AN F-SPACE (PRELIMINARY VERSION)

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Abstract. We present an example of a zero-dimensional F-space that is not strongly zero-dimensional.

Introduction

The question whether zero-dimensionality implies strong zero-dimensionality has a long history, a summary of which can be found in [4, Section 6.2]. We define a space to be zero-dimensional if it is a T_1-space and its clopen subsets form a base for the topology. We exploit the fact that zero-dimensional spaces are Tychonoff spaces to define a space to be strongly zero-dimensional if its Čech-Stone compactification is zero-dimensional.

There are by now many examples of zero-dimensional spaces that are not strongly zero-dimensional, even metrizable ones, see [8], but the authors are not aware of an F-space of this nature. Indeed, a question on MathOverFlow that asks explicitly for such an example has remained unanswered for five years, see [7]. Recently Ali Reza Olfati raised this question with the first author in a different context.

The reason that there might not be an F-space example is that there are characterizations of F-spaces that seem to imply strong zero-dimensionality but do not: a Tychonoff space X is an F-space iff for every continuous function $f : X \to \mathbb{R}$ there is another continuous function $h : X \to \mathbb{R}$ with the property that $|f| = f \cdot h$; so h is constant on the sets $\{x : f(x) > 0\}$ and $\{x : f(x) < 0\}$ with values 1 and -1 respectively. Although h seems to split X into two clopen pieces it does not, its existence merely shows that $\{x : f(x) = 0\}$ has a sizable interior.

In the next section we shall construct a zero-dimensional F-space that is not strongly zero-dimensional. Its Čech-Stone compactification contains non-trivial compact connected subsets and these are even connected F-spaces.

1. A zero-dimensional F-space that is not strongly zero-dimensional

The construction in this section is inspired by an answer, by the second author, to a question on MathOverFlow, see [1], which in turn was inspired by Dowker's example M in [3]. The latter is a subspace of $\omega_1 \times [0, 1]$; the example on MathOverFlow is a quotient of $\omega_1 \times \mathbb{A}$, where $\mathbb{A}$ is Alexandroff's split interval.

We replace the ordinal space ω_1 by the G_δ-modification of the ordinal space ω_2, which we denote $(\omega_2)_\delta$; likewise $(\omega_2 + 1)_\delta$ denotes the G_δ-modification of $\omega_2 + 1$. We replace $\mathbb{A}$ by the split interval over a suitable ordered continuum.

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We shall use an ordered continuum K with a dense subset D that can be enumerated as $\langle d_\alpha : \alpha \in \omega_2\rangle$ in such a way that every tail set $T_\alpha = \{d_\beta : \beta > \alpha\}$ is dense in K.

Example 1. If CH fails then we can take $K = [0,1]$ and, like Dowker did, choose $\aleph_2$ many distinct cosets of $\mathbb{Q}$, say $\{\mathbb{Q}_\alpha : \alpha \in \omega_2\}$, and enumerate their union D as $\langle d_\alpha : \alpha \in \omega_2\rangle$ in such a way that $\langle d_{\omega\cdot\alpha+n} : n \in \omega\rangle$ enumerates $\mathbb{Q}_\alpha \cap (0,1)$.

Example 2. For a ZFC example let L be the tree $(\omega_2^\star + \omega_2)^{<\omega}$, ordered by

$$s \prec t \text{ if } \begin{cases} s \subset t \text{ and } t\bigl(|s|\bigr) \in \omega_2, \text{ or} \\ t \subset s \text{ and } s\bigl(|t|\bigr) \in \omega_2^\star, \text{ or} \\ (\exists k)\bigl(s \restriction k = t \restriction k \land s(k) < t(k)\bigr) \end{cases}$$

The linear order $\prec$ has the property that every interval has cardinality $\aleph_2$.

We let K be the (connected) Dedekind completion of L; the set L itself is the desired dense set, under any enumeration.

We need the following Lemma, which is a variation of a result of Van Douwen, see [4, Problem 3.12.20.(c)].

Lemma 1. *Let X be a compact Hausdorff space. The product $(\omega_2)_\delta \times X$ is C-embedded in $(\omega_2+1)_\delta \times X$.*

Proof. Let $f : (\omega_2)_\delta \times X \to \mathbb{R}$ be continuous.

