ON THE DUALITY PRINCIPLE IN OPTION PRICING: SEMIMARTINGALE SETTING

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Abstract. The purpose of this paper is to describe the appropriate mathematical framework for the study of the *duality principle* in option pricing. We consider models where prices evolve as general exponential semimartingales and provide a complete characterization of the dual process under the dual measure. Particular cases of these models are the ones driven by Brownian motions and by Lévy processes, which have been considered in several papers.

Generally speaking, the duality principle states that the calculation of the price of a call option for a model with price process $S = \mathrm{e}^H$ (w.r.t. the measure P) is equivalent to the calculation of the price of a put option for a suitable dual model $S' = \mathrm{e}^{H'}$ (w.r.t. the dual measure P'). More sophisticated duality results are derived for a broad spectrum of exotic options.

1. Introduction

Consider an investor who buys a call option in the foreign exchange markets, for example the Euro/Dollar market. Owning the call option, she has the right to buy euros for a strike rate K. In case she exercises the option, she has to pay in dollars. Therefore, the right to buy euros is at the same time a right to sell dollars at the inverse rate $K' = 1/K$. Thus, the call option on the Euro/Dollar rate is equivalent to a put option on the Dollar/Euro rate. The prices of these options determine each other. This obvious fact was already noted by Grabbe (1983).

Nevertheless, behind this simple observation lies a much deeper result, which we call the *duality principle*. Assuming some dynamics about the evolution of the exchange rate (or even the price process of a stock or an index), what are the dynamics of the reciprocal rate? The duality principle provides the answer to this natural question. We model asset prices as general exponential semimartingales, hence we work in the widest possible framework, as far as arbitrage theory is concerned. The appropriate tool to express this answer turned out to be the triplet of predictable characteristics of a semimartingale (cf. Jacod 1979). Indeed, the central result provides the explicit form of the triplet of predictable characteristics of the dual process under the dual martingale measure.

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The most standard application of the duality principle relates the value of a European call option to the value of a European put option. Carr (1994) derived a put-call duality for the Black and Scholes (1973) model and more general diffusion models. Chesney and Gibson (1995) considered a two-factor diffusion model and Bates (1997) considered diffusion and jump-diffusion models. Schroder (1999) investigated duality results for a number of payoffs in diffusion and jump-diffusion models. Fajardo and Mordecki (2006b) considered Lévy processes, whereas Eberlein and Papapantoleon (2005b) obtained a number of results for time-inhomogeneous Lévy processes.

Eberlein, Kluge, and Papapantoleon (2006) studied duality results in the context of fixed income markets. They derived results analogous to those in foreign exchange and equity markets, for interest rate options such as caps, floors and options on bonds.

Duality type results were used to derive static hedging strategies for some exotic derivatives, using standard European options as hedging instruments; see e.g. Carr, Ellis, and Gupta (1998). They were also used by Bates (1997), and more recently by Fajardo and Mordecki (2006a), to calculate the so-called "skewness premium" from observed market prices of vanilla options.

Naturally, once the duality for European options was derived, researchers looked into analogous results for American options. The duality between American call and put options is even more interesting than its European counterpart, since for American options the put-call *parity* holds only as an *inequality*. Carr and Chesney (1996) proved the put-call duality for American options for diffusion models, Detemple (2001) studied dualities for American options with general payoffs in diffusion models, while Fajardo and Mordecki (2006b) proved analogous results in Lévy models.

The duality principle demonstrates its full strength when considering exotic derivatives. In certain cases – a typical example is the *Russian option* – it allows to reduce a problem involving *two random variables* – for example, the asset price and its supremum – to a problem involving just *one random variable* – in this example, the supremum – under a dual measure. The computation of expected values (option prices) based on a joint distribution of two variables is typically very time-consuming; therefore, the evaluation of an expectation based on a univariate variable speeds up the numerical computation considerably.

Henderson and Wojakowski (2002) showed an equivalence between floating and fixed strike Asian options in the Black–Scholes model. Vanmaele et al. (2006) extended those results to forward-start Asian options in the Black–Scholes model. Eberlein and Papapantoleon (2005a) extended the results of Henderson and Wojakowski by using Lévy processes as the driving force for the asset price; they also proved analogous equivalence results for floating and fixed strike lookback options.

Večeř (2002) and Večeř and Xu (2004) used this change of measure to derive a one-dimensional partial (integro-)differential equation for floating and fixed strike Asian options in the Black–Scholes and a general semimartingale model. Andreasen (1998) had used this change of measure to derive a one-dimensional partial (integro-)differential equation for floating and fixed strike lookback options in the Black–Scholes and in a jump-diffusion model.

The connection between the choice of an appropriate numeraire and a subsequent change of measure has been beautifully described in Geman, El Karoui, and Rochet (1995). The change of measure method itself has already been used in earlier work, see e.g. Shepp and Shiryaev (1994) and Shiryaev et al. (1994).

Finally, it is interesting to mention the put-call duality of Peskir and Shiryaev (2002), who used *negative* volatility.

2. Exponential semimartingale models

1. Let $\mathcal{B}_T = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a stochastic basis (Jacod and Shiryaev 2003, p. 2) with $\mathcal{F}_T = \mathcal{F}$ and $S = (S_t)_{0\le t\le T}$ be an exponential semimartingale, i.e. a stochastic process with representation

$$S_t = \mathrm{e}^{H_t}, \quad 0 \le t \le T \tag{2.1}$$

(shortly: $S = \mathrm{e}^H$), where $H = (H_t)_{0\le t\le T}$ is a semimartingale, $H_0 = 0$.

The process S is interpreted as the price process of a financial asset, e.g. a stock or an FX rate. Together with the *compound interest* representation (2.1) for (positive) prices S, which is appropriate for the statistical analysis of S, the following *simple interest* representation

$$S_t = \mathcal{E}(\widetilde{H})_t, \quad 0 \le t \le T, \tag{2.2}$$

with some suitable semimartingale $\widetilde{H} = (\widetilde{H}_t)_{0\le t\le T}$, is convenient for the study of the process S by martingale methods; see details in Shiryaev (1999).

In (2.2) we used the standard notation $\mathcal{E}(X) = (\mathcal{E}(X)_t)_{0\le t\le T}$ for the stochastic exponential of a semimartingale, defined* as the unique strong solution of the stochastic differential equation

$$\mathrm{d}\mathcal{E}(X)_t = \mathcal{E}(X)_{t-}\mathrm{d}X_t. \tag{2.3}$$

From (2.1) and (2.2) it follows that the process $\widetilde{H}$ should satisfy the equation

$$\mathrm{e}^{H_t} = \mathcal{E}(\widetilde{H})_t, \quad 0 \le t \le T \tag{2.4}$$

which implies $\Delta\widetilde{H} > -1$. In other words

$$H_t = \log \mathcal{E}(\widetilde{H})_t, \quad 0 \le t \le T \tag{2.5}$$

and vice versa

$$\widetilde{H}_t = \mathcal{L}\mathrm{og}(\mathrm{e}^{H_t}), \quad 0 \le t \le T \tag{2.6}$$

where $\mathcal{L}\mathrm{og}\, X$ denotes the stochastic logarithm of a positive process $X = (X_t)_{0\le t\le T}$:

$$\mathcal{L}\mathrm{og}\, X_t = \int_0^t \frac{\mathrm{d}X_s}{X_{s-}}. \tag{2.7}$$

Note that for a positive process X with $X_0 = 1$ we have for $\mathcal{L}\mathrm{og}\, X$

$$\mathcal{L}\mathrm{og}\, X = \log X + \frac{1}{2X_-^2} \cdot \langle X^c \rangle - \sum_{0<s\le \cdot} \left(\log\left(1 + \frac{\Delta X_s}{X_{s-}}\right) - \frac{\Delta X_s}{X_{s-}} \right). \tag{2.8}$$

For details see Kallsen and Shiryaev (2002) or Jacod and Shiryaev (2003).

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*For notions of the general theory of stochastic processes and notation in general, we rely on the monograph Jacod and Shiryaev (2003).

If $\mu^H = \mu^H(\omega; \mathrm{d}s, \mathrm{d}x)$ and $\mu^{\widetilde{H}} = \mu^{\widetilde{H}}(\omega; \mathrm{d}s, \mathrm{d}x)$ are the random measures of jumps of H and $\widetilde{H}$ then from (2.4)–(2.7) one gets the following useful formulae

$$\widetilde{H} = H + \frac{1}{2}\langle H^c \rangle + (\mathrm{e}^x - 1 - x) * \mu^H \tag{2.9}$$

and

$$H = \widetilde{H} - \frac{1}{2}\langle \widetilde{H}^c \rangle + (\log(1+x) - x) * \mu^{\widetilde{H}} \tag{2.10}$$

where $W * \mu$, for $W = W(\omega; s, x)$ and the integer-valued measure $\mu = \mu(\omega; \mathrm{d}s, \mathrm{d}x)$, $s \in [0, T]$, $x \in E$, means the integral

$$\int_0^{\cdot} \int_E W(\omega; s, x)\mu(\omega; \mathrm{d}s, \mathrm{d}x)$$

often written as $\int_0^{\cdot} \int_E W \mathrm{d}\mu$.

