EQUIVALENCE OF FLOATING AND FIXED STRIKE ASIAN AND LOOKBACK OPTIONS

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Abstract. We prove a symmetry relationship between floating-strike and fixed-strike Asian options for assets driven by general Lévy processes using a change of numéraire and the characteristic triplet of the dual process. We apply the same technique to prove a similar relationship between floating-strike and fixed-strike lookback options.

1. Introduction

The aim of this paper is to prove a useful symmetry between floating and fixed strike Asian and floating and fixed strike lookback options for Lévy driven assets. We extend the results of Henderson and Wojakowski (2002) in two directions; firstly, by considering a general Lévy process as the driving process of the underlying and secondly by applying the same technique to prove a symmetry result for lookback options. A change of numéraire and a representation for the characteristic triplet of the dual of a Lévy process are the main tools used for the proof. Moreover, stationarity of the increments plays a crucial role.

Lévy processes have attracted much interest for financial applications lately, since they exhibit certain features of the market that diffusion models cannot capture, both in the real and in the risk-neutral world. For the merits of Lévy modelling in finance and some further applications see Eberlein (2001) and references therein. Carr et al. (2002) also provide empirical evidence supporting the use of –pure jump– Lévy processes for financial modelling.

There are several symmetry results known in option pricing theory, relating various types of options and payoffs. These results become more important when considering exotic options, since closed-form solutions might not exist for certain payoffs. This is the case for Asian options, although there are more results available for the fixed than the floating strike case.

Symmetries are even more important when we depart from Brownian motion models and consider more general driving processes, such as Lévy processes. In that case, closed form solutions are not available for several types of payoffs, such as lookback or Asian options, or are available only for one type of payoff, most probably fixed strike options, as in Borovkov and Novikov (2002) or Benhamou (2002).

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Moreover, the cost in computational time of calculating a joint density could be significantly higher than that of a single one. Hence, we aim to unify the treatment of floating and fixed strike Asian and lookback options in order to be able to transfer knowledge from one case to the other.

The results of Henderson and Wojakowski have been generalized to forward-start Asian options and discrete averaging by Vanmaele et al. (2006). Similar symmetry results have been obtained by Hoogland and Neumann (2000) using local scale invariance. The same change of numéraire has been applied to Asian options by Večeř (2002) for a Brownian motion model and Večeř and Xu (2004) for a special semimartingale model, in order to obtain a one-dimensional PDE (PIDE respectively) for both fixed and floating strike Asian options. Nielsen and Sandmann (1999) describe an analogous change of numéraire for a Brownian motion model with stochastic interest rates. This change of numéraire has also been applied by Andreasen (1998) to derive one dimensional PDEs for floating and fixed strike lookback options.

2. Model and Payoffs

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis, i.e. the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions. We model the asset price process as an exponential Lévy process

$$S_t = S_0 \exp L_t \tag{2.1}$$

where the Lévy process L satisfies Assumption ($\mathbb{M}$), which is given below. In that case, L is a special semimartingale and has the canonical decomposition (cf. Jacod and Shiryaev 2003, II.2.38)

$$L_t = bt + \sigma W_t + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x) \tag{2.2}$$

where the drift term b equals the expectation of L_1 and can be written as

$$b = r - \delta - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x)\lambda(\mathrm{d}x). \tag{2.3}$$

Here $r \geq 0$ is the (domestic) risk-free interest rate, $\delta \geq 0$ the continuous dividend yield (or foreign interest rate), $\sigma \geq 0$ the diffusion coefficient and W a standard Brownian motion under $\mathbb{P}$. μ^L is the random measure of jumps of the process L and $\nu^L(\mathrm{d}t, \mathrm{d}x) = \lambda(\mathrm{d}x)\mathrm{d}t$ is the compensator of the jump measure μ^L, where λ is the Lévy measure of L_1. The Lévy process L has the Lévy triplet (b, σ^2, λ).

We assume that $\mathbb{P}$ is a risk neutral measure, i.e. the asset price has mean rate of return $\mu \triangleq r - \delta$ and the auxiliary process $\widehat{S}_t = e^{\delta t} S_t$, once discounted, is a martingale under $\mathbb{P}$. In general, markets modelled by exponential Lévy processes, as defined in (2.1)–(2.3), are incomplete and there exists a large class of risk neutral measures. We refer to Eberlein and Jacod (1997) for a characterization of the class of equivalent martingale measures. Moreover, note that finiteness of $\mathbb{E}[\widehat{S}_t]$ is ensured by Assumption ($\mathbb{M}$).