Take $\alpha \in \omega_2$ of cofinality $\aleph_1$. For every $x \in X$ and $n \in \omega$ one can find $\beta(x,n) < \alpha$ and an open set $U(x,n)$ in X such that $x \in U(x,n)$ and

$$f\bigl[(\beta(x,n),\alpha] \times U(x,n)\bigr] \subseteq \bigl(f(\alpha,x) - 2^{-n}, f(\alpha,x) + 2^{-n}\bigr)$$

By compactness we can take a finite subcover $\{U(x,n) : x \in F_n\}$ of the cover $\{U(x,n) : x \in X\}$. Let $\beta_n = \max\{\beta(x,n) : x \in F_n\}$, then for all $x \in X$ and $\gamma \in (\beta_n,\alpha]$ we have $\bigl|f(\gamma,x) - f(\alpha,x)\bigr| < 2^{-n+1}$.

Next let $\beta_\alpha = \sup\{\beta_n : n \in \omega\}$, then $\beta_\alpha < \alpha$ and f is constant on each horizontal line $(\beta_\alpha,\alpha] \times \{x\}$.

The Pressing-Down Lemma now gives us a single β such that f is constant on $(\beta,\omega_2) \times \{x\}$ for all x. Those constant values give us our continuous extension of f to $(\omega_2+1)_\delta \times X$. $\square$

Note that the proof also shows that $(\alpha)_\delta \times X$ is C-embedded in $(\alpha+1)_\delta \times X$ whenever α has uncountable cofinality.

Split intervals. Using the continuum K and the dense set $\{d_\alpha : \alpha \in \omega_2\}$ we create a sequence $\langle K_\alpha : \alpha \leqslant \omega_2\rangle$ of ordered compacta, as follows:

$$K_\alpha = \{\langle x,i\rangle \in K \times 2 : \text{if } x \notin \{d_\beta : \beta \geqslant \alpha\} \text{ then } i = 0\}$$

ordered lexicographically. Thus K_α is a split interval over K, where all points d_β with $\beta \geqslant \alpha$ are split in two; if $\alpha = \omega_2$ then no points are split and K_{ω_2} is just K itself.

There are obvious maps $q_{\alpha,\beta} : K_\alpha \to K_\beta$ when $\alpha < \beta$, defined by

$$q_{\alpha,\beta}(x,i) = \langle x,0\rangle \text{ when } x \notin \{d_\gamma : \gamma \geqslant \beta\}$$
$$q_{\alpha,\beta}(d_\gamma,i) = \langle d_\gamma,i\rangle \text{ when } \gamma \geqslant \beta.$$

We abbreviate the maps $q_{0,\alpha}$ by q_α.

Using remainders. To get to our F-space we let $X_\alpha = (\omega\times K_\alpha)^*$, the Čech-Stone remainder. The maps $q_{\alpha,\beta}$ induce maps from X_α to X_β when $\alpha<\beta$; we denote these by $\boldsymbol{q}_{\alpha,\beta}$ and $\boldsymbol{q}_\alpha$.

We consider the product $(\omega_2)_\delta\times X_0$ and the union

$$X=\bigcup\bigl\{\{\alpha\}\times X_\alpha:\alpha<\omega_2\bigr\}$$

as well as $(\omega_2+1)_\delta\times X_0$ and $X^+ = X\cup(\{\omega_2\}\times X_{\omega_2})$.

We shall topologize X^+ in such a way that

(1) X is a zero-dimensional F-space, and
(2) X is C-embedded in X^+

The latter condition implies that X is not strongly zero-dimensional as we will have $\beta X=\beta X^+$ and the latter space contains X_{ω_2}, which is one-dimensional.

The topology. We define $\boldsymbol{q}:(\omega_2+1)_\delta\times X_0\to X^+$ as expected:

$$\boldsymbol{q}(\alpha,x)=\bigl\langle\alpha,\boldsymbol{q}_\alpha(x)\bigr\rangle$$

We give X^+ the quotient topology determined by $\boldsymbol{q}$. We show $\boldsymbol{q}$ is a closed map.

To begin note that for each α the set $\{\alpha\}\times X_\alpha$ is closed and gets the topology it already had as a Čech-Stone remainder. Also, if α has countable cofinality then $\{\alpha\}\times X_0$ is clopen, hence so is $\{\alpha\}\times X_\alpha$.