It is useful to note that discrete time sequences $H = (H_n)_{n \geq 0}$ with $H_0 = 0$ and $\mathcal{F}_n$-measurable random variables H_n can be considered as a semimartingale $\overline{H} = (\overline{H}_t)_{t \geq 0}$ in continuous time, where $\overline{H}_t = H_n$ for $t \in [n, n+1)$, given on the stochastic basis $\overline{\mathcal{B}} = (\Omega, \mathcal{F}, (\overline{\mathcal{F}}_t)_{t \geq 0}, P)$ with $\overline{\mathcal{F}}_t = \mathcal{F}_n$ for $t \in [n, n+1)$.

In the discrete time setting $S = (S_n)_{n \geq 0}$ has a *compound interest* representation

$$S_n = \mathrm{e}^{H_n}, \quad H_n = h_1 + \cdots + h_n, \quad n \geq 1, \tag{2.11}$$

$S_0 = 1$, $h_0 = 0$ and the analogue of the *simple interest* representation has the form

$$S_n = \mathcal{E}(\widetilde{H})_n = \prod_{0 \leq k \leq n} (1 + \widetilde{h}_k) \tag{2.12}$$

with $\widetilde{h}_k = \mathrm{e}^{h_k} - 1$, $\widetilde{H}_k = \widetilde{h}_1 + \cdots + \widetilde{h}_k$, $k \geq 1$, $\widetilde{H}_0 = 0$. We see that

$$\Delta S_n = S_{n-1} \Delta \widetilde{H}_n$$

where $\Delta S_n = S_n - S_{n-1}$, $\Delta \widetilde{H}_n = \widetilde{H}_n - \widetilde{H}_{n-1} = \widetilde{h}_n$ (compare with (2.3)).

2. It is known that every semimartingale $H = (H_t)_{0 \leq t \leq T}$ admits a canonical representation

$$H = H_0 + B + H^c + h(x) * (\mu - \nu) + (x - h(x)) * \mu \tag{2.13}$$

where

a) $h = h(x)$ is a *truncation function*; a canonical choice of h is $h(x) = xI_{\{|x| \leq 1\}}$ where I_A, denotes the indicator of the set A;
b) $B = (B_t)_{0 \leq t \leq T}$ is a predictable process of bounded variation;
c) $H^c = (H^c_t)_{0 \leq t \leq T}$ is the continuous martingale part of H;
d) $\nu = \nu(\omega; \mathrm{d}s, \mathrm{d}x)$ is the predictable compensator of the random measure of jumps $\mu = \mu(\omega; \mathrm{d}s, \mathrm{d}x)$ of H; for clarity we write also ν^H and μ^H instead of ν and μ.

The continuous martingale H^c has a predictable quadratic characteristic $\langle H^c \rangle$ which will be denoted C $(= (C_t)_{0 \leq t \leq T})$. For the processes B, C, and the measure ν we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which will be called the *triplet of predictable characteristics* of the semimartingale H with respect to the probability measure P.

The concepts of predictability and predictable characteristics played and play a crucial role in the development of the general theory of Stochastic Processes, the theory of Semimartingales and their applications, including Mathematical Finance. Recall that the financial notion of a *portfolio* is based on the concept of predictability (non-anticipating). For more details, we refer to Jacod and Shiryaev (2003), especially the Bibliographical Comments at the end of the book.

3. From formulae (2.9) and (2.10) it is not difficult to find the relationships between the triplets $\mathbb{T}(H|P) = (B, C, \nu)$ and $\mathbb{T}(\widetilde{H}|P) = (\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ (with respect to the same truncation function h):

$$\begin{aligned} \widetilde{B} &= B + \frac{C}{2} + (h(\mathrm{e}^x - 1) - h(x)) * \nu \\ \widetilde{C} &= C \\ I_A(x) * \widetilde{\nu} &= I_A(\mathrm{e}^x - 1) * \nu, \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} B &= \widetilde{B} - \frac{\widetilde{C}}{2} + (h(\log(1+x)) - h(x)) * \widetilde{\nu} \\ C &= \widetilde{C} \\ I_A(x) * \nu &= I_A(\log(1+x)) * \widetilde{\nu}, \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}) \end{aligned} \tag{2.15}$$

(see more details in Kallsen and Shiryaev (2002) and Jacod and Shiryaev (2003, p. 66)).

4. It is important to note that the canonical representation (2.13) depends on the selected *truncation function* $h = h(x)$. However, the characteristics C and ν do not depend on the choice of h while $B = B(h)$ does. If h and h' are two truncation functions then $B(h) - B(h') = (h - h') * \nu$ (see Jacod and Shiryaev 2003, p. 81).

In the sequel the function $h = h(x)$ will be assumed to satisfy the *antisymmetry property*:

$$h(-x) = -h(x).$$

We will see that this choice of the truncation function simplifies many formulae. Note that the canonical choice $h(x) = xI_{\{|x|\leq 1\}}$ satisfies this property.

3. Martingale measures and dual martingale measures

1. Let $\mathcal{M}_{\mathrm{loc}}(P)$ be the class of all local martingales on the given stochastic basis $\mathcal{B}_T = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$. It follows from the canonical representation (2.13) that if $\mathbb{T}(H|P) = (B, C, \nu)$ then

$$H \in \mathcal{M}_{\mathrm{loc}}(P) \Leftrightarrow B + (x - h(x)) * \nu = 0. \tag{3.1}$$

Similarly, for the process $\widetilde{H} = \mathcal{L}\mathrm{og}(\mathrm{e}^H)$ we have

$$\widetilde{H} \in \mathcal{M}_{\mathrm{loc}}(P) \Leftrightarrow \widetilde{B} + (x - h(x)) * \widetilde{\nu} = 0. \tag{$\widetilde{3.1}$}$$

In the sequel, we will assume that the following condition is in force.

Assumption $\mathbb{ES}$. *The process $I_{\{x>1\}}\mathrm{e}^x * \nu$ has bounded variation.*

Under assumption $\mathbb{ES}$ the property $(\widetilde{3.1})$ can be rewritten, taking into account (2.14), in the following form:

$$\widetilde{H} \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu = 0. \tag{3.2}$$

Remark 3.1. The assumption that the process $I_{\{x>1\}}\mathrm{e}^x * \nu$ has bounded variation, is equivalent by Kallsen and Shiryaev (2002, Lemma 2.13) to the assumption that the semimartingale H is *exponentially special*, i.e., the price process $S = \mathrm{e}^H$ is a special semimartingale. This justifies to call it *Assumption* $\mathbb{ES}$.

Since $\widetilde{H} \in \mathcal{M}_{\text{loc}}(P)$ iff $\mathcal{E}(\widetilde{H}) \in \mathcal{M}_{\text{loc}}(P)$ we get from (3.2)

$$\mathcal{E}(\widetilde{H}) \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu = 0, \tag{3.3}$$

and, therefore, using (2.4)

$$S = \mathrm{e}^H \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu = 0. \tag{3.4}$$

2. For the purpose of the next theorem (Theorem 3.4), we shall assume that S is not only a local martingale but also a *martingale* ($S \in \mathcal{M}(P)$) on $[0, T]$. Thus $ES_T = 1$ which allows us to define on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T})$ a new probability measure P' with

$$\mathrm{d}P' = S_T \mathrm{d}P. \tag{3.5}$$

Since S is a martingale

$$\mathrm{d}(P'|\mathcal{F}_t) = S_t\, \mathrm{d}(P|\mathcal{F}_t), \quad 0 \leq t \leq T \tag{3.6}$$

and since $S > 0$ (P-a.s.), we have $P \ll P'$ and

$$\mathrm{d}P = \frac{1}{S_T}\mathrm{d}P'. \tag{3.7}$$

Let us introduce the process

$$S' = \frac{1}{S}. \tag{3.8}$$

Then with $H' = -H$

$$S' = \mathrm{e}^{H'}. \tag{3.9}$$

The following simple but, as we shall see, useful lemma plays a crucial role in the *problem of duality* between call and put options. It also explains the name of *dual martingale measure* for the measure P'.

Lemma 3.2. *Suppose $S = \mathrm{e}^H \in \mathcal{M}(P)$ i.e. S is a P-martingale. Then the process $S' \in \mathcal{M}(P')$ i.e. S' is a P'-martingale.*

Proof. The proof follows directly from Proposition 3.8 on page 168 in Jacod and Shiryaev (2003) which states that if $Z = \frac{\mathrm{d}P'}{\mathrm{d}P}$ then $S' \in \mathcal{M}(P')$ iff $S'Z$ is a P-martingale. In our case $Z = S$ and $S'S \equiv 1$. Thus $S' \in \mathcal{M}(P')$. □

3. The next theorem is crucial for all calculations of option prices on the basis of the duality principle (see Section 4). We prepare the proof of the theorem with the following lemma, which is of interest on its own.

Lemma 3.3. *Let f be a predictable, bounded process. The triplet of predictable characteristics of the stochastic integral process $J = \int_0^{\cdot} f\,\mathrm{d}H$, denoted by $\mathbb{T}(J|P) = (B_J, C_J, \nu_J)$, is*

$$B_J = f \cdot B + [h(fx) - fh(x)] * \nu \tag{3.10}$$

$$C_J = f^2 \cdot C \tag{3.11}$$

$$I_A(x) * \nu_J = I_A(fx) * \nu, \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}). \tag{3.12}$$

Proof. The last two statements follow directly from the properties of the stochastic integral $J = f \cdot H$:

$$J^c = f \cdot H^c \tag{3.13}$$

and

$$\Delta J = f \Delta H. \tag{3.14}$$

Indeed, (3.11) follows directly from (3.13) and Jacod and Shiryaev (2003, I.4.41):

$$C_J = \langle J^c \rangle = f^2 \cdot \langle H^c \rangle = f^2 \cdot C.$$

From (3.14) we deduce

$$I_A(x) * \mu_J = I_A(fx) * \mu^H, \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}) \tag{3.15}$$

which gives for ν_J, the compensator of the random measure of jumps μ_J of J, the relation (3.12).

