Analysis

Traveling Waves for the Discrete Nagumo Equations

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In this project, the discrete FitzHugh-Nagumo equations

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \qquad n \in \mathbb{Z} \tag{1}$$

are considered, where $d > 0$ is a constant and f can be thought of as a cubic polynomial of the form

$$f(x) = x(x-a)(1-x), \qquad a \in \left(0, \frac{1}{2}\right). \tag{2}$$

Equation (1) models the voltage in a myelinated nerve axon as a function of time $t \geq 0$ and discrete position $n \in \mathbb{Z}$. A myelinated axon is covered in segments by meylin, which accelerates the propagation of signals compared to a naked axon. The discrete Nagumo equations are reminiscent of their partial differential equation (PDE) counterpart, the Nagumo equation

$$\partial_t u = D, \qquad \partial_x^2 u + f(u), \tag{3}$$

where $D > 0$ and f is as before. Equation (3) models the voltage in an unmyelinated nerve axon as a function of time $t \geq 0$ and now continuous position $n \in \mathbb{R}$. It admits for any $D > 0$ traveling wave solutions of the form

$$u(t,x) = U\left(\frac{x}{\sqrt{D}} + ct\right), \tag{4}$$

where $c\sqrt{D} > 0$ is the wave speed and U is twice continuously differentiable with $U(-\infty) = 0$, $U(\infty) = 1$ (traveling front, see [2]). For the discrete counterpart (1), a corresponding result has been an open problem until the work of [1], where existence of traveling front solutions like in (4) was proved for $d > 0$ sufficiently large (for $d > 0$ small non-existence was proved in [3]. The proof of Zinner [1] uses Brouwer's fixed-point theorem in combination with a subtle approximation argument. The goal of the project is to understand this proof and the necessary mathematical prerequisites, in particular Brouwer's fixed-point theorem and some elements of index theory. Interested students may also acquire knowledge on the modelling of (1) or (3) and its relevance in neuroscience.

Reference for colloquium:

[1] B. Zinner. Existence of traveling wavefront solutions for the discrete Nagumo equation. J. Differential Equations, 96(1):1–27, 1992.

Other references:

[2] P.C. Fife and J.B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling front solutions. Arch. Rational Mech. Anal., 65(4):335–361, 1977.

[3] J.P. Keener. Propagation and its failure in the discrete Nagumo equation. In Ordinary and partial differential equations (Dundee, 1986), volume 157 of Pitman Res. Notes Math. Ser., pages 95–112. Longman Sci. Tech., Harlow, 1987.

Analysis

Verschillen tussen $\aleph_0$ en $\aleph_1$

K. P. Hart

Het eerste oneindige kardinaalgetal $\aleph_0$ heeft vele mooie eigenschappen. Voorbeelden daarvan zijn de stelling van Ramsey over het kleuren van oneindige grafen en het oneindigheidslemma van Kőnig dat gebruikt wordt bij compactheidsargumenten van de vorm “als elke eindige deelverzamelingnetjes is dan is de hele verzameling ook netjes”. Veel van die stellingen laten zich niet makkelijk generaliseren naar het eerstvolgende oneindige kardinaalgetal $\aleph_1$. Doel van dit project is een aantal van dit soort stellingen te bestuderen, te begrijpen en te zien wat misgaat bij $\aleph_1$.

De nodige voorkennis kan worden opgedaan bij het bestuderen van de stellingen.

Reference for colloquium:

K. Kunen, Combinatorics, Handbook of Mathematical Logic.

Other references:

K. Kunen, Set Theory (an introduction to independence proofs).

Analysis

Hardy spaces and Toeplitz operators

Martijn Caspers
Mario Klisse

In complex analysis, Hardy spaces are certain spaces of holomorphic functions on the unit disc. By compression of multiplication operators on the circle to the Hardy space $\mathbf{H}^2$, one can define so-called Toeplitz operators. These are continuous linear maps which have a number of interesting and useful properties. For instance, it is possible to formulate a “baby version” of the famous Atiyah-Singer index theorem in the context of Toeplitz operators. This version states that the Fredholm index of a Toeplitz operator is equal to the winding number of its underlying function on the circle.

The aim of this project is to introduce necessary functional analytic notions, define Hardy spaces and to study properties of Toeplitz operators, the main goal being a proof of the “baby” index theorem.

Prerequisites: Analysis, Linear algebra. Basic knowledge in functional analysis is also useful but not necessary.

Reference for colloquium:

G. Murphy, C*-algebras and operator theory, Academic Press, Inc., Boston, MA, 1990. x+286 pp.

Other references:

To be determined during the project, but Murphy will be the main reference.