Assumption ($\mathbb{M}$). The Lévy measure λ of the distribution of L_1 is assumed to satisfy the following integrability conditions:

$$\int_{\{x<-1\}} |x|\lambda(\mathrm{d}x) < \infty \qquad \text{and} \qquad \int_{\{x>1\}} xe^x\lambda(\mathrm{d}x) < \infty.$$

Remark 2.1 (Assumption ($\mathbb{M}$))**.** For large x and $\varepsilon > 0$, we have that $x < e^{\varepsilon x}$ hence, we could merge the above assumptions and use the following *stronger* condition: Assume there exists a constant $M > 1$, such that

$$\int_{\{|x|>1\}} \exp(ux)\lambda(\mathrm{d}x) < \infty, \qquad \forall |u| < M; \tag{2.4}$$

since $g(x) = \exp(x)$ is submultiplicative, we see that using Theorem 25.3 in Sato (1999), (2.4) is equivalent to

$$\mathbb{E}\left[\exp(uL_1)\right] < \infty, \qquad \forall |u| < M.$$

We denote by $-\lambda$ the (non-negative) Lévy measure defined by

$$-\lambda([a,b]) := \lambda([-b,-a]) \tag{2.5}$$

for $a, b \in \mathbb{R}$, $a < b$. Thus, $-\lambda$ is the mirror image of the original measure with respect to the vertical axis. Whenever we use the symbol "$-$" in front of a Lévy measure, we will refer to the Lévy measure defined as above.

Next we provide a useful lemma, which describes the characteristic triplet of the *dual* of a Lévy process in terms of the characteristic triplet of the original process.

Lemma 2.2 (dual characteristics)**.** *Let L be a Lévy process with Lévy triplet (b, c, λ). Then $L^\star := -L$ is again a Lévy process with Lévy triplet $(b^\star, c^\star, \lambda^\star)$, where $b^\star = -b$, $c^\star = c$ and $\lambda^\star = -\lambda$.*

Proof. From the Lévy-Khintchine representation we know that

$$\varphi_{L_t}(u) = \mathbb{E}\left[e^{iuL_t}\right] = \exp\left\{t\Big(ibu - \frac{c}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\lambda(\mathrm{d}x)\Big)\right\}.$$

We get immediately

$$\begin{aligned}\varphi_{-L_t}(u) &= \varphi_{L_t}(-u) \\ &= \exp\left\{t\Big(ib(-u) - \frac{c}{2}u^2 + \int_{\mathbb{R}} (e^{i(-u)x} - 1 - i(-u)x)\lambda(\mathrm{d}x)\Big)\right\} \\ &= \exp\left\{t\Big(i(-b)u - \frac{c}{2}u^2 + \int_{\mathbb{R}} (e^{iu(-x)} - 1 - iu(-x))\lambda(\mathrm{d}x)\Big)\right\}.\end{aligned}$$

Hence, we can conclude that $L^\star$ is also a Lévy process and has characteristics $b^\star = -b$, $c^\star = c$ and $\lambda^\star = -\lambda$. □

Define as Σ_D the *arithmetic* average of the process S –either continuously or discretely observed– during a time interval of length D. More precisely, let $T_1 < T_2 < \cdots < T_n$ be equidistant time points such that $D = T_n - T_1$; then, for a continuously observed process we define $\Sigma_D = \frac{1}{D}\int_{T_1}^{T_n} S_u du$, whereas for a discretely observed process we define instead $\Sigma_D = \frac{1}{n}\sum_{i=1}^{n} S_{T_i}$.

Option type	Asian Payoff	Lookback payoff
Fixed Strike call	$(\Sigma_D - K)^+$	$(M_D - K)^+$
Fixed Strike put	$(K - \Sigma_D)^+$	$(K - N_D)^+$
Floating Strike call	$(\theta S_T - \Sigma_D)^+$	$(\theta S_T - N_D)^+$
Floating Strike put	$(\Sigma_D - \theta S_T)^+$	$(M_D - \theta S_T)^+$

TABLE 2.1. Types of payoffs for Asian and Lookback options

Similarly, define N_D and M_D to be the minimum and maximum of the process S –either continuously or discretely observed– during a time interval of length D, that is, in the discrete case we define $N_D = \min_{T_1 \le T_i \le T_n} S_{T_i}$ and $M_D = \max_{T_1 \le T_i \le T_n} S_{T_i}$.