For the proof of relation (3.10) we recall the *canonical representation* of the semimartingale H:

$$H = H_0 + B + M + (x - h(x)) * \mu^H \tag{3.16}$$

where M is a local martingale (in fact $M = H^c + h(x) * (\mu^H - \nu)$) and the canonical representation of the semimartingale J:

$$J = J_0 + B_J + J^c + h(y) * (\mu_J - \nu_J) + (y - h(y)) * \mu_J. \tag{3.17}$$

From the definition $J = f \cdot H$ and the representation (3.16) we get

$$J = f \cdot B + f \cdot M + (fx - fh(x)) * \mu^H \tag{3.18}$$

which gives, together with (3.15), the following formula:

$$\begin{aligned} J - (y - h(y)) * \mu_J &= \\ &= f \cdot B + f \cdot M + (fx - fh(x)) * \mu^H - (fx - h(fx)) * \mu^H \\ &= f \cdot B + f \cdot M + (h(fx) - fh(x)) * \mu^H. \end{aligned} \tag{3.19}$$

The process $J - (y - h(y)) * \mu_J$ has bounded jumps. Hence this process is a special semimartingale (Jacod and Shiryaev 2003, Lemma 4.24, p. 44) and by Proposition 4.23(iii), again from Jacod and Shiryaev (2003, p. 44), we conclude that the process $f \cdot B + (h(fx) - fh(x)) * \mu^H) \in \mathcal{A}_{\text{loc}}$, i.e. it is a process with locally integrable variation. Note now that the process $f \cdot B$ belongs also to the class $\mathcal{A}_{\text{loc}}$ since it is a predictable process of locally bounded variation (Jacod and Shiryaev 2003, Lemma 3.10, p. 29). Hence the process $(h(fx) - fh(x)) * \mu^H \in \mathcal{A}_{\text{loc}}$ and using Jacod and Shiryaev (2003, Theorem 3.18, p. 33) there exists a compensator of this process given by the formula $(h(fx) - fh(x)) * \nu$ (Jacod and Shiryaev 2003, Theorem 1.8, pp. 66–67). As a result we get from (3.19) that

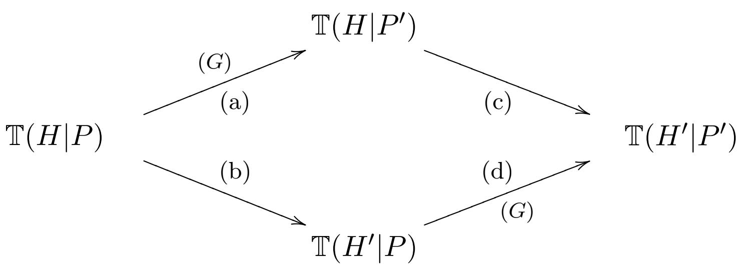
$$\begin{aligned} J - (y - h(y)) * \mu_J &= [f \cdot B + (h(fx) - fh(x)) * \nu] \\ &\qquad + [f \cdot M + (h(fx) - fh(x)) * (\mu^H - \nu)] \\ &= f \cdot B + (h(fx) - fh(x)) * \nu \\ &\qquad + f \cdot H^c + h(fx) * (\mu^H - \nu). \end{aligned} \tag{3.20}$$

Comparing the decomposition (3.20) of the special semimartingale $J - (y - h(y)) * \mu_J$ with the representation of $J - (y - h(y)) * \mu_J$ from the canonical representation (3.17) we conclude, by the uniqueness of the representation of a special semimartingale (Jacod and Shiryaev 2003, I.4.22), that the processes B_J and $f \cdot B + (h(fx) - fh(x)) * \nu$ are indistinguishable; cf. Jacod and Shiryaev (2003, p. 3). Therefore, formula (3.10) is proved. □

Theorem 3.4. *The triplet* $\mathbb{T}(H'|P') = (B', C', \nu')$ *can be expressed via the triplet* $\mathbb{T}(H|P) = (B, C, \nu)$ *by the following formulae:*

$$\begin{aligned} B' &= -B - C - h(x)(\mathrm{e}^x - 1) * \nu \\ C' &= C \\ I_A(x) * \nu' &= I_A(-x)\mathrm{e}^x * \nu, \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}). \end{aligned} \tag{3.21}$$

Proof. We give two proofs which are of interest here since these proofs contain some additional useful relationships between different triplets. The structure of these proofs can be represented by the following diagram:



(3.22)

where $\xrightarrow{(G)}$ means that we use Girsanov's theorem for calculating the *right side* triplet from the *left side* one.

(a) $\mathbb{T}(H|P) \xrightarrow{(G)} \mathbb{T}(H|P')$.

For the calculation of the triplet $\mathbb{T}(H|P') = (B^+, C^+, \nu^+)$ from the triplet $\mathbb{T}(H|P) = (B, C, \nu)$, we use Girsanov's theorem for semimartingales (Jacod and Shiryaev 2003, pp. 172–173) which states that

$$B^+ = B + \beta^+ \cdot C + h(x)(Y^+ - 1) * \nu \tag{3.23}$$

$$C^+ = C \tag{3.24}$$

$$\nu^+ = Y^+ \cdot \nu. \tag{3.25}$$

Here $\beta^+ = \beta_t^+(\omega)$ and $Y^+ = Y^+(\omega; t, x)$ are defined by the following formulae (see Jacod and Shiryaev 2003, p. 173):

$$\langle S^c, H^c \rangle = (S_- \beta^+) \cdot C \tag{3.26}$$

and

$$Y^+ = M^P_{\mu^H}\Big(\frac{S}{S_-}|\widetilde{\mathcal{P}}\Big). \tag{3.27}$$

In (3.27) $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ is the σ-field of predictable sets in $\widetilde{\Omega} = \Omega \times [0,T] \times \mathbb{R}$, $M^P_{\mu^H} = \mu^H(\omega; \mathrm{d}t, \mathrm{d}x)P(\mathrm{d}\omega)$ is the positive measure on $(\widetilde{\Omega}, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}))$ defined by

$$M^P_{\mu^H}(W) = E(W * \mu^H)_T \tag{3.28}$$

for measurable nonnegative functions $W = W(\omega; t, x)$ given on $\Omega \times [0,T] \times \mathbb{R}$.

The *conditional expectation* $M^P_{\mu^H}\Big(\frac{S}{S_-}|\widetilde{\mathcal{P}}\Big)$ is, by definition, the $M^P_{\mu^H}$-a.s. unique $\widetilde{\mathcal{P}}$-measurable function Y^+ with the property

$$M^P_{\mu^H}\Big(\frac{S}{S_-}U\Big) = M^P_{\mu^H}(Y^+U) \tag{3.29}$$

for all nonnegative $\widetilde{\mathcal{P}}$-measurable functions $U = U(\omega; t, x)$.

We show that in our special case $S = \mathrm{e}^H$, where evidently $\frac{S}{S_-} = \mathrm{e}^{\Delta H}$, one may take the following versions of β^+ and Y^+:

$$\beta^+ \equiv 1 \quad \text{and} \quad Y^+ = \mathrm{e}^x. \tag{3.30}$$

Indeed, in (3.26), for $S = \mathrm{e}^H$, we get applying Itô's formula to e^H, that

$$(\mathrm{e}^H)^c = \int_0^\cdot \mathrm{e}^{H_{s-}} \mathrm{d}H^c_s$$

and, therefore,

$$\begin{aligned} \langle S^c, H^c\rangle &= \langle (\mathrm{e}^H)^c, H^c\rangle = \Big\langle \int_0^\cdot \mathrm{e}^{H_-}\mathrm{d}H^c, H^c\Big\rangle \\ &= \int_0^\cdot \mathrm{e}^{H_-}\mathrm{d}\langle H^c\rangle = \int_0^\cdot \mathrm{e}^{H_-}\mathrm{d}C = S_- \cdot C \end{aligned} \tag{3.31}$$

and

$$(S_-\beta^+) \cdot C = \int_0^\cdot \mathrm{e}^{H_-}\beta^+ \mathrm{d}C. \tag{3.32}$$

From this formula and the equality (3.26) we see that one may take $\beta^+ \equiv 1$.

For the proof that one may choose $Y^+ = \mathrm{e}^x$ we need to verify (3.29) with this version of Y^+.

