MSc thesis projects – Martijn Caspers

Below you can find a list of possible MSc topics. All topics have a literature component and a (modest but genuinely interesting) research component. In the literature component you learn about a certain more specialized subject. Collecting and exposing the literature about a certain subject in an original way can surely contribute to a Master thesis or sometimes it can even constitute a whole Master thesis. In the research component you will also try to solve a small problem. Solving such a problem should typically use variations of existing methods (at least I believe so), but behind the horizon there are also many open problems (that would go beyond the scope of a thesis).

Prerequisites. There is a very diverse set of possible projects in the list below. For these projects it is *necessary* that you have taken the course on Applied Functional Analysis in Delft plus ideally a follow-up course that treats some aspect of C*-algebras or harmonic analysis (Spectral Theory of Linear Operators, Harmonic Analysis, or in Mastermath Operator Algebras, Functional Analysis). Feel free to contact me at an early stage of your Master in case you wish advice in selecting courses and prospective Master projects.

Other topics. There are several other topics available on request taking into account the set of courses you have followed. Feel free to inform!

Multi-linear harmonic analysis. This topic concerns a multi-linear versions of Fourier multipliers. In the most classical version, an L_p-Fourier multiplier with $1 \leq p < \infty$, is a function $m : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{F}_2 \circ m \circ \mathcal{F}_2^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ extends to a bounded map $L_p(\mathbb{R}) \to L_p(\mathbb{R})$. Morally such a map is of the form

$$\left(x \mapsto e^{irx}\right) \mapsto \left(x \mapsto m(r)e^{irx}\right).$$

This is only morally, since $x \mapsto e^{irx}$ is not an L_2-function, but if you decompose an L_2-function as an integral of trigonometric functions, then this is what happens to the integrants. The theory of L_p-multipliers is already vast and very powerful, see [Gra08]. The idea of this project is to analyse some theorems here in the multilinear case, where one takes a function $m : \mathbb{R}^2 \to \mathbb{R}$ and and looks at maps $L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R}) \to L_q(\mathbb{R})$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}$ that are formally of the form

$$\left(x \mapsto e^{irx}\right) \times \left(x \mapsto e^{isx}\right) \mapsto \left(x \mapsto m(r,s)e^{i(r+s)x}\right).$$

There are several options to study problems here. On of them would be to construct multipliers on Heisenberg groups of dimension $2n+1$ or nilpotent (homogeneous) Lie groups.

Techniques: Fourier analysis, Harmonic analysis.

Literature: [Gra08] and more depending on the direction.

Non-commutative differentiation and estimates on operator integrals. In [PoSu11] the following result was proved. Let $p \in (1, \infty)$. There exists a constant $C_p > 0$ such that for any $A, B \in B(H)$ self-adjoint and any Lipschitz function $f : \mathbb{R} \to \mathbb{C}$ with Lipschitz constant ≤ 1 we have

$$\| f(A) - f(B) \|_p \leq \| A - B \|_p.$$

Here the norm is defined as $\|X\|_p = \mathrm{Tr}(|X|^p)^{\frac{1}{p}}$ (if you want it is the ℓ_p-norm of the eigenvalues (or singular values) of X). The crucial part of the proof relies on the fact that the *entry-wise* matrix multiplication $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with a matrix of the form

$$\left(\frac{f(i) - f(j)}{i - j} \right)_{i,j \in F}, \qquad F \subseteq \mathbb{R}, |F| = n.$$

is bounded if $M_n(\mathbb{C})$ is equipped with the $\| \, \|_p$-norm. A special case of this estimate that is widely applied in quantum information is the entropy estimate

$$\| \rho \log(\rho) - \sigma \log(\sigma) \|_p \leq \| \rho - \sigma \|_p.$$

The aim would be to study this proof. Ideally you would also look at the higher order analogue of the problem studied in [PSS13], [CSZ20] and try to sharpen current estimates (I have concrete suggestions how to do that which can be part of the project).

Techniques: Fourier analysis, Functional Analysis.

Literature: [PoSu11], [PSS13], [CSZ20].

Optimizing Kazhdan constants through semi-definite optimization algorithms. The Kazhdan constant is a certain constant that one can associate to a pair (Γ, S) of a group Γ that is generated by a finite set S. The constant is given by the spectral gap of the Laplace operator on the Cayley graph of the group (details omitted). Some people view it as a degree for the connectivity of the Cayley graph; such a graph or communication network is highly connected if the Kazhdan constant is non-zero. A group is said to have Kazhdan's property (T) if the Kazhdan constant is non-zero. The stereotype groups with property (T) are $SL(n, \mathbb{Z}), n \geq 3$ - groups of $n \times n$-matrices with integer coefficients. In a number of recent papers (see literature below) the Kazhdan constants of specific groups were estimated. The idea of the proof is roughly to reduce the problem of finding Kazhdan constants to a semi-definite optimization problem, and then run an algorithm on a computer (sometimes supercomputer). In this project you will learn these techniques and can try to: (1) improve the current Kazhdan constant, (2) estimate the Kazhdan constant of other groups that are similar to the known cases.

Techniques: Elementary algebra (groups, algebras), semi-definite optimization, algorithms.

Literature: Algemeen over property (T): boek van Bekka, de la Harpe, Valette; arXiv papers: 2207.02783, 1812.03456, 1712.07167 (use google to find them).

Uniqueness of C*-norms on $*$-algebras. This project deals with the following question. Given a $*$-algebra A, how many norms can one put on A such that its completion is a C*-algebra? The question has in particular been studied for the case that A is the group algebra $\mathbb{C}[\Gamma]$ of a discrete group Γ. Here the group algebra $\mathbb{C}[\Gamma]$ is defined as follows. As a (complex) vector space it has basis $e_\gamma, \gamma \in \Gamma$ and the multiplication is given by $e_\gamma e_\mu = e_{\gamma\mu}$ which can be extended linearly. By now there are examples where $\mathbb{C}[\Gamma]$ has a unique C*-norm and there are whole classes of groups for which it does not have a unique C*-norm. The idea is to look for C*-uniqueness for other natural classes of algebras. Most natural examples come from crossed products, but also other constructions can be taken into account.

Techniques: C*-algebras.

Literature: [KeAl19], [CaSk19], [Sca20].

References

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