There exist fixed and floating strike Asian and lookback options. The payoff of fixed strike options depends on the difference between an average or an extreme value of the underlying and a fixed strike. The payoff of the floating strike option depends on the difference between an average or an extreme value of the underlying and the value of the asset at maturity; hence, in that case, the average or the extreme value plays the role of the strike. More generally, one can consider θS_T instead of the value at maturity S_T, for some constant $\theta \in \mathbb{R}_+$. The different types of payoffs of the Asian and lookback option are summarized in Table 2.1, where $x^+ = \max\{x, 0\}$.

According to the duration of the averaging period, there exist different variants of the Asian option. Let $[0, T]$ be a time interval, where the option starts at time $t \in [0, T)$ and matures at time T. If the averaging starts at time $T_+ > t$, we have a *forward-start* option, at time t we have the *standard* option and if the averaging starts at time $T_- < t$ we have an *in-progress* Asian option. In-progress Asian options can be re-written in terms of standard Asian options, see Večeř (2002), but floating strike options become a mixture of floating and fixed strike Asian options, because of the additional term that corresponds to the averaging up to time t. Here, we concentrate on forward-start Asian options, treating standard options as a special case of them.

A variant of floating strike lookback options are *partial* lookback options, with payoff

$$(S_T - \alpha N_D)^+ \quad \text{and} \quad (\alpha M_D - S_T)^+$$

for call and put options respectively, where the constant $\alpha \ge 1$, respectively $\alpha \in (0, 1]$, denotes the degree of partiality.

By arbitrage arguments, the price of an option is equal to its discounted expected payoff under an equivalent martingale measure. We introduce the following notation for the floating strike forward-start Asian call option

$$V_c(\theta S_T, \Sigma_D, r; b, \sigma^2, \lambda; T_1, T_n) = V_{flc} = e^{-rT} \mathbb{E}\left[(\theta S_T - \Sigma_D)^+\right]$$

where T_1 and T_n denote the first and last value of the equidistant time points of an interval of length D; the asset is modelled as an exponential Lévy process according to (2.1)–(2.3) and the option starts at time 0 and matures at T. For the fixed strike forward-start Asian put option we set

$$V_p(K, \Sigma_D, r; b, \sigma^2, \lambda; T_1, T_n) = V_{fxp} = e^{-rT} \mathbb{E}\left[(K - \Sigma_D)^+\right].$$

Similar notation will be used for the other types of Asian options.

3. An equivalence result for Asian options

In this section we state and prove the main result that shows an equivalence relationship between floating and fixed strike Asian options.

Theorem 3.1. *Assuming that the asset price evolves as an exponential Lévy process according to equations (2.1)–(2.3), we can relate the floating and fixed strike forward-start Asian option via the following symmetry:*

$$V_c\left(\theta S_T, \Sigma_D, r; b, \sigma^2, \lambda; T_1, T\right) = V_p\left(\theta S_0, \Sigma_D, \delta; b^*, \sigma^2, -f\lambda; 0, T-T_1\right)$$

$$V_p\left(\Sigma_D, \theta S_T, r; b, \sigma^2, \lambda; T_1, T\right) = V_c\left(\Sigma_D, \theta S_0, \delta; b^*, \sigma^2, -f\lambda; 0, T-T_1\right)$$

where $D = T - T_1$, $b^* = \delta - r - \frac{\sigma^2}{2} - \int_{\mathbb{R}}(e^{-x} - 1 + x)e^x\lambda(\mathrm{d}x)$ *and* $f(x) = e^x$.

Proof. We will prove the result –for simplicity and without loss of generality– in the case of discrete averaging for the floating strike call. The case of continuous averaging and the floating strike put can be proved in an analogous way.