We have, using that μ^H is the random measure of jumps of H:

$$\begin{aligned} M^P_{\mu^H}(\mathrm{e}^x U) &= E\Big[\int_0^T \int_{\mathbb{R}} \mathrm{e}^x U(\omega; t, x)\mu^H(\omega; \mathrm{d}t, \mathrm{d}x)\Big] \\ &= E\Big[\sum_{0<t\le T} \mathrm{e}^{\Delta H_t(\omega)} U(\omega; t, \Delta H_t(\omega)) I_{(\Delta H_t(\omega)\neq 0)}\Big] \\ &= E\Big[\int_0^T \int_{\mathbb{R}} \frac{S_t(\omega)}{S_{t-}(\omega)} U(\omega; t, x)\mu^H(\omega; \mathrm{d}t, \mathrm{d}x)\Big] \\ &= M^P_{\mu^H}\Big(\frac{S}{S_-}U\Big). \end{aligned} \tag{3.33}$$

Consequently in (3.23)–(3.25) one may put $\beta^+ \equiv 1$ and $Y^+ = \mathrm{e}^x$ which gives the following result:

$$\begin{aligned} B^+ &= B + C + h(x)(\mathrm{e}^x - 1) * \nu \\ C^+ &= C \\ \nu^+ &= \mathrm{e}^x \cdot \nu. \end{aligned} \tag{3.34}$$

Remark 3.5. It is useful to note that for the *discrete time* case the relation $\mathrm{d}\nu^+ = \mathrm{e}^x \mathrm{d}\nu$ can be proved (with the obvious notation) in the following simple way.

Let $h_n = \Delta H_n$ and $\mu_n = \mu_n(\omega; \cdot)$ be the random measure of jumps in time n, i.e.

$$\mu_n(\omega; A) = I_A(h_n(\omega)) \quad \text{for } A \in \mathcal{B}(\mathbb{R}\backslash\{0\}).$$

The compensator $\nu_n = \nu_n(\omega; \cdot)$ of $\mu_n(\omega; \cdot)$ has here the simple form $\nu_n(\omega; A) = P(h_n \in A | \mathcal{F}_{n-1})(\omega)$ (see Jacod and Shiryaev (2003, p. 92) for the definition of the compensator in the case of discrete time). If $\nu_n^+(\omega; A) = P'(h_n \in A | \mathcal{F}_{n-1})(\omega)$ then from the already used Proposition 3.8 on page 168 in Jacod and Shiryaev (2003) or, equivalently, from Bayes' formula (also called the *conversion formula*; see Shiryaev (1999, p. 438)) we find that

$$\begin{aligned} \nu_n^+(\omega; A) &= E'[I_A(h_n) | \mathcal{F}_{n-1}](\omega) \\ &= E[I_A(h_n)\mathrm{e}^{h_n} | \mathcal{F}_{n-1}](\omega) = \int_A \mathrm{e}^x \nu_n(\omega; \mathrm{d}x). \end{aligned}$$

Therefore,

$$\nu_n^+ \ll \nu_n \quad \text{and} \quad \frac{\mathrm{d}\nu_n^+}{\mathrm{d}\nu_n}(\omega; x) = \mathrm{e}^x \qquad (\nu_n\text{-a.e.}).$$

(b) $\mathbb{T}(H|P) \longrightarrow \mathbb{T}(H'|P)$.

Because $H' = -H$ the triplet $\mathbb{T}(H'|P) = \mathbb{T}(-H|P)$. Denote $\mathbb{T}(-H|P) = (B^-, C^-, \nu^-)$. Now, it suffices to apply Lemma 3.3 to the process $J = -H$, i.e. $f \equiv -1$, and use the assumption $h(-x) = -h(x)$, to obtain that

$$\begin{aligned} B^- &= -B \\ C^- &= C \\ I_A(x) * \nu^- &= I_A(-x) * \nu, \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}). \end{aligned} \tag{3.35}$$

(c) $\mathbb{T}(H|P') \longrightarrow \mathbb{T}(H'|P')$.

The triplet $\mathbb{T}(H|P') = (B^+, C^+, \nu^+)$ is given by the formulae (3.34). Then from (3.35) (with evident changes in the notation) we get

$$\begin{aligned} B' &= -B^+ = -B - C - h(x)(\mathrm{e}^x - 1) * \nu \\ C' &= C^+ = C \\ I_A(x) * \nu' &= I_A(-x) * \nu^+ = I_A(-x)\mathrm{e}^x * \nu, \end{aligned} \tag{3.36}$$

so, the proof using steps (a) and (c) leads to the formulae (3.21).

(d) $\mathbb{T}(H'|P) \xrightarrow{(G)} \mathbb{T}(H'|P')$.

Here $\mathbb{T}(H'|P) = \mathbb{T}(-H|P) = (B^-, C^-, \nu^-)$ and $\mathbb{T}(H'|P') = \mathbb{T}(-H|P') = (B', C', \nu')$. Similarly to the case (a) we have the following formulae (compare with (3.23)):

$$B' = B^- + \beta^- \cdot C^- + h(x)(Y^- - 1) * \nu^- \tag{3.37}$$

$$C' = C^- \tag{3.38}$$

$$\nu' = Y^- \cdot \nu^- \tag{3.39}$$

where $\beta^- = \beta_t^-(\omega)$ and $Y^- = Y^-(\omega; t, x)$ are given by the formulae (compare with (3.26) and (3.27))

$$\langle S^c, (-H)^c \rangle = (S_- \beta^-) \cdot C^- \tag{3.40}$$

and

$$Y^- = M^P_{\mu^{-H}}\Big(\frac{S}{S_-}|\widetilde{\mathcal{P}}\Big). \tag{3.41}$$

Since

$$\begin{aligned}\langle S^c, (-H)^c \rangle &= \big\langle (\mathrm{e}^H)^c, -H^c \big\rangle \\ &= \Big\langle \int_0^\cdot \mathrm{e}^{H_-} \mathrm{d}H^c, -H^c \Big\rangle = -\int_0^\cdot \mathrm{e}^{H_-} \mathrm{d}\langle H^c \rangle \\ &= -\int_0^\cdot \mathrm{e}^{H_-} \mathrm{d}\langle (-H)^c \rangle = -\int_0^\cdot \mathrm{e}^{H_-} \mathrm{d}C^- \\ &= (-S_-) \cdot C^-, \end{aligned} \tag{3.42}$$

comparing (3.40) and (3.42) we see that one may take $\beta^- \equiv -1$.

Similarly to the calculations in (a) we derive

$$\begin{aligned} M^P_{\mu^{-H}}(\mathrm{e}^{-x}U) &= E\Big[\int_0^T \int_{\mathbb{R}} \mathrm{e}^{-x} U(\omega; t, x) \mu^{-H}(\omega; \mathrm{d}t, \mathrm{d}x)\Big] \\ &= E\Big[\sum_{0<t\le T} \mathrm{e}^{-\Delta(-H_t(\omega))} U(\omega; t, \Delta(-H_t(\omega))) I_{(\Delta(-H_t(\omega))\neq 0)}\Big] \\ &= E\Big[\sum_{0<t\le T} \mathrm{e}^{\Delta(H_t(\omega))} U(\omega; t, \Delta(-H_t(\omega))) I_{(\Delta(-H_t(\omega))\neq 0)}\Big] \\ &= E\Big[\int_0^T \int_{\mathbb{R}} \frac{S_t(\omega)}{S_{t-}(\omega)} U(\omega; t, x) \mu^{-H}(\omega; \mathrm{d}t, \mathrm{d}x)\Big] \\ &= M^P_{\mu^{-H}}\Big(\frac{S}{S_-}U\Big). \end{aligned} \tag{3.43}$$

Therefore one may take $Y^- = \mathrm{e}^{-x}$ in (3.41) and from (3.37)–(3.39) and (3.35) we find that

$$B' = -B - C + h(x)\big(\mathrm{e}^{-x} - 1\big) * \nu^- \tag{3.44}$$

$$C' = C \tag{3.45}$$

$$\nu' = \mathrm{e}^{-x} \cdot \nu^- \tag{3.46}$$

where ν^- is such that $I_A(x) * \nu^- = I_A(-x) * \nu$, $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$. Hence, as one easily sees

$$I_A(x) * \nu' = I_A(x) * \big(\mathrm{e}^{-x} \cdot \nu^-\big) = I_A(-x)\mathrm{e}^x * \nu. \tag{3.47}$$

In addition, if $h(-x) = -h(x)$

$$\begin{aligned} h(x)\big(\mathrm{e}^{-x} - 1\big) * \nu^- &= h(-x)(\mathrm{e}^{x} - 1) * \nu \\ &= -h(x)(\mathrm{e}^{x} - 1) * \nu. \end{aligned} \tag{3.48}$$

From (3.44)–(3.48) we find that the triplet $\mathbb{T}(H'|P') = (B', C', \nu')$, obtained using (b), (d), is given by formulae (3.21). Theorem 3.4 is proved. □

Remark 3.6. Note that under assumption $\mathbb{ES}$ we can conclude from formulae (3.21) that $(|x|^2 \wedge 1) * \nu' \in \mathcal{A}_{\mathrm{loc}}$ because

$$(|x|^2 \wedge 1) * \nu' \le K|x|^2 I_{\{|x|\le 1\}} * \nu + I_{\{x<-1\}} * \nu + \mathrm{e}^x I_{\{x>1\}} * \nu.$$

Here K is a constant and the processes on the right-hand side are predictable processes of bounded variation, hence belong to $\mathcal{A}_{\mathrm{loc}}$ (cp. Jacod and Shiryaev 2003, Lemma I.3.10). Similarly, we get that ν' satisfies assumption $\mathbb{ES}$, because

$$I_{\{x>1\}}\mathrm{e}^x * \nu' = I_{\{x<-1\}}\mathrm{e}^{-x}\mathrm{e}^x * \nu = I_{\{x<-1\}} * \nu$$

and $I_{\{x<-1\}} * \nu \in \mathcal{A}_{\mathrm{loc}}$.

