ and $\frac{s_n}{v-1} = (v+1)n\pi i$ and so

$$\operatorname{Res}(H(x,s), s_n) = \frac{\exp(-x(v-1)n\pi i) - \exp(-x(v+1)n\pi i)}{-\frac{1}{v+1}\exp(-(v-1)n\pi i) + \frac{1}{v-1}\exp(-(v+1)n\pi i)}$$

# Residues

The denominator becomes ($e^{n\pi i} = (-1)^n$):

$$(-1)^n e^{-vn\pi i}\left(-\frac{1}{v+1}+\frac{1}{v-1}\right)=\frac{(-1)^n 2e^{-vn\pi i}}{v^2-1}$$

We find

$$\begin{aligned}\operatorname{Res}\left(\frac{H(x,s)}{s^2}e^{st}, s_n\right) &= \frac{\exp(-x(v-1)n\pi i)-\exp(-x(v+1)n\pi i)}{(-1)^{n+1}2n^2\pi^2(v^2-1)e^{-vn\pi i}}e^{s_n t}\\ &= \frac{(-1)^{n+1}e^{vn\pi i}e^{s_n t}\left(e^{-x(v-1)n\pi i}-e^{-x(v+1)n\pi i}\right)}{2n^2\pi^2(v^2-1)}\end{aligned}$$

where $e^{s_n t} = e^{t(v^2-1)n\pi i}$

# Residues

Finally, at 0 the residu that we need is the partial derivative $\frac{\partial}{\partial s}H(x,s)e^{st}$ evaluated at 0. This comes out as

$$\frac{x}{v^2-1}\bigl(v - xv + t(v^2-1)\bigr)$$

Now collect everything in one giant sum.

# The final form

We get the inverse transform of $H(x,s)/s^2$:

$$\frac{x}{v^2-1}\left(v - xv + t(v^2-1)\right) - \sum_{n\neq 0} \frac{(-1)^n e^{vn\pi i} e^{s_n t}\left(e^{-x(v-1)n\pi i} - e^{-x(v+1)n\pi i}\right)}{2n^2\pi^2(v^2-1)}$$

For more details see the original paper (link on BlackBoard).

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# What to do?

Now go forth and study!