Analysis

Bilinear Schur multipliers: generating conjectures using computer approximations

Martijn Caspers

Schur multiplication is entry-wise matrix multiplication. So for every $A \in M_n(\mathbb{C})$ we analyse the map $T_A^{(1)}: M_n(\mathbb{C}) \to M_n(\mathbb{C}): (x_{i,j})_{i,j} \mapsto (A_{i,j}\, x_{i,j})_{i,j}$.
There are natural bilinear versions of these maps. Namely if $A = (A_{i_0,i_1,i_2})_{i_0,i_1,i_2}$ is a set of coefficients in $\mathbb{C}$ then we may look at the map

$$T_A^{(2)}: M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C}): (x_1, x_2, x_3) \mapsto \left(\sum_{i_1=1}^{n} A_{i_0,i_1,i_2} x_{i_0,i_1} x_{i_1,i_2} \right)_{i_0,i_2}.$$

An easy example occurs when $A_{i_0,i_1,i_2} = 1$ for all indices so that $T_A^{(2)}$ is just the matrix multiplication map. A rather non-trivial example occurs when A is

$$A_{i_0,i_1,i_2} = \mathrm{sign}\big((i_1 - i_0) - (i_2 - i_1)\big). \tag{1}$$

Where $\mathrm{sign}(\xi) = 1$ if ξ is positive and $\mathrm{sign}(\xi) = -1$ otherwise.
The aim of this project is to make a guess for the norms of these maps. More precisely we may put two norms on $M_n(\mathbb{C})$ namely

$$\|x\|_2 = \mathrm{Tr}(x^*x)^{\frac{1}{2}} = \left(\sum_{i,j=1}^{n} |x_{i,j}|^2 \right)^{\frac{1}{2}} \text{ and } \|x\|_1 = \mathrm{Tr}\left((x^*x)^{\frac{1}{2}}\right),$$

where $(x^*x)^{\frac{1}{2}}$ is defined by diagonalizing x^*x and taking the roots of the (positive) eigenvalues.
Now for A as in (1) let $C_n > 0$ be the smallest constant such that for all $x_1, x_2 \in M_n(\mathbb{C})$ we have $\|T_A(x_1, x_2)\|_1 \leq C_n \|x_1\|_2 \|x_2\|_2$. Such C_n always exists, but it seems to be an open question whether $\sup_n C_n < C$ for a uniform constant $C > 0$.
Answering this question would have a number of important consequences in harmonic analysis, functional analysis and operator theory. There are reasons to believe that $C > 0$ exists and there are reasons to believe it cannot be found (my personal conjecture is that $C_n > \log(n)$, i.e. it cannot be found). The aim of this project is to approximate C_n using computer algorithms.
In the project you will learn about basics of functional analysis (norms, continuity) as well as non-commutative L_p-spaces of matrix algebras (especially the special cases $p = 1$, $p = 2$ and $p = \infty$). You will learn the arguments in favor and against why a constant $C > 0$ may exist. Finally, and most importantly, the constants C_n can be approximated using a computer. The aim is to determine C_n for small n and make a conjecture on its growth. If you really get far perhaps there is time to do an attempt on proving the conjecture. I would be quite grateful if this project gets picked and could be carried out!
Prerequisites: Linear algebra, analysis (norms, continuity), programming skills (python or any of your favorite languages).

Reference for colloquium:

M. Caspers, G. Wildschut, On the complete bounds of L_p-Schur multipliers, Arch. Math. (Basel) 113 (2019), no. 2, 189-200.

Other references:

To be determined during the project and depending on the prerequisites.

Analysis

Ultrafilters

K. P. Hart

Ultrafilters worden in veel delen van de wiskunde gebruikt. In de logica, de analyse, de algebra, de topologie, de verzamelingenleer, … Het project zou kunnen gaan over

- ultrafilters als punten in uitbreidingen van structuren
- ultrafilters als een manier om verzamelingen groot en klein te kunnen noemen
- het tellen en onderscheiden van ultrafilters
- hun gebruik in de analyse

en nog veel meer.

Reference for colloquium:

Comfort and Negrepontis: The Theory of Ultrafilters

Other references:

J. van Mill, An introduction to $\beta\omega$

Analysis

The order and topology of L^0

Mark Veraar

$L^0(\Omega)$ is the space of measurable functions from Ω into $\mathbb{R}$. On L^0 we can consider a topology induced by convergence in measure, and an order induced by the order of the real numbers. In this project you are supposed to formulate an overview on the structure of L^0 on the level of a bachelor student. In particular, algebraic properties and analytic properties such as completeness, and order completeness will be investigated, and applications in probability theory will be considered.

The project requirements are the courses:

Real Analysis and Advanced Probability

Further information can be asked by email.

Reference for colloquium:

Fremlin measure theory volume 2

Other references:

Karatzas and Shreve, Methods of Mathematical finance.

Analysis

Growth of A^n and $\exp(tA)$ in finite and infinite dimensions under resolvent conditions.

Mark Veraar

The Kreiss resolvent condition states that

$|(z - A)^{-1}| \leq \frac{K}{|z|-1}$ whenever $z \in \mathbb{C}$ satisfies $|z| > 1$. Here A is a given $(N+1) \times (N+1)$-matrix and K is a constant independent of z. Under this condition one can show that $|A^n| \leq Ke \min\{n+1, N+1\}$. These bounds end their analogues for $\exp(tA)$ under a related assumption, are important in some parts of numerical analysis and partial differential equations. In the project you are supposed to create an overview of the theory connected to these bounds, discuss applications, and explain why they are sharp. There are several open problem related to these bounds which provide interesting research questions, which could also be addressed .