Remark 3.7 (Symmetry)**.** In case the original market (S, P) and the dual market (S', P') satisfy the property

$$\mathrm{Law}(S|P) = \mathrm{Law}(S'|P') \tag{3.49}$$

we say that these markets are *symmetric*. For processes $S = \mathrm{e}^H$ and $S' = \mathrm{e}^{H'}$ where the laws $\mathrm{Law}(S|P)$ and $\mathrm{Law}(S'|P')$ are completely determined by the corresponding triplets $\mathbb{T}(H|P)$ and $\mathbb{T}(H'|P')$ – this is, for example, the case for Lévy processes H and H' – the symmetry property (3.49) holds iff $\nu' = \nu$. This is clear since by (3.21) $C' = C$, and by (3.21) and (3.4)

$$\begin{aligned} B' &= -B - C - h(x)(\mathrm{e}^x - 1) * \nu \\ &= \frac{C}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu - C - h(x)(\mathrm{e}^x - 1) * \nu \\ &= -\frac{C}{2} - (-\mathrm{e}^x + 1 + h(x)\mathrm{e}^x) * \nu \\ &= -\frac{C}{2} - (\mathrm{e}^x - 1 - h(x)) * \nu' = B. \end{aligned}$$

This means $\mathbb{T}(H|P) = \mathbb{T}(H'|P')$. The case of Lévy processes has been discussed in this context by Fajardo and Mordecki (2003).

Corollary 3.8. *Suppose that H is a P-Lévy process with local characteristics (b, c, F). Then the process H' is a P'-Lévy process with local characteristics (b', c', F') given by the formulae (we take $h(-x) = -h(x)$):*

$$\begin{aligned} b' &= -b - c - \int_{\mathbb{R}} h(x)(\mathrm{e}^x - 1)F(\mathrm{d}x) \\ c' &= c \\ F'(A) &= \int_{\mathbb{R}} I_A(-x)\mathrm{e}^x F(\mathrm{d}x), \quad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}). \end{aligned}$$

Proof. The proof follows from Theorem 3.4 and Jacod and Shiryaev (2003, Corollary 4.19, p. 107). □

Remark 3.9. The formulae (3.21) provide a simple way to see that the process $S' = \mathrm{e}^{H'} \in \mathcal{M}_{\mathrm{loc}}(P')$.

Indeed, by (3.4) it is sufficient to check that

$$B' + \frac{C'}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu' = 0. \tag{3.50}$$

From (3.21) (with $-h(-x) = h(x)$)

$$\begin{aligned} B' &+ \frac{C'}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu' \\ &= \big[- B - C - h(x)(\mathrm{e}^x - 1) * \nu\big] + \frac{C}{2} + [\mathrm{e}^{-x} - 1 - h(-x)]\mathrm{e}^x * \nu \\ &= -\bigg[B + \frac{C}{2} + (\mathrm{e}^x - 1 - h(x)) * \nu\bigg] = 0 \end{aligned}$$

where the last equality follows from the assumption $S = \mathrm{e}^H \in \mathcal{M}_{\mathrm{loc}}(P)$ and criterion (3.4).

4. Now we consider some examples that show how to calculate the triplet $\mathbb{T}(H'|P')$ from the triplet $\mathbb{T}(H|P)$ and for which particular models in finance assumption $\mathbb{ES}$ is satisfied.

Example 3.10 (Diffusion models)**.** The dynamics of the asset price process are described by the stochastic differential equation

$$\mathrm{d}S_t = S_t\sigma(t, S_t)\mathrm{d}W_t, \qquad S_0 = 1, \tag{3.51}$$

where $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion. If the local volatility function $\sigma : [0, T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is Lipschitz, i.e. satisfies the conditions

(a) $|\sigma(t, x) - \sigma(t, y)| \le K|x - y|$, $\forall t \in [0, T]$, K constant
(b) $t \mapsto \sigma(t, x)$ is right continuous with left limits, $\forall x \in \mathbb{R}_+$,

then the SDE (3.51) has a unique strong solution (cp. Protter 2004, Theorem V.6), for which

$$\begin{aligned} S_t &= \mathcal{E}\Big(\int_0^{\cdot} \sigma(u, S_u)\mathrm{d}W_u\Big)_t \\ &= \exp\Big(\int_0^t \sigma(u, S_u)\mathrm{d}W_u - \frac{1}{2}\Big\langle \int_0^{\cdot} \sigma(u, S_u)\mathrm{d}W_u\Big\rangle_t \Big) \\ &= \exp\Big(\int_0^t \sigma(u, S_u)\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(u, S_u)\mathrm{d}u\Big). \end{aligned}$$

This class of diffusion models, which are also known as *local volatility models*, were introduced by Dupire (1994); we refer to Skiadopoulos (2001) for a survey of this literature.

Now, assuming the *canonical setting* (Jacod and Shiryaev 2003, p. 154), these models fit in the general exponential semimartingale framework with driving process $H = (H_t)_{0\le t\le T}$, where

$$H_t = \int_0^t \sigma(u, \mathrm{e}^{H_u})\mathrm{d}W_u - \frac{1}{2}\int_0^t \sigma^2(u, \mathrm{e}^{H_u})\mathrm{d}u, \tag{3.52}$$

and triplet $\mathbb{T}(H|P) = (B, C, \nu)$ where

$$\begin{aligned} B &= -\frac{1}{2}\int_0^{\cdot} \sigma^2(u, \mathrm{e}^{H_u})\mathrm{d}u \\ C &= \int_0^{\cdot} \sigma^2(u, \mathrm{e}^{H_u})\mathrm{d}u \\ \nu &\equiv 0, \end{aligned}$$

and, of course, $S = \mathrm{e}^H \in \mathcal{M}_{\mathrm{loc}}(P)$. If $S = \mathrm{e}^H \in \mathcal{M}(P)$, which holds if, for example, Novikov's condition is satisfied, then applying Theorem 3.4, we get that $\mathbb{T}(H'|P') = (B', C', \nu')$, where

$$\begin{aligned} B' &= -B - C = -\frac{1}{2}\int_0^{\cdot} \sigma^2(u, \mathrm{e}^{H_u})\mathrm{d}u \\ C' &= \int_0^{\cdot} \sigma^2(u, \mathrm{e}^{H_u})\mathrm{d}u \\ \nu' &\equiv 0. \end{aligned}$$

In particular, if the volatility parameter σ is constant, then the SDE takes the form $\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t$ and we recover the model in the seminal paper of Black and Scholes (Samuelson model). The solution of this SDE has the simple form

$$S_t = \mathrm{e}^{\sigma W_t - \frac{\sigma^2}{2}t},$$

i.e. $H_t = \sigma W_t - \frac{\sigma^2}{2}t$ and hence $B_t = -\frac{\sigma^2}{2}t$ and $C_t = \sigma^2 t$. Evidently $B + \frac{C}{2} = 0$ which implies by (3.4) that $S \in \mathcal{M}_{\mathrm{loc}}(P)$ and, in fact, $S \in \mathcal{M}(P)$.

The process $S' = \mathrm{e}^{H'} = \mathrm{e}^{-H}$ has stochastic differential

$$\mathrm{d}S'_t = -\sigma S'_t(\mathrm{d}W_t - \sigma \mathrm{d}t). \tag{3.53}$$

Since $S' \in \mathcal{M}_{\mathrm{loc}}(P')$ from formula (3.53) one can deduce that the process $W'_t = W_t - \sigma t$, $0 \le t \le T$ is a P'-local martingale. This is a particular case of the classical Girsanov theorem which can be easily checked directly using the fact (already mentioned before in Lemma 3.2) that $W' \in \mathcal{M}_{\mathrm{loc}}(P')$ iff $W'S \in \mathcal{M}_{\mathrm{loc}}(P)$. The last property follows from calculating $\mathrm{d}(W'S)$ by Itô's formula. The fact that W' is a P'-Brownian motion follows also from Lévy's characterization of a Brownian motion (Revuz and Yor 1999, Theorem IV.(3.6)). So, $\mathrm{d}S'_t = -\sigma S'_t \mathrm{d}W'_t$.

Example 3.11 (Poisson model)**.** Consider $S = \mathrm{e}^H$ with

$$H_t = \alpha \pi_t - \lambda(\mathrm{e}^{\alpha} - 1)t, \quad \alpha \neq 0 \tag{3.54}$$

where $\pi = (\pi_t)_{0 \le t \le T}$ is a Poisson process with parameter $\lambda > 0$ ($E\pi_t = \lambda t$). Take $h(x) \equiv 0$. Then the corresponding triplet (B, C, ν) has the following form:

$$\begin{aligned} B_t &= -\lambda(\mathrm{e}^{\alpha} - 1)t \\ C_t &= 0 \\ \nu(\mathrm{d}t, \mathrm{d}x) &= \lambda I_{\{\alpha\}}(\mathrm{d}x)\mathrm{d}t. \end{aligned} \tag{3.55}$$

By (3.4) $S \in \mathcal{M}_{\mathrm{loc}}(P) \Leftrightarrow B + (\mathrm{e}^x - 1) * \nu = 0$. With the process given in (3.54)

$$B_t + (\mathrm{e}^x - 1) * \nu_t = -\lambda(\mathrm{e}^{\alpha} - 1)t + \lambda(\mathrm{e}^{\alpha} - 1)t = 0.$$

So, $S \in \mathcal{M}_{\text{loc}}(P)$ and even $S \in \mathcal{M}(P)$ and with respect to the measure P' the process S' is a local martingale. This follows directly from criterion (3.4)

$$B' + (\mathrm{e}^x - 1) * \nu' = 0. \tag{3.56}$$

By Theorem 3.4

$$B'_t = \lambda(\mathrm{e}^\alpha - 1)t$$

and

$$(\mathrm{e}^x - 1) * \nu'_t = (\mathrm{e}^{-x} - 1)\mathrm{e}^x * \nu_t = \lambda(1 - \mathrm{e}^\alpha)t.$$

Hence, the property (3.56) does hold and $S' \in \mathcal{M}_{\text{loc}}(P')$.