Hence to finish the proof that q is closed we let α be of cofinality $\aleph_1$, take $x\in X_\alpha$ and an open set O in $(\omega_2+1)_\delta\times X_0$ such that $\boldsymbol{q}^\leftarrow(\alpha,x)=\{\alpha\}\times\boldsymbol{q}_\alpha^\leftarrow(x)\subseteq O$. By compactness there are V open in X_0 and $\beta<\alpha$ such that

$$\{\alpha\}\times\boldsymbol{q}_\alpha^\leftarrow(x)\subseteq(\beta,\alpha]\times V\subseteq O$$

Because $\boldsymbol{q}_\alpha:X_0\to X_\alpha$ is closed there is an open set U in X_α such that $\boldsymbol{q}_\alpha^\leftarrow[U]\subseteq V$. Then $(\beta,\alpha]\times\boldsymbol{q}_\alpha^\leftarrow[U]$ is an open subset of $(\omega_2+1)_\delta\times X_0$. For $\gamma\in(\beta,\alpha)$ we have $\boldsymbol{q}_\alpha=\boldsymbol{q}_{\gamma,\alpha}\circ\boldsymbol{q}_\gamma$, hence $\boldsymbol{q}_\alpha^\leftarrow[U]=\boldsymbol{q}_\gamma^\leftarrow\bigl[\boldsymbol{q}_{\gamma,\alpha}^\leftarrow[U]\bigr]$.

It follows that $\boldsymbol{q}^\leftarrow[W]=(\beta,\alpha]\times\boldsymbol{q}_\alpha^\leftarrow[U]$, where

$$W=\bigcup\bigl\{\{\gamma\}\times\boldsymbol{q}_{\gamma,\alpha}^\leftarrow[U]:\beta<\gamma\leqslant\alpha\bigr\}$$

The set W is open and $\boldsymbol{q}^\leftarrow[W]\subseteq O$.

Zero-dimensional. Here is where we use that every tail set $T_\alpha=\{d_\beta:\beta\geqslant\alpha\}$ is dense in K. This implies that each K_α and hence each X_α, when $\alpha<\omega_2$, is zero-dimensional: the family $\mathcal{B}_\alpha$ of all clopen intervals of the form $\bigl[\min K,\langle e,0\rangle\bigr]$, $\bigl[\langle d,1\rangle,\langle e,0\rangle\bigr]$, and $\bigl[\langle d,1\rangle,\max K\bigr]$, where $d,e\in T_\alpha$, is base for the topology of K_α. Also, when one interprets one of the intervals above, I say, in K_0 then it satisfies $q_\beta^\leftarrow\bigl[q_\beta[I]\bigr]=I$ whenever $\beta\leqslant\alpha$.

We shall refine the argument used to show that $\boldsymbol{q}$ is a closed map and use notation $\operatorname{Ex}U=\beta X\setminus\operatorname{cl}(X\setminus U)$ for the largest open set in βX whose intersection with X is the given open set U.

Let F and G be disjoint closed subsets of $\beta(\omega\times K_\alpha)$. A standard argument, see for example [5, Section 3], produces a sequence $\langle\mathcal{I}_n:n\in\omega\rangle$ finite pairwise disjoint subfamilies of $\mathcal{B}_\alpha$ such that the clopen set $\operatorname{Ex}I$ separates F and G, where $I=\bigcup\bigl\{\{n\}\times\bigcup\mathcal{I}_n:n\in\omega\bigr\}$. The important point is that $\operatorname{Ex}I$ has the same property as the members of $\mathcal{B}_\alpha$: $X_0\cap\operatorname{Ex}I=\boldsymbol{q}_\beta^\leftarrow\bigl[\boldsymbol{q}_\beta[X_0\cap\operatorname{Ex}I]\bigr]$ whenever $\beta\leqslant\alpha$.

Let U be an open neighbourhood of a point $\langle\alpha, x\rangle$ in X. In case α is of countable cofinality we can take a clopen neighbourhood of x in X_α such that $\{\alpha\}\times V\subseteq U$; this suffices because $\{\alpha\}\times V$ is clopen in X.

If α has uncountable cofinality then we first take $\beta<\alpha$ and an open set W in X_0 such that $\{\alpha\}\times \boldsymbol{q}_\alpha(x)\subseteq(\beta,\alpha]\times W\subseteq \boldsymbol{q}^\leftarrow[U]$. The argument given above now gives us a clopen subset C of X_0 such that $\boldsymbol{q}^\leftarrow(x)\subseteq C\subseteq W$ and $\boldsymbol{q}_\gamma^\leftarrow\bigl[\boldsymbol{q}_\gamma[C]\bigr]=C$ whenever $\gamma\leqslant\alpha$. Then $C^+=(\beta,\alpha]\times C$ is clopen,

$$\{\alpha\}\times \boldsymbol{q}_\alpha(x)\subseteq C^+\subseteq \boldsymbol{q}^\leftarrow[U]$$

and $\boldsymbol{q}^\leftarrow\bigl[\boldsymbol{q}[C^+]\bigr]=C^+$. Hence $\boldsymbol{q}[C^+]$ is clopen and $\langle\alpha,x\rangle\in \boldsymbol{q}[C^+]\subseteq U$.