The project requirements are the courses:

Real Analysis and Fourier Analysis

Reference for colloquium:

Eisner, T., & Zwart, H. (2006). Continuous-time Kreiss resolvent condition on infinite-dimensional spaces. Mathematics of computation, 75(256), 1971-1985.

Other references:

Kraaijevanger, J. F. B. M. (1994). Two counterexamples related to the Kreiss matrix theorem. BIT Numerical Mathematics, 34(1), 113-119.

Spijker, M., Tracogna, S., & Welfert, B. (2003). About the sharpness of the stability estimates in the Kreiss matrix theorem. Mathematics of computation, 72(242), 697-713.

Analysis

Characterizations of admissible dilation groups for continuous wavelet transforms

Martijn Caspers
Jordy van Velthoven

The construction of wavelet inversion formula for $L^2(\mathbb{R}^d)$ requires the existence of a square-integrable function whose Fourier transform satisfies a Calderón-type identity, a so-called admissible wavelet. In dimension $d = 1$, such wavelets are easy to construct. However, in $d > 1$ the question which dilation groups $H \leq \mathrm{GL}(d, \mathbb{R})$ admit admissible wavelets is highly non-trivial. Matrix groups admitting admissible wavelets are also called admissible.

The admissible dilation groups can be completely characterized in terms of orbits and stabilizers associated to the matrix group's (adjoint) action on $\mathbb{R}^d$. It is the aim of the project to provide a self-contained overview of the (sharp) sufficient and necessary conditions for the existence of admissible wavelets. In addition, numerous examples of admissible and non-admissible dilation groups will be studied.

Reference for colloquium:

R. S. Laugesen, N. Weaver, G. L. Weiss, and E. N. Wilson. A characterization of the higher dimensional groups associated with continuous wavelets. J. Geom. Anal., 12(1):89–102, 2002.

Other references:

D. Bernier and K. Taylor. Wavelets from square-integrable representations, SIAM J. Math. Anal. 27 (1996), 594-608.

H. Führ. Generalized Calderón conditions and regular orbit spaces. Colloq. Math., 120(1):103–126, 2010.

Analysis

Orthogonal polynomials on an exponential lattice

Christophe Smet

Given a certain weight function w on a real interval I, a sequence of orthogonal polynomials $\{P_n\}$ satisfies the property $\int_I P_n(x)P_m(x)w(x)dx = 0$ if $m \neq n$.

These polynomials have been extensively studied for a wide range of weights. Examples are the so-called classical orthogonal polynomials: Hermite, Laguerre and Jacobi polynomials. Results include a strong characterization of the recurrence coefficients which link polynomials of different degree: these sequences satisfy Painlevé equations.

The goal in this project is to study orthogonal polynomials on an exponential lattice $\{\pm q^n\}$ where q is a real number in $(0,1)$: what kinds of weights make sense in this setting, what can we say about those recurrence coefficients, where are the zeros of these polynomials located and similar questions.

Reference for colloquium:

W. Van Assche, *Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials*, Proceedings of the international conference on Difference Equations, Special Functions and Orthogonal Polynomials, World Scientific (2007), 687-725.

Other references:

F. W. Nijhoff, *On a q-deformation of the discrete Painlevé I equation and q-orthogonal polynomials*, Lett. Math. Phys. **30** (1994), 327-336.

R. Koekoek, R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Reports of the faculty of Technical Mathematics and Informatics no. 98-17, Delft University of Technology, 1998

Analysis

Minimal surfaces

Bart van den Dries

Consider the following problem: given a closed curve C in space, find a surface S with C as boundary and surface area as small as possible. Such a surface is called a minimal surface, and the problem of finding such a surface given the boundary is called Plateau's problem.

This problem has a rich history. Starting with Lagrange in the 18th century, the problem has been studied by many mathematicians from the point of view of differential geometry, (complex) analysis and numerical analysis. Minimal surfaces have found many applications, ranging from physics (soap films are minimal surfaces!) to biology and architecture.

The aim of this research project is to understand the paper "Computing minimal surfaces with differential forms" by Wang et al. ([1]). This paper proposes a numerical method to find minimal surfaces, by exploiting properties of differential forms.

The authors combine techniques from differential geometry and real analysis (and to some extent optimization). If you want to do this project, it is important that you are interested in these topics and that you followed the corresponding courses. For a general introduction to differential forms and their properties, you can study chapter 14 in "Introduction to smooth manifolds" by Lee ([2]).

For questions, you can contact Bart van den Dries (B.vandendries@tudelft.nl).

Reference for colloquium:

[1] Stephanie Wang et al, Computing minimal surfaces with differential forms, ACM Transactions on Graphics (2021)

Other references:

[2] John M. Lee, Introduction to smooth manifolds, Second edition, Graduate texts in Mathematics, Springer (2013)