Example 3.12 (Discrete time, CRR-model)**.** In the Cox–Ross–Rubinstein model (CRR-model) asset prices are modeled by $S_n = \mathrm{e}^{H_n}$, with $H_n = h_1 + \cdots + h_n$, $n \geq 1$, $H_0 = 0$, where $(h_n)_{n\geq 1}$ is a P-iid sequence of random variables which have only two values.

If $\widetilde{h}_n = \mathrm{e}^{h_n} - 1$ then $S_n = \prod_{k\leq n}(1 + \widetilde{h}_k)$ and $S_n = (1 + \widetilde{h}_n)S_{n-1}$, $n \geq 1$, with $S_0 = 0$. For simplicity let us assume that the random variables h_n take the values $\ln\lambda$ and $\ln\frac{1}{\lambda}$ with $\lambda > 1$. So

$$\widetilde{h}_n = \begin{cases} a = \lambda^{-1} - 1, \\ b = \lambda - 1. \end{cases}$$

If the probability measure P is such that

$$P\Big(h_n = \ln\frac{1}{\lambda}\Big) = P(\widetilde{h}_n = a) = \frac{b}{b-a} = \frac{\lambda}{1+\lambda} \tag{3.57}$$

and

$$P(h_n = \ln\lambda) = P(\widetilde{h}_n = b) = \frac{-a}{b-a} = \frac{1}{1+\lambda} \tag{3.58}$$

then we find that

$$E\mathrm{e}^{h_n} = E(1 + \widetilde{h}_n) = 1.$$

This means that the measure P is a *martingale measure* for the sequence $S = (S_n)_{n\geq 0}$. Indeed it is the *unique* martingale measure for the CRR-model; see Shiryaev (1999, Example 2 on pp. 477–480).

With the truncation function $h(x) = x$ and the martingale measure P we easily find that the triplet $\mathbb{T}(H|P) = (B, 0, \nu)$ where (with $\Delta B_n = B_n - B_{n-1}$)

$$\Delta B_n = Eh_n = \frac{1-\lambda}{1+\lambda}\ln\lambda \tag{3.59}$$

and (with $\nu_n(A) = \nu(\{n\} \times A)$)

$$\begin{aligned} \nu_n(\{\ln\lambda\}) &= P(h_n = \ln\lambda) &&= \frac{1}{1+\lambda} \\ \nu_n\Big(\Big\{\ln\frac{1}{\lambda}\Big\}\Big) &= P\Big(h_n = \ln\frac{1}{\lambda}\Big) &&= \frac{\lambda}{1+\lambda}. \end{aligned} \tag{3.60}$$

Note that from (3.59) and (3.60) we find $\Delta B_n + (\mathrm{e}^x - 1 - x) * \nu_n = 0$ which is another derivation of the martingale property for S under the measure P given by (3.57) and (3.58).

Based on formulae (3.21) we find directly that

$$\Delta B'_n = \Delta B_n, \quad \nu'_n = \nu_n \tag{3.61}$$

and from the previous note and (3.4) it follows that $S' \in \mathcal{M}_{\text{loc}}(P')$ (in fact $S' \in \mathcal{M}(P')$).

Example 3.13 (Purely discontinuous Lévy models)**.** In this class of models, asset prices are modeled as $S = \mathrm{e}^H$, where $H = (H_t)_{0\leq t\leq T}$ is a *purely discontinuous* Lévy process with triplet $\mathbb{T}(H|P) = (B, 0, \nu)$. We can also work with the triplet of *local characteristics* denoted by $(b, 0, F)$ which, using Jacod and Shiryaev (2003, II.4.20), is related for our case to the triplet of semimartingale characteristics via

$$B_t(\omega) = bt, \quad \nu(\omega; \mathrm{d}t, \mathrm{d}x) = \mathrm{d}tF(\mathrm{d}x).$$

Let $S = \mathrm{e}^H \in \mathcal{M}_{\mathrm{loc}}(P)$, then the characteristic b takes the form

$$b = -\int_{\mathbb{R}} (\mathrm{e}^x - 1 - h(x))F(\mathrm{d}x).$$

and criterion (3.4) is satisfied. For Lévy processes H, $S = \mathrm{e}^H \in \mathcal{M}_{\mathrm{loc}}(P)$ implies $S \in \mathcal{M}(P)$; see, e.g. Lemma 4.4. in Kallsen (2000). Therefore, we can apply Theorem 3.4 and the triplet $\mathbb{T}(H'|P') = (B', 0, \nu')$ is given by

$$I_A(x) * \nu' = I_A(-x)\mathrm{e}^x * \nu \tag{3.62}$$

and $B' = -B - h(x)(\mathrm{e}^x - 1) * \nu = -(\mathrm{e}^x - 1 - h(x)) * \nu'$. Therefore $S' = \mathrm{e}^{H'} \in \mathcal{M}_{\mathrm{loc}}(P')$.

When considering parametric models it is very convenient to represent the Lévy measure $F = F(\mathrm{d}x)$ in the form

$$F(\mathrm{d}x) = \mathrm{e}^{\vartheta x} f(x)\mathrm{d}x \tag{3.63}$$

where $\vartheta \in \mathbb{R}$ and f is an even function, i.e. $f(x) = f(-x)$. In that case, the triplet of local characteristics of the dual process H' is $(b', 0, F')$ where

$$\int I_A(x)F'(\mathrm{d}x) = \int I_A(-x)\mathrm{e}^{(1+\vartheta)x} f(x)\mathrm{d}x$$

and, of course, $b' = -\int_{\mathbb{R}}(\mathrm{e}^x - 1 - h(x))F'(\mathrm{d}x)$.

Examples of parametric models are:

Example 3.13.1 (Generalized hyperbolic model)**.** Let $H = (H_t)_{0\leq t\leq T}$ be a generalized hyperbolic process with $\mathrm{Law}(H_1|P) = GH(\lambda, \alpha, \beta, \delta, \mu)$, cp. Eberlein (2001, p. 321) or Eberlein and Prause (2002). Then the Lévy measure of H admits the representation (3.63) with parameters $\vartheta = \beta$, $0 \leq |\beta| < \alpha$ and

$$f(x) = \frac{1}{|x|}\int_0^\infty \frac{\exp(-\sqrt{2y + \alpha^2}\,|x|)}{\pi^2 y (J^2_{|\lambda|}(\delta\sqrt{2y}) + Y^2_{|\lambda|}(\delta\sqrt{2y}))}\mathrm{d}y + \lambda \mathrm{e}^{-\alpha|x|} I_{\{\lambda>0\}},$$

where $\alpha > 0$, $\delta > 0$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, cp. Eberlein (2001, p. 323). Here J_λ and Y_λ are the modified Bessel functions of first and second kind respectively. The moment generating function exists for $u \in (-\alpha - \beta, \alpha - \beta)$, hence, assumption $\mathbb{ES}$ is satisfied. The class of generalized hyperbolic distributions contains several other distributions as subclasses, for example hyperbolic distributions (Eberlein and Keller 1995), normal inverse Gaussian distributions (Barndorff-Nielsen 1998) or limiting classes (e.g. variance gamma). We refer to Eberlein and v. Hammerstein (2004) for an extensive survey.

Example 3.13.2 (CGMY model)**.** Let $H = (H_t)_{0\leq t\leq T}$ be a CGMY Lévy process, cp. Carr, Geman, Madan, and Yor (2002); another name for this process is (generalized) tempered stable process. The Lévy measure of this process admits the representation (3.63) with the following parameters

$$\vartheta = \begin{cases} G, & x < 0 \\ -M, & x > 0 \end{cases} \quad \text{and} \quad f(x) = \frac{C}{|x|^{1+Y}},$$

where $C > 0$, $G > 0$, $M > 0$, and $Y < 2$.

The CGMY processes are closely related to stable processes; in fact, the function f coincides with the Lévy measure of the stable process with index $Y = \alpha \in (0, 2)$, cp. Samorodnitsky and Taqqu (1994, Def. 1.1.6). Due to the exponential tempering of the Lévy measure, the CGMY distribution has finite moments of all orders. Moreover, the moment generating function exists, hence assumption $\mathbb{ES}$ is satisfied. Again, the class of CGMY distributions contains several other distributions as subclasses, for example the variance gamma distribution (Madan and Seneta 1990) and the bilateral gamma distribution (Küchler and Tappe 2008).