The price of the floating strike option expressed in units of the *numéraire* yields

$$\begin{aligned}\widetilde{V}_{flc} := \frac{V_{flc}}{S_0} &= \frac{e^{-rT}}{S_0}\mathbb{E}\left[(\theta S_T - \Sigma_D)^+\right] \\ &= e^{-\delta T}\mathbb{E}\left[\frac{e^{-rT}S_T}{e^{-\delta T}S_0}\frac{(\theta S_T - \Sigma_D)^+}{S_T}\right]. \end{aligned} \tag{3.1}$$

Define a new measure $\widetilde{\mathbb{P}}$ via its Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \frac{e^{-rT}S_T}{e^{-\delta T}S_0} = \eta_T \tag{3.2}$$

and using S as the numéraire, the valuation problem (3.1) becomes

$$\widetilde{V}_{flc} = e^{-\delta T}\widetilde{\mathbb{E}}\left[(\theta - \widetilde{\Sigma}_D)^+\right]. \tag{3.3}$$

Here

$$\widetilde{\Sigma}_D := \frac{\Sigma_D}{S_T} = \frac{1}{n}\sum_{i=1}^{n}\frac{S_{T_i}}{S_T} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{S}_{T_i}$$

and $\widetilde{S}$ is defined as $\widetilde{S}_{T_i} := \frac{S_{T_i}}{S_T}$.

Because the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are related via the adapted and positive density process η, we immediately conclude that $\widetilde{\mathbb{P}} \overset{\text{loc}}{\sim} \mathbb{P}$ and we can apply Girsanov's theorem for semimartingales; we refer to Jacod and Shiryaev (2003, III.3.24). The density process corresponding to that change of measure can be represented in the usual form

$$\begin{aligned}\eta_t = \mathbb{E}\left[\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{F}_t\right] &= \frac{e^{-rt}S_t}{e^{-\delta t}S_0} \\ &= \exp\left\{\sigma W_t + \int_0^t\int_{\mathbb{R}} x(\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x) - \left(\frac{\sigma^2}{2} + \int_{\mathbb{R}}(e^x - 1 - x)\lambda(\mathrm{d}x)\right)t\right\}\end{aligned}$$

therefore we can identify the tuple (β, Y) of predictable processes (functions)

$$\beta(t) = 1 \qquad \text{and} \qquad Y(t,x) = \exp(x)$$

that characterize the change of measure.

Combining Girsanov's theorem with Theorem II.4.15 and Corollary II.4.19 in Jacod and Shiryaev (2003), we deduce that a Lévy process (PIIS) remains a Lévy process under the measure $\widetilde{\mathbb{P}}$, because the processes β and Y are deterministic and Y is independent of t.

As a consequence of Girsanov's theorem for semimartingales we infer that $\widetilde{W}_t = W_t - \sigma t$ is a $\widetilde{\mathbb{P}}$-Brownian motion and $\widetilde{\nu}^L = Y\nu^L$ is the $\widetilde{\mathbb{P}}$ compensator of the random measure of jumps μ^L. Whence, we define $\widetilde{\lambda}(\mathrm{d}x) = e^x\lambda(\mathrm{d}x)$.

Note that this change of measure can also be interpreted as an *Esscher transformation*; we refer to Shiryaev (1999) for more on the Esscher transformation and Eberlein and Keller (1995) for an application in Lévy processes and finance.

Therefore, we have that

$$\begin{aligned}
\widetilde{S}_{T_i} &= \frac{S_{T_i}}{S_T} \\
&= \exp\Bigg\{\Bigg(r - \delta - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x)\lambda(\mathrm{d}x)\Bigg)(T_i - T) + \sigma\left(W_{T_i} - W_T\right) \\
&\qquad + \int_0^{T_i}\int_{\mathbb{R}} x(\mu^L - \nu^L)(\mathrm{d}s,\mathrm{d}x) - \int_0^{T}\int_{\mathbb{R}} x(\mu^L - \nu^L)(\mathrm{d}s,\mathrm{d}x)\Bigg\} \\
&= \exp\Bigg\{\Bigg(r - \delta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^{-x} - 1 + x)\widetilde{\lambda}(\mathrm{d}x)\Bigg)(T_i - T) + \sigma\big(\widetilde{W}_{T_i} - \widetilde{W}_T\big) \\
&\qquad + \int_0^{T_i}\int_{\mathbb{R}} x(\mu^L - \widetilde{\nu}^L)(\mathrm{d}s,\mathrm{d}x) - \int_0^{T}\int_{\mathbb{R}} x(\mu^L - \widetilde{\nu}^L)(\mathrm{d}s,\mathrm{d}x)\Bigg\} \\
&= \exp\Bigg\{\Bigg[\Big(r - \delta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^{-x} - 1 + x)\widetilde{\lambda}(\mathrm{d}x)\Big)T_i + \sigma\widetilde{W}_{T_i} \\
&\qquad\quad + \int_0^{T_i}\int_{\mathbb{R}} x(\mu^L - \widetilde{\nu}^L)(\mathrm{d}s,\mathrm{d}x)\Bigg] \\
&\qquad - \Bigg[\Big(r - \delta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^{-x} - 1 + x)\widetilde{\lambda}(\mathrm{d}x)\Big)T + \sigma\widetilde{W}_{T} \\
&\qquad\quad + \int_0^{T}\int_{\mathbb{R}} x(\mu^L - \widetilde{\nu}^L)(\mathrm{d}s,\mathrm{d}x)\Bigg]\Bigg\}.
\end{aligned}$$