F-space. To see that X is an F-space let $f:X\to\mathbb{R}$ be continuous. We seek a continuous function $h:X\to\mathbb{R}$ such that $|f|=h\cdot f$.

To this end we first extend f to $f^+\to X^+\to\mathbb{R}$, while also remembering the ordinal α with the property that $f\circ q$ is constant on all horizontal lines that start at α. Since X_{ω_2} is an F-space we get a continuous function $g:X_{\omega_2}\to\mathbb{R}$ such that $g(x)\cdot f^+(\omega_2,x)=\bigl|f^+(\omega_2,x)\bigr|$. For all $\beta>\alpha$ we define h on $\{\beta\}\times X_\beta$ by $h(\beta,x)=g(\boldsymbol{q}_{\beta,\omega_2}(x))$, and h^+ on $\{\beta\}\times X_0$ by $h(\beta,x)=g(\boldsymbol{q}_{\omega_2}(x))$. Then h^+ is continuous and $h^+=h\circ q$ on $(\alpha,\omega_2]\times X_0$, so that h is continuous as well.

To find h on the rest of X, that is $\bigcup\{\{\beta\}\times X_\beta:\beta\leqslant\alpha\}$, we apply [6, Theorem 2.1] to see that that union is an F-space and hence the desired function exists.

C-embedding. To show that X is C-embedded in X^+ we let $f:X\to\mathbb{R}$ be continuous. The proof of Lemma 1 produces an $\alpha<\omega_2$ such that $f\circ\boldsymbol{q}$ is constant on $(\alpha,\omega_2)\times\{x\}$ for all $x\in X$, which then determines the (unique) extension $g:(\omega_2+1)_\delta\times X_0\to\mathbb{R}$ of $f\circ\boldsymbol{q}$.

We show that $g(\omega_2,x)=g(\omega_2,y)$ whenever $\boldsymbol{q}_{\omega_2}(x)=\boldsymbol{q}_{\omega_2}(y)$; for then g determines a continuous extension of f to X^+. We assume $x\neq y$ of course and take disjoint neighbourhoods U and V of x and y in $\beta(\omega\times K_0)$. The argument used in the proof of zero-dimensionality now produces two sequences $\langle\mathcal{I}_n:n\in\omega\rangle$ and $\langle\mathcal{J}_n:n\in\omega\rangle$ of finite families of clopen intervals in K_0 such that the clopen sets $I=\bigcup\bigl\{\{n\}\times\bigcup\mathcal{I}_n:n\in\omega\bigr\}$ and $J=\bigcup\bigl\{\{n\}\times\bigcup\mathcal{J}_n:n\in\omega\bigr\}$ satisfy

- $I\in x$ and $J\in y$ (x and y are ultrafilters of closed sets), and
- $I\subseteq U$ and $J\subseteq V$.

For each n let E_n be the set of points in K that occur as first coordinates of endpoints of one of the intervals in $\mathcal{I}_n$ and $\mathcal{J}_n$. The union, E, of these sets is countable. Therefore there is a $\beta\geqslant\alpha$ such that $E\cap\{d_\gamma:\gamma\geqslant\beta\}=\emptyset$. This means that for $\gamma\geqslant\beta$ the restriction $q_\gamma\restriction E$ is injective.

Because $\boldsymbol{q}_{\omega_2}(x)=\boldsymbol{q}_{\omega_2}(y)$ the intersection of $q_{\omega_2}[I]$ and $q_{\omega_2}[J]$ is not compact and this intersection is contained in $F=\bigcup\bigl\{\{n\}\times E_n:n\in\omega\bigr\}$.

It follows that for all $\gamma\geqslant\beta$ we have $\boldsymbol{q}_\gamma(x)=\boldsymbol{q}_\gamma(y)$ and therefore

$$g(\gamma,x)=f(\gamma,\boldsymbol{q}_\gamma(x))=f(\gamma,\boldsymbol{q}_\gamma(y))=g(\gamma,y)$$

and this implies $g(\omega_2,x)=g(\omega_2,y)$, as desired.

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