Example 3.13.3 (Meixner model)**.** Let $H = (H_t)_{0 \le t \le T}$ be a Meixner process with $\text{Law}(H_1|P) = \text{Meixner}(\alpha, \beta, \delta)$, $\alpha > 0$, $-\pi < \beta < \pi$, $\delta > 0$, cp. Schoutens and Teugels (1998) and Schoutens (2002). The Lévy measure of the Meixner process admits the representation (3.63) with $\vartheta = \frac{\beta}{\alpha}$ and

$$f(x) = \frac{\delta}{x \sinh(\frac{\pi x}{\alpha})}.$$

The Meixner distribution possesses finite moments of all orders. Moreover, the moment generating function exists, hence, assumption $\mathbb{ES}$ is again satisfied.

Remark 3.14. Notice that in all the cases considered H' is a P'-Lévy process from the same class of processes, just with a new parameter ϑ in representation (3.63).

Remark 3.15. Theorem 3.4 cannot be applied, for example, to *stable processes*, because they do not satisfy Assumption $\mathbb{ES}$. In fact, stable processes may not even have finite first moment, cp. Samorodnitsky and Taqqu (1994, Property 1.2.16). This fact makes them particularly unsuitable for option pricing, although these models are applied for risk management purposes, cp. Rachev (2003).

Example 3.16 (Stochastic volatility Lévy models)**.** This class of models was proposed by Carr, Geman, Madan, and Yor (2003) and further investigated in Schoutens (2003).

Let $X = (X_t)_{0 \le t \le T}$ be a pure jump Lévy process and $Y = (Y_t)_{0 \le t \le T}$ be an increasing process, independent of X. The process Y acts as a *stochastic clock* measuring activity in business time and has the form

$$Y_t = \int_0^t y_s \mathrm{d}s$$

where $y = (y_s)_{0 \le s \le T}$ is a positive process. Carr, Geman, Madan, and Yor (2003) consider the CIR process as a candidate for y, i.e. the solution of the stochastic differential equation

$$\mathrm{d}y_t = K(\eta - y_t)\mathrm{d}t + \lambda y_t^{\frac{1}{2}} \mathrm{d}W_t,$$

where $W = (W_t)_{0\leq t\leq T}$ is a standard Brownian motion. For other choices of Y see Schoutens (2003).

The stochastic volatility Lévy process is defined by time-changing the Lévy process X with the increasing process Y, that is

$$Z_t = X_{Y_t}.$$

The process Z is a pure jump semimartingale with canonical decomposition

$$Z_t = Z_0 + B^Z + h(x) * (\mu^Z - \nu^Z) + (x - h(x)) * \mu^Z,$$

where the compensator of the random measure of jumps of Z has the form $\nu^Z(\mathrm{d}s, \mathrm{d}x) = y(s)\nu^X(\mathrm{d}x)\mathrm{d}s$, where ν^X denotes the Lévy measure of X.

Asset prices are modeled as $S = \mathrm{e}^H$, where H is a semimartingale such that $\nu^H(\mathrm{d}s, \mathrm{d}x) := \nu^Z(\mathrm{d}s, \mathrm{d}x) = y(s)\nu^X(\mathrm{d}x)\mathrm{d}s$ and $S \in \mathcal{M}_{\mathrm{loc}}(P)$, therefore, $\mathbb{T}(H|P) = (B, 0, \nu^H)$, where

$$B = -(\mathrm{e}^x - 1 - h(x)) * \nu^H.$$

If $S \in \mathcal{M}(P)$ – for sufficient conditions, see Kallsen (2006) – then applying Theorem 3.4, we get that $\mathbb{T}(H'|P') = (B', 0, \nu')$ with

$$\begin{aligned} I_A(x) * \nu' &= I_A(-x)\mathrm{e}^x * \nu^H \\ &= \int I_A(-x)\mathrm{e}^x \nu^X(\mathrm{d}x) y(s)\mathrm{d}s \end{aligned}$$

and $B' = -(\mathrm{e}^x - 1 - h(x)) * \nu'$.

4. The "call-put duality" in option pricing

Let $S = (S_t)_{0\leq t\leq T}$ be the price process as given in (2.1) and $f_T = f_T(S)$ the payoff of the option. Here $f_T(S) = f_T(S_t, 0\leq t\leq T)$ is an $\mathcal{F}_T^S$-measurable functional, where $\mathcal{F}_T^S = \sigma(S_t, 0\leq t\leq T)$. In order to simplify the notation we assume that the current interest rate is zero. For detailed formulae in the case of a positive interest rate (and dividend yield) we refer to Eberlein and Papapantoleon (2005b).

As is well known, in a complete market, where the martingale measure P is *unique*, the rational (or arbitrage-free) price of the option is given by $Ef_T (= E_P f_T)$. In incomplete markets one has to choose an equivalent martingale measure. In this paper we do not discuss the problem of the choice of a *reasonable* martingale measure, for example, in the sense of minimization of a *distance* (L^2-distance, Hellinger distance, entropy minimization, etc.) from the given measure P or in the sense of constructing the simplest possible measure (e.g. Esscher transformation). The practitioners' point of view is that the choice of this measure is the result of a calibration to market prices of plain vanilla options. We will assume that the initial measure P is a *martingale measure* and all our calculations of $E_P f_T$ will be done with respect to this measure P. In the case of an incomplete market this option price $E_P f_T$ could be called a *quasi rational* option price.

A. European call and put options. In case of a standard *call* option the payoff function is

$$f_T = (S_T - K)^+, \quad K > 0, \tag{4.1}$$

whereas for a *put* option it is

$$f_T = (K - S_T)^+, \quad K > 0. \tag{4.2}$$

The corresponding option prices are given by the formulae

$$\mathbb{C}_T(S; K) = E(S_T - K)^+ \tag{4.3}$$

and

$$\mathbb{P}_T(K; S) = E(K - S_T)^+ \tag{4.4}$$

where E is the expectation operator with respect to the initial martingale measure P. From (4.3) for $S = \mathrm{e}^H$ we get

$$\begin{aligned} \mathbb{C}_T(S; K) &= E\Big[S_T \frac{f_T}{S_T}\Big] = E'\Big[\frac{f_T}{S_T}\Big] = E'(1 - KS'_T)^+ \\ &= KE'\Big(\frac{1}{K} - S'_T\Big)^+ = KE'(K' - S'_T)^+ \end{aligned} \tag{4.5}$$

where $K' = \frac{1}{K}$. Comparing here the right hand side with (4.4) we find the following result.

Theorem 4.1. *For standard call and put options the option prices satisfy the following duality relations:*

$$\frac{1}{K}\mathbb{C}_T(S; K) = \mathbb{P}'_T(K'; S')$$

and

$$\frac{1}{K}\mathbb{P}_T(K; S) = \mathbb{C}'_T(S'; K')$$

where $K' = \frac{1}{K}$, $\mathbb{P}'_T(K'; S')$ *and* $\mathbb{C}'_T(S', K')$ *are the corresponding option prices for puts and calls with* S' *as underlying price process, computed with respect to the dual measure* P'.

Corollary 4.2. *Call and put prices in markets* (S, P) *and* (S', P') *which satisfy the duality relation, are connected by the following "call-call parity"*

$$\mathbb{C}_T(S; K) = K\mathbb{C}'_T(S'; K') + 1 - K$$

and the following "put-put parity"

$$\mathbb{P}_T(K; S) = K\mathbb{P}'_T(K'; S') + K - 1.$$

Proof. From the identity $(S_T - K)^+ = (K - S_T)^+ + S_T - K$ we get, taking expectations with respect to the measure P, the well-known *call-put parity*:

$$\mathbb{C}_T(S; K) = \mathbb{P}_T(K; S) + 1 - K.$$

The result follows from the duality relations in Theorem 4.1. □

B. Standard call and put options of American type. The general theory of pricing of American options (see, for example, Shiryaev (1999, Chapters VI and VIII)) states that, for payoff functions described by the process $\mathrm{e}^{-\lambda t} f_t$, $0 \le t \le T$, $\lambda \ge 0$, the price $\widehat{\mathbb{V}}_T(S)$ of the American option is given by the formula

$$\widehat{\mathbb{V}}_T(S) = \sup_{\tau \in \mathcal{M}_T} E \mathrm{e}^{-\lambda\tau} f_\tau, \tag{4.6}$$

where $\mathcal{M}_T$ is the class of stopping times τ such that $0 \le \tau \le T$.

For a standard call option $f_\tau = (S_\tau - K)^+$ and for a standard put option $f_\tau = (K - S_\tau)^+$ where $K > 0$ is a constant strike.