We now define the Lévy process $\widetilde{L}$ via

$$\widetilde{L}_t = \widetilde{b}t + \sigma\widetilde{W}_t + \int_0^t \int_{\mathbb{R}} x(\mu^L - \widetilde{\nu}^L)(\mathrm{d}s, \mathrm{d}x) \tag{3.4}$$

where

$$\widetilde{b} = r - \delta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^{-x} - 1 + x)\widetilde{\lambda}(\mathrm{d}x). \tag{3.5}$$

Notice that using Assumption ($\mathbb{M}$) and Theorem 25.3 in Sato (1999), we can easily deduce that $\widetilde{\mathbb{E}}\big[|\widetilde{L}_1|\big] < \infty$, hence, $\widetilde{L}$ is a $\widetilde{\mathbb{P}}$-special semimartingale.

The characteristic triplet of $\widetilde{L}$ is $(\widetilde{b}, \sigma^2, \widetilde{\lambda})$ and the part without drift of $\widetilde{L}$ is a martingale under the measure $\widetilde{\mathbb{P}}$. Since $\widetilde{L}$ is a Lévy process, the dual process defined as $\widetilde{L}^* := -\widetilde{L}$ is also a Lévy process and from Lemma 2.2 we deduce that its Lévy triplet is $(-\widetilde{b}, \sigma^2, -\widetilde{\lambda})$. This simplifies the expression for $\widetilde{S}$

$$\begin{aligned} \widetilde{S}_{T_i} &= \exp\left\{\widetilde{L}_{T_i} - \widetilde{L}_T\right\} \\ &\overset{\mathrm{d}}{=} \exp\left\{\widetilde{L}^*_{T-T_i}\right\} \end{aligned}$$

where stationarity of $\widetilde{L}$ is used for the last equation and $\overset{\mathrm{d}}{=}$ denotes equality in law.

As a result we have that

$$\begin{aligned} \widetilde{\Sigma}_D &= \frac{1}{n}\sum_{i=1}^n \widetilde{S}_{T_i} \\ &\overset{\mathrm{d}}{=} \frac{1}{n}\sum_{i=1}^n \exp\left\{\widetilde{L}^*_{T-T_i}\right\}. \end{aligned}$$

Reversing the time index via the substitution $\widehat{T}_j = T - T_{n-j+1}$ we get

$$\widetilde{\Sigma}_D \overset{\mathrm{d}}{=} \frac{1}{n}\sum_{j=1}^n \exp\left\{\widetilde{L}^*_{\widehat{T}_j}\right\} =: \widetilde{\Sigma}^*_D.$$

Hence, we can conclude the proof

$$\begin{aligned} \widetilde{V}_{flc} &= e^{-\delta T}\widetilde{\mathbb{E}}\left[(\theta - \widetilde{\Sigma}_D)^+\right] \\ &= e^{-\delta T}\widetilde{\mathbb{E}}\left[(\theta - \widetilde{\Sigma}^*_D)^+\right] \end{aligned}$$

where, in the last expression we consider the expectation with respect to $\widetilde{\mathbb{P}}$, under which the Lévy process $\widetilde{L}^*$ has the triplet $(-\widetilde{b}, \sigma^2, -\widetilde{\lambda})$. Notice that $e^{rt}\widetilde{S}^*_t$ once discounted at the rate δ, is a $\widetilde{\mathbb{P}}$-martingale. □

Remark 3.2. The equivalence results do not hold in the case of *in-progress* Asian options, because of the additional term created in floating strike options from averaging up to time t, when the option starts.

Remark 3.3. The equivalence results hold in the case of an option on the *geometric* or *harmonic average*, that is when the average is of the form

$$\Gamma_D = \Big(\prod_{i=1}^{n} S_{T_i}\Big)^{\frac{1}{n}} \qquad \text{and} \qquad A_D = \frac{n}{\sum_{i=1}^{n}\frac{1}{S_{T_i}}} \tag{3.6}$$

respectively.

