

# SPECTRAL MAPPING THEOREMS FOR HOLOMORPHIC FUNCTIONAL CALCULI

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## ABSTRACT

A spectral inclusion theorem and a spectral mapping theorem are proved for the functional calculus for sectorial operators. Most proofs are generic, so that similar results can be obtained for other functional calculi. Applications are a new proof for the spectral mapping theorem for fractional powers and the identity  $\sigma(\log A) = \log(\sigma(A) \setminus \{0\})$  for any injective sectorial operator  $A$ .

### 1. Introduction

For a bounded operator  $A \in \mathcal{L}(X)$  on a Banach space  $X$ , the well known *Riesz–Dunford* calculus allows  $f(A)$  to be defined if  $f$  is a holomorphic function on a neighborhood of the spectrum  $\sigma(A)$ . It is also well known that one has

$$f(\sigma(A)) = \sigma(f(A)) \tag{1}$$

in that case (see [9]). We call this identity a *spectral mapping theorem*. As pioneered by Hille [17], Bade [4], McIntosh [19] and deLaubenfels [6], functional calculi have also been constructed for *unbounded* operators  $A$  (see also [12, 18, 20, 21]). If, for example,  $A$  is the generator of a  $C_0$ -semigroup  $T = (T(t))_{t \geq 0}$ , then one can read  $T(t) = f(A)$  with  $f(z) = e^{tz}$ , and a spectral mapping theorem such as (1) has important consequences for the asymptotic behaviour of the semigroup (see [10, Section VI.3, Lemma V.1.9]). In recent years, the holomorphic functional calculus for sectorial operators has been studied and used by many authors because of its importance in the field of evolution equations. See [2] and the references therein.

It is well known that general spectral mapping theorems for unbounded functional calculi may fail, but there are positive results in certain special situations (see [5; 10, IV.3.10; 18, Theorem 4.3.1]). In particular, if  $A$  is a sectorial operator, then (1) holds for the *fractional powers*  $f(z) = z^\alpha$  ( $\operatorname{Re} \alpha > 0$ ). The first proof for this was given by Balakrishnan using Gelfand theory. (In [18], this proof and a new one are given.)

Recently, Dore has, in unpublished notes [8], given a more detailed account of the functional calculus for sectorial operators. He also obtained partial results towards a spectral mapping theorem. However, a reasonably general spectral mapping (or even inclusion) theorem for the functional calculus for sectorial operators seems to have been missing up to now.

This paper fills this gap. Moreover, the construction of the functional calculus is carried out by a generic method (see [16]), and most proofs presented here are generic, so that it will be easy to transfer them to holomorphic functional calculi for other types of operators.

Actually, (1) is not the ‘right’ formulation for the spectral mapping theorem if  $A$  is unbounded. Consider for example the ‘easiest’ spectral mapping theorem for a closed operator  $A$ , namely

$$\sigma(R(\lambda, A)) = \frac{1}{\lambda - \sigma(A)} := \left\{ \frac{1}{\lambda - \mu} \mid \mu \in \sigma(A) \right\} \tag{2}$$

for  $\lambda \in \varrho(A)$ . In the case when  $A$  is unbounded, this is obviously false, since 0 is then a member of the left-hand set. Thus there are two possibilities: either make 0 a special case (as in [10, Theorem IV.1.13]) or, which is what we will do, include  $\infty$  in our considerations, that is, view the spectrum as essentially a subset of the Riemann sphere  $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ . Therefore, we define

$$\tilde{\sigma}(A) := \begin{cases} \sigma(A) & A \in \mathcal{L}(X) \\ \sigma(A) \cup \{\infty\} & A \notin \mathcal{L}(X) \end{cases}$$

and call it the *extended spectrum* of  $A$ . The extended spectrum is never empty and it is a compact subset of the Riemann sphere. Moreover, (2) holds when ‘spectrum’ is replaced by ‘extended spectrum’, at least for  $\lambda \in \varrho(A)$ . (Actually, this is true for every  $\lambda \in \mathbb{C}$  when  $R(\lambda, A) := (\lambda - A)^{-1}$  is defined in the sense of multivalued operators. Even  $A$  can be multivalued itself. Moreover, this ‘spectral mapping theorem for the resolvent’ is even true when ‘spectrum’ is replaced by ‘extended point spectrum’, ‘extended approximate point spectrum’ or ‘extended residual spectrum’. Of course, these notions have to be defined appropriately. See [15, Appendix A].)

We thus arrive at the formulation

$$f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A)), \tag{3}$$

which we want to prove for as many functions  $f$  as possible.

Let us fix some notation. For any open set  $\Omega \subset \mathbb{C}$ , we denote by  $\mathcal{O}(\Omega)$  ( $\mathcal{M}(\Omega)$ ,  $H^\infty(\Omega)$ ) the space of all holomorphic (respectively meromorphic, bounded holomorphic) functions on  $\Omega$ . Some special choices of  $\Omega$  will be important in the sequel, namely sectors and strips.

For  $0 < \omega \leq \pi$ , let  $S_\omega := \{z \in \mathbb{C} \mid z \neq 0 \text{ and } |\arg z| < \omega\}$  denote the open sector symmetric about the positive real axis with opening angle  $\omega$ . To cover also the case  $\omega = 0$ , we define  $S_0 := (0, \infty)$ .

Given  $\omega > 0$ , we denote by  $H_\omega := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \omega\}$  the open horizontal strip symmetric to the real axis. To cover also the case  $\omega = 0$ , we define  $H_0 := \mathbb{R}$ .

For an operator  $A$  on a Banach space  $X$ , we denote by  $\mathcal{N}(A)$  its kernel and denote by  $\mathcal{R}(A)$  its range. We denote by  $\mathcal{L}(X)$  the set of all bounded, fully defined operators on  $X$ .

For any operator  $A$ , we denote by  $P\sigma(A) := \{\lambda \in \mathbb{C} \mid \mathcal{N}(\lambda - A) \neq \{0\}\}$  its point spectrum. The resolvent set  $\varrho(A)$  of  $A$  is defined as  $\varrho(A) := \{\lambda \in \mathbb{C} \mid \lambda \notin P\sigma(A), (\lambda - A)^{-1} \in \mathcal{L}(X)\}$ , and we write  $R(\lambda, A) := (\lambda - A)^{-1}$  for the resolvent of  $A$ . The set  $\sigma(A) := \mathbb{C} \setminus \varrho(A)$  is the spectrum of  $A$ . If  $A \in \mathcal{L}(X)$ , the number  $r(A) := \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$  is called the spectral radius of  $A$ . [10] can be consulted for other notation and the basic results of spectral theory for unbounded operators.

If  $A$  and  $B$  are any operators on  $X$ , then we define their product  $BA$  as  $(BA)x := B(Ax)$  for  $x \in \mathcal{D}(BA) := \{x \in \mathcal{D}(A) \mid Ax \in \mathcal{D}(B)\}$ . Also we use the notation  $A \subset B$  as an abbreviation for  $\mathcal{D}(A) \subset \mathcal{D}(B)$  and  $Bx = Ax$  for all  $x \in \mathcal{D}(A)$ . If  $A$  is any operator on  $X$  and  $T \in \mathcal{L}(X)$ , then we say that  $A$  and  $T$  commute if

$TA \subset AT$ . (In the case  $\varrho(A) \neq \emptyset$ , this is equivalent to saying that  $T$  commutes with the resolvent of  $A$ ; see [15, Appendix A].)

2. The functional calculus

We recall the construction of the functional calculus for sectorial operators, but this will be more general than other accounts; cf. [1, 16, 15, 19].

DEFINITION 2.1. Let  $0 \leq \omega < \pi$ . An operator  $A$  on a Banach space  $X$  is called sectorial of angle  $\omega$  ( $A \in \text{Sect}(\omega)$ ) if  $\sigma(A) \subset \overline{S_\omega}$  and

$$M(A, \omega') := \sup\{\|\lambda R(\lambda, A)\| \mid \lambda \notin \overline{S_{\omega'}}\} < \infty$$

for all  $\omega < \omega' < \pi$ . The number

$$\omega_A := \min\{\omega \mid A \in \text{Sect}(\omega)\}$$

is called the spectral angle (or sectoriality angle) of  $A$ .

In contrast to other common definitions of the term ‘sectorial operator’, we do not suppose that  $A$  is densely defined or that  $A$  has dense range.

An operator  $A$  on a Banach space  $X$  is sectorial of some angle if and only if  $(-\infty, 0) \subset \varrho(A)$  and  $M(A) := \sup_{t>0} \|t(t + A)^{-1}\| < \infty$  (see [18, Proposition 1.2.1]). If  $A \in \text{Sect}(\omega)$  is injective, then also  $A^{-1} \in \text{Sect}(\omega)$ . This is due to the fundamental identity

$$\lambda(\lambda + A^{-1})^{-1} = I - \frac{1}{\lambda} \left( \frac{1}{\lambda} + A \right)^{-1}, \tag{4}$$

which holds for all  $0 \neq \lambda \in \mathbb{C}$  (in the sense of multivalued operators). For other basic properties of sectorial operators, see [18] or [15].

We shall first construct an ‘elementary’ functional calculus for sectorial operators. Let us define

$$\mathcal{E}_0(S_\varphi) := \left\{ f \in H^\infty(S_\varphi) \mid \int_{\partial S_{\omega'}} |f(z)| \frac{|dz|}{|z|} < \infty \right. \\ \left. \text{for all } 0 \leq \omega' < \varphi \text{ and } \lim_{z \rightarrow 0, \infty} f(z) = 0 \right\}$$

and

$$H_0^\infty(S_\varphi) := \{f \in \mathcal{O}(S_\varphi) \mid \exists C, s > 0 \text{ s.t. } |f(z)| \leq C \min(|z|^s, |z|^{-s})\}.$$

Obviously  $H_0^\infty(S_\varphi) \subset \mathcal{E}_0(S_\varphi)$ , and both are ideals in the algebra  $H^\infty(S_\varphi)$ . If  $A \in \text{Sect}(\omega)$ ,  $\omega < \varphi < \pi$  and  $f \in \mathcal{E}_0(S_\varphi)$ , then we define

$$f(A) := \Phi(f) := \frac{1}{2\pi i} \int_\Gamma f(z)R(z, A) dz, \tag{5}$$

where  $\Gamma$  is the positively oriented boundary of a sector  $S_{\omega'}$ , with  $\omega < \omega' < \varphi$  being arbitrary. (Cauchy’s theorem shows that  $f(A)$  is independent of  $\omega'$ .)

Since we want the ‘elementary’ functional calculus to contain resolvents, we increase the algebra  $\mathcal{E}_0(S_\varphi)$  by the function  $(1 + z)^{-1}$ . It is easy to see that

$$\mathcal{E}(S_\varphi) := \mathcal{E}_0(S_\varphi) \oplus \mathbb{C} \frac{1}{1 + z}$$

is in fact an algebra.

We can extend  $\Phi$  from  $\mathcal{E}_0(S_\varphi)$  to  $\mathcal{E}(S_\varphi)$  simply by defining  $\Phi(1/(1+z)) := (1+A)^{-1}$ . Then  $\Phi: \mathcal{E}(S_\varphi) \rightarrow \mathcal{L}(X)$  is an algebra homomorphism and  $(\mathcal{E}(S_\varphi), \mathcal{M}(S_\varphi), \Phi)$  is an abstract functional calculus (see [16] for terminology and the missing computations). A function  $f \in \mathcal{M}(S_\varphi)$  is *regularizable* by  $\mathcal{E}(S_\varphi)$  if there is  $e \in \mathcal{E}(S_\varphi)$  such that  $ef \in \mathcal{E}(S_\varphi)$  and  $e(A)$  is injective. In this case  $f(A)$  is defined by

$$f(A) := e(A)^{-1}(ef)(A).$$

(This is independent of the ‘regularizer’  $e$ ; see [16].) Note that  $f(A)$  is always a closed operator, which, however, need not be densely defined. We denote by

$$\mathcal{M}(S_\varphi)_A := \{f \in \mathcal{M}(S_\varphi) \mid f \text{ is regularizable by } \mathcal{E}(S_\varphi)\}$$

the set of regularizable elements. Additionally, we define

$$H(A) := \{f \in \mathcal{M}(S_\varphi)_A \mid f(A) \in \mathcal{L}(X)\}.$$

Then we obtain the following theorem.

**THEOREM 2.2.** *Let  $0 \leq \omega < \varphi < \pi$ ,  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}(S_\varphi)_A$ . Then the following assertions hold.*

(a) *If  $T \in \mathcal{L}(X)$  commutes with  $A$ , that is, if  $TA \subset AT$ , then  $T$  also commutes with  $f(A)$ , that is, one has  $Tf(A) \subset f(A)T$ . If  $f(A) \in \mathcal{L}(X)$ , then  $f(A)$  commutes with  $A$ .*

(b)  $\mathbf{1}(A) = I$  and  $(z)(A) = A$ .

(c) Also let  $g \in \mathcal{M}(S_\varphi)_A$ . Then

$$f(A) + g(A) \subset (f + g)(A) \quad \text{and} \quad f(A)g(A) \subset (fg)(A).$$

Furthermore,  $\mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)) = \mathcal{D}(f(A)g(A))$ , and one has equality in the above relations if  $g(A) \in \mathcal{L}(X)$ .

(d) The mapping  $(f \mapsto f(A)) : H(A) \rightarrow \mathcal{L}(X)$  is a homomorphism of algebras.

(e) One has  $f(A) = g(A)^{-1}f(A)g(A)$  if  $g(A)$  is bounded and injective.

(f) Let  $\lambda \in \mathbb{C}$ . Then

$$\frac{1}{\lambda - f(z)} \in \mathcal{M}(S_\varphi)_A \iff \lambda - f(A) \text{ is injective.}$$

In this case,  $(\lambda - f(z))^{-1}(A) = (\lambda - f(A))^{-1}$ . In particular,  $\lambda \in \varrho(f(A))$  if and only if  $(\lambda - f(z))^{-1}(A) \in \mathcal{L}(X)$ .

See [16] for a proof. Since  $\varphi > \omega$  was arbitrary, we can form the inductive limits

$$\mathcal{E}[S_\omega] := \bigcup_{\omega < \varphi < \pi} \mathcal{E}(S_\varphi) \quad \text{and} \quad \mathcal{M}[S_\omega] := \bigcup_{\omega < \varphi < \pi} \mathcal{M}(S_\varphi).$$

Then we define  $\mathcal{M}[S_\omega]_A := \bigcup_{\omega < \varphi < \pi} \mathcal{M}(S_\varphi)_A$ . Note that  $(\lambda - z)^{-1} \in \mathcal{M}[S_\omega]_A$  if and only if  $\lambda \notin P\sigma(A)$ .

**DEFINITION 2.3.** Let  $A \in \text{Sect}[S_\omega]$  and  $f \in \mathcal{M}[S_\omega]$ . If  $f \in \mathcal{M}[S_\omega]_A$ , that is, if  $f$  is regularizable by  $\mathcal{E}[S_\omega]$ , then we say that  $f(A)$  is defined by the natural functional calculus (NFC) for sectorial operators.

The following lemma is of fundamental importance for the proof of the spectral inclusion theorem Theorem 4.1.

LEMMA 2.4. Let  $f \in \mathcal{E}[S_\omega]$  and  $0 \neq \lambda \in \overline{S_\omega}$ . Then

$$\frac{f(z) - f(\lambda)}{\lambda - z} \in \mathcal{E}[S_\omega].$$

*Proof.* Obviously  $h := (f(z) - f(\lambda))/(\lambda - z) \in H^\infty[S_\omega]$ . One has to consider  $h(z) - h(0)(1+z)^{-1}$  and check the integrability condition as well as the limits at 0 and  $\infty$ . This is easy.  $\square$

REMARK 2.5. If the operator  $A$  is injective, then every function in  $\mathcal{E}[S_\omega]$  is already regularized by  $z(1+z)^{-2} \in H_0^\infty[S_\omega]$ . Hence the algebra  $H_0^\infty[S_\omega]$  ‘generates’ the functional calculus (cf. [16]). This is no longer true if  $A$  fails to be injective. In fact, an easy computation shows that  $f(A)x = 0$  for all  $x \in \mathcal{N}(A)$  and all  $f \in \mathcal{E}_0[S_\omega]$ . Hence if  $A$  is not injective, then *no* function from  $\mathcal{E}_0$  can act as a regularizer and every regularizer necessarily has a nonzero limit at 0. This readily implies that in this case, each  $f \in \mathcal{M}[S_\omega]_A$  has a finite limit at 0. Moreover,  $f(A)x = f(0)x$  for all  $x \in \mathcal{N}(A)$ ,  $f \in \mathcal{M}[S_\omega]_A$ .

### 3. The spectral inclusion theorem for the point spectrum

We begin with the most elementary part, namely a spectral inclusion theorem for the point spectrum.

PROPOSITION 3.1. Let  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]_A$ . If  $0 \neq \lambda \in P\sigma(A)$ , then  $f(\lambda) \in \mathbb{C}$ . Also, if  $0 \in P\sigma(A)$ , then  $\lim_{z \rightarrow 0} f(z) =: f(0)$  exists in  $\mathbb{C}$ . Moreover, we have

$$f(A)x = f(\lambda)x$$

for all  $\lambda \in P\sigma(A)$  and all  $x \in \mathcal{N}(\lambda - A)$ .

*Proof.* Let  $\lambda \in P\sigma(A)$ . For  $0 \neq x \in \mathcal{N}(\lambda - A)$ , we clearly have  $R(z, A)x = (1/(z - \lambda))x$  for all  $z \in \rho(A)$ . This yields  $e(A)x = e(\lambda)x$  for all  $e \in \mathcal{E}[S_\omega]$ . Now, let  $e \in \mathcal{E}$  be a regularizer for  $f$ . Then  $e(A)$  is injective and  $ef \in \mathcal{E}$ . By what we have shown,  $e(\lambda) \neq 0$ . Hence  $f(\lambda) \neq \infty$  if  $\lambda \neq 0$  and  $\lim_{z \rightarrow 0} f(z)$  exists in  $\mathbb{C}$  if  $\lambda = 0$ . Moreover,  $f(A)x = e(A)^{-1}(ef)(A)x = e(A)^{-1}(ef)(\lambda)x = f(\lambda)e(A)^{-1}e(\lambda)x = f(\lambda)x$ .  $\square$

REMARK 3.2. Note that in the case  $0 \in P\sigma(A)$ ,  $f$  approaches its limit at 0 in an integrable fashion, that is,  $f$  is ‘regular’ at 0 in the sense of Definition 6.1. This is due to the fact that  $f$  has the same limit behavior at 0 as its regularization  $ef$ .

The next statement follows immediately from Proposition 3.1.

THEOREM 3.3 (spectral inclusion theorem for the point spectrum). Let  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]_A$ . Then

$$f(P\sigma(A)) \subset P\sigma(f(A)).$$

## 4. The spectral inclusion theorem

In this section we consider the statement  $f(\tilde{\sigma}(A)) \subset \tilde{\sigma}(f(A))$  for  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]_A$ . Clearly there is an immediate difficulty, namely that  $f$  need not be defined at 0 or at  $\infty$ . This will be the issue in Section 5. For now we are going to prove the following weaker result.

**THEOREM 4.1** (spectral inclusion theorem). *Let  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]_A$ . Then  $f(\sigma(A) \setminus \{0\}) \subset \tilde{\sigma}(f(A))$ .*

For the proof we need some auxiliary results. We shall use Theorem 2.2 without further reference.

**LEMMA 4.2.** *Let  $A \in \text{Sect}(\omega)$ ,  $e \in H(A)$  and  $\lambda \in \mathbb{C}$ , with  $\lambda - A$  being injective. Assume that there is  $0 \neq c \in \mathbb{C}$  such that*

$$f(z) := \frac{e(z) - c}{\lambda - z} \in H(A).$$

Then  $e(A)(\lambda - A)^{-1} = (\lambda - A)^{-1}e(A)$ .

*Proof.* Since  $e(A)(\lambda - A)^{-1} \subset (\lambda - A)^{-1}e(A)$  is always true, we only have to prove the inclusion  $e(A)(\lambda - A)^{-1} \supset (\lambda - A)^{-1}e(A)$ . Take  $x \in X$  such that  $e(A)x \in \mathcal{D}((\lambda - A)^{-1})$ . Then there is  $z \in \mathcal{D}(A)$  with  $e(A)x = (\lambda - A)z$