Denote

$$\widehat{\mathbb{C}}_T(S;K) = \sup_{\tau \in \mathcal{M}_T} E \mathrm{e}^{-\lambda\tau} (S_\tau - K)^+ \tag{4.7}$$

and

$$\widehat{\mathbb{P}}_T(K;S) = \sup_{\tau \in \mathcal{M}_T} E \mathrm{e}^{-\lambda\tau} (K - S_\tau)^+. \tag{4.8}$$

Similarly to the case of European options we find that for $f_\tau = (S_\tau - K)^+$

$$\begin{aligned}
\widehat{\mathbb{C}}_T(S;K) &= \sup_{\tau \in \mathcal{M}_T} E\Big[\mathrm{e}^{-\lambda\tau} f_\tau \frac{S_T}{S_T}\Big] = \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} \frac{f_\tau}{S_T}\Big] \\
&= \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} f_\tau S'_T\Big] = \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} f_\tau E'(S'_T|\mathcal{F}_\tau)\Big] \\
&= \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} f_\tau S'_\tau\Big] \\
&= \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} (S_\tau - K)^+ S'_\tau\Big] \\
&= \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} (1 - K S'_\tau)^+\Big] \\
&= K \sup_{\tau \in \mathcal{M}_T} E'\Big[\mathrm{e}^{-\lambda\tau} (K' - S'_\tau)^+\Big] \\
&= K\, \widehat{\mathbb{P}}'_T(K';S').
\end{aligned}$$

Thus, similarly to the statements in Theorem 4.1 we have for American options the following duality relations:

$$\frac{1}{K} \widehat{\mathbb{C}}_T(S;K) = \widehat{\mathbb{P}}'_T(K';S')$$

and also

$$\frac{1}{K} \widehat{\mathbb{P}}_T(K;S) = \widehat{\mathbb{C}}'_T(S';K').$$

C. Floating strike lookback call and put options. Suppose $S \in \mathcal{M}(P)$ and let α be such that $\alpha \ge 1$. For a floating strike lookback call option we get

$$\begin{aligned}\mathbb{C}_T(S;\alpha\inf S) &= E(S_T-\alpha\inf_{t\le T}S_t)^+\\ &= E\Big[S_T\Big(1-\frac{\alpha\inf_{t\le T}S_t}{S_T}\Big)^+\Big]\\ &= E'\big(1-\alpha \mathrm{e}^{\inf_{t\le T}H_t-H_T}\big)^+\\ &= E'\big(1-\alpha \mathrm{e}^{H'_T-\sup_{t\le T}H'_t}\big)^+\\ &= \alpha E'\Big(\frac{1}{\alpha}-\mathrm{e}^{H'_T-\sup_{t\le T}H'_t}\Big)^+. \end{aligned} \tag{4.9}$$

In order to simplify the last expression further, let us assume that the process $H'=(H'_t)_{0\le t\le T}$ satisfies the following *reflection principle*:

$$\mathrm{Law}(\sup_{t\le T}H'_t-H'_T|P')=\mathrm{Law}(-\inf_{t\le T}H'_t|P'). \tag{4.10}$$

This property, of course, holds if the process H' is a Lévy process with respect to P' (see e.g. Kyprianou 2006, Lemma 3.5). From (4.9) and (4.10) we find that

$$\begin{aligned}\frac{1}{\alpha}\mathbb{C}_T(S;\alpha\inf S) &= E'\Big(\frac{1}{\alpha}-\mathrm{e}^{\inf_{t\le T}H'_t}\Big)^+\\ &= E'\Big(\frac{1}{\alpha}-\inf_{t\le T}S'_t\Big)^+=\mathbb{P}'_T\Big(\frac{1}{\alpha};\inf S'\Big). \end{aligned} \tag{4.11}$$

Similarly, assuming the following *reflection principle*

$$\mathrm{Law}(H'_T-\inf_{t\le T}H'_t|P')=\mathrm{Law}(\sup_{t\le T}H'_t|P') \tag{4.12}$$

which again holds, for example, for Lévy processes (Kyprianou 2006, Lemma 3.5) we get

$$\frac{1}{\beta}\mathbb{P}_T(\beta\sup S;S)=\mathbb{C}'_T\Big(\sup S';\frac{1}{\beta}\Big), \tag{4.13}$$

where $0<\beta\le 1$. Hence we have the following results.

Theorem 4.3. *Let H satisfy* (4.10)*, then the calculation of the price of the lookback call option $\mathbb{C}_T(S;\alpha\inf S)$ $(\alpha\ge 1)$ with floating strike $\alpha\inf S$ can be reduced via formulae* (4.11) *to the calculation of the price $\mathbb{P}'_T(\frac{1}{\alpha};\inf S')$ which is the price of a lookback put option with fixed strike.*

Theorem 4.4. *Let H satisfy* (4.12)*, then the calculation of the price of the lookback put option $\mathbb{P}_T(\beta\sup S;S)$ $(0<\beta\le 1)$ with floating strike $\beta\sup S$ can be reduced via formulae* (4.13) *to the calculation of the price $\mathbb{C}'_T(\sup S';\frac{1}{\beta})$ which is the price of a lookback call option with fixed strike.*

D. Floating strike Asian options. Suppose again $S \in \mathcal{M}(P)$ and consider the price

$$
\begin{aligned}
\mathbb{C}_T\Big(S; \frac{1}{T}\int S\Big) &= E\Big(S_T - \frac{1}{T}\int_0^T S_t \mathrm{d}t\Big)^+ \\
&= E\Big[S_T\Big(1 - \frac{1}{T}\int_0^T \frac{S_t}{S_T}\mathrm{d}t\Big)^+\Big] \\
&= E'\Big(1 - \frac{1}{T}\int_0^T \frac{S'_T}{S'_t}\mathrm{d}t\Big)^+ \\
&= E'\Big(1 - \frac{1}{T}\int_0^T \mathrm{e}^{H'_T - H'_t}\mathrm{d}t\Big)^+ \\
&= E'\Big(1 - \frac{1}{T}\int_0^T \mathrm{e}^{H'_T - H'_{(T-u)-}}\mathrm{d}u\Big)^+; \qquad (4.14)
\end{aligned}
$$

the last equality follows from a change of variables and the properties of the Lebesgue integral for càdlàg functions.

Now, assume that the following property holds:

$$
\mathrm{Law}(H'_T - H'_{(T-t)-}; 0 \le t < T | P') = \mathrm{Law}(H'_t; 0 \le t < T | P'). \qquad (4.15)
$$

It is well-known that (4.15) holds if H' is a Lévy process; see e.g. Kyprianou (2006, Lemma 3.4).

From (4.14) and (4.15) we conclude

$$
\begin{aligned}
\mathbb{C}_T\Big(S; \frac{1}{T}\int S\Big) &= E'\Big(1 - \frac{1}{T}\int_0^T \mathrm{e}^{H'_u}\mathrm{d}u\Big)^+ \\
&= E'\Big(1 - \frac{1}{T}\int_0^T S'_u\mathrm{d}u\Big)^+ = \mathbb{P}'_T\Big(1; \frac{1}{T}\int S'\Big). \qquad (4.16)
\end{aligned}
$$

Similarly

$$
\mathbb{P}_T\Big(\frac{1}{T}\int S; S\Big) = \mathbb{C}'_T\Big(\frac{1}{T}\int S', 1\Big). \qquad (4.17)
$$

Therefore we have the following result.

Theorem 4.5. *Let H satisfy* (4.15), *then the calculation of prices of Asian call and put options $\mathbb{C}_T(S; \frac{1}{T}\int S)$ and $\mathbb{P}_T(\frac{1}{T}\int S; S)$ with floating strikes $\frac{1}{T}\int S$ can be reduced via formulae* (4.16) *and* (4.17) *to the calculation of the prices $\mathbb{P}'_T(1; \frac{1}{T}\int S')$ and $\mathbb{C}'_T(\frac{1}{T}\int S'; 1)$ of fixed strike Asian put and call options.*

Remark 4.6. Similar duality results can be obtained for Asian options on the geometric and the harmonic average (cf. Papapantoleon 2006)

E. Forward-start options. Suppose $S \in \mathcal{M}(P)$. The payoff of a forward-start call option is $(S_T - S_t)^+$, where $t \in (0, T)$. Similarly, the payoff of the forward-start put option is $(S_t - S_T)^+$. The price of the forward-start call option is given by the formula

$$\begin{aligned}\mathbb{C}_{T,t}(S;S) &= E(S_T - S_t)^+ = E\Big[S_T \frac{(S_T - S_t)^+}{S_T}\Big] \\ &= E'\Big(1 - \frac{S_t}{S_T}\Big)^+ = E'\Big(1 - \frac{S'_T}{S'_t}\Big)^+ \\ &= E'(1 - \mathrm{e}^{H'_T - H'_t})^+.\end{aligned}$$

Now, assume that the following property holds:

$$\mathrm{Law}(H'_T - H'_t | P') = \mathrm{Law}(H'_{T-t} | P'). \tag{4.18}$$

This property is satisfied if H' is a Lévy process, but also holds for other classes of processes.

Appealing then to (4.18) we have the following relation between a forward start call option and a plain vanilla put option

$$\begin{aligned}\mathbb{C}_{T,t}(S;S) &= E'(1 - \mathrm{e}^{H'_{T-t}})^+ \\ &= \mathbb{P}'_{T-t}(1;S'),\end{aligned} \tag{4.19}$$

Similarly, we get a relationship between a forward-start put option and a European plain vanilla call option

$$\mathbb{P}_{t,T}(S;S) = \mathbb{C}'_{T-t}(S';1). \tag{4.20}$$

Therefore, we have the following result.

Theorem 4.7. *Let H satisfy* (4.18)*, then the calculation of the prices of forward-start call and put options $\mathbb{C}_{T,t}(S;S)$ and $\mathbb{P}_{t,T}(S;S)$ can be reduced via formulae* (4.19) *and* (4.20)*, to the calculation of the prices of plain vanilla European put and call options $\mathbb{P}'_{T-t}(1;S')$ and $\mathbb{C}'_{T-t}(S';1)$.*

Remark 4.8. It is an interesting problem to investigate which processes satisfy (4.10), (4.12) and/or (4.15) beyond Lévy processes.

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