Proof. It suffices to notice that, if we apply the same change of numéraire as in (3.1), the resulting averaging term is of the same form as in (3.6) and we can define a random variable $\widetilde{S}$ in an analogous way. We have for the geometric and harmonic average respectively

$$\begin{aligned}\widetilde{\Gamma}_D = \frac{\Gamma_D}{S_T} = \frac{\big(\prod_{i=1}^{n} S_{T_i}\big)^{\frac{1}{n}}}{S_T} &= \Big(\prod_{i=1}^{n}\frac{S_{T_i}}{S_T}\Big)^{\frac{1}{n}} \\ &= \Big(\prod_{i=1}^{n}\widetilde{S}_{T_i}\Big)^{\frac{1}{n}}\end{aligned}$$

and

$$\begin{aligned}\widetilde{A}_D = \frac{A_D}{S_T} = \frac{\frac{n}{\sum_{i=1}^{n}\frac{1}{S_{T_i}}}}{S_T} &= \frac{n}{\sum_{i=1}^{n}\frac{S_T}{S_{T_i}}} \\ &= \frac{n}{\sum_{i=1}^{n}\frac{1}{\widetilde{S}_{T_i}}}\end{aligned}$$

where $\widetilde{S}_{T_i} = \frac{S_{T_i}}{S_T}$. The remaining part follows along the same lines as the proof of Theorem 3.1. □

4. An equivalence result for Lookback options

In this section we use the techniques applied in the previous section to prove an equivalence relationship between floating and fixed strike lookback options.

We introduce the following notation for the floating strike forward-start lookback call option

$$V_c(\theta S_T, N_D, r; b, \sigma^2, \lambda; T_1, T_n) = V_{flc} = e^{-rT}\mathbb{E}\left[(\theta S_T - N_D)^+\right]$$

where T_1 and T_n denote the first and last value of the equidistant time points of an interval of length D; the asset is modelled as an exponential Lévy process according to (2.1)–(2.3) and the option starts at time 0 and matures at T. For the fixed strike forward-start lookback put option we set

$$V_p(K, N_D, r; b, \sigma^2, \lambda; T_1, T_n) = V_{fxp} = e^{-rT}\mathbb{E}\left[(K - N_D)^+\right]$$

Similar notation will be used for the other types of lookback options.

Theorem 4.1. *Assuming that the asset price evolves as an exponential Lévy process according to equations (2.1)–(2.3), we can relate the floating and* *fixed strike forward-start lookback option via the following symmetry:*

$$V_c(\theta S_T, N_D, r; b, \sigma^2, \lambda; T_1, T) = V_p(\theta S_0, N_D, \delta; b^*, \sigma^2, -f\lambda; 0, T-T_1)$$

$$V_p(M_D, \theta S_T, r; b, \sigma^2, \lambda; T_1, T) = V_p(M_D, \theta S_0, \delta; b^*, \sigma^2, -f\lambda; 0, T-T_1)$$

where $D = T - T_1$, $b^* = \delta - r - \frac{\sigma^2}{2} - \int_{\mathbb{R}}(e^{-x} - 1 + x)e^x\lambda(\mathrm{d}x)$ *and* $f(x) = e^x$*.*

Proof. We consider a floating strike lookback put option and the case of a floating strike lookback call can be proved in an analogous way. Applying the same change of numéraire as in (3.1), the valuation problem is equivalent to the one in (3.3)

$$\widetilde{V}_{flp} = e^{-\delta T}\widetilde{\mathbb{E}}\left[(\widetilde{M}_D - \theta)^+\right].$$

Here

$$\widetilde{M}_D := \frac{M_D}{S_T} = \frac{\max_{T_1 \le T_i \le T} S_{T_i}}{S_T} = \max_{T_1 \le T_i \le T} \widetilde{S}_{T_i}$$

where $\widetilde{S}$ is defined as $\widetilde{S}_{T_i} = \frac{S_{T_i}}{S_T}$. Following the steps of the proof of Theorem 3.1 we complete the result. □

Remark 4.2. Because the pricing function is homogeneous of degree one, i.e.

$$\varrho\mathbb{E}\left[(M_T - K)^+\right] = \mathbb{E}\left[(\varrho M_T - \varrho K)^+\right]$$

where $\varrho \in \mathbb{R}_+$, for a suitable choice of α and ϱ we can apply the symmetry result of Theorem 4.1 for pricing partial lookback options when a method for fixed strike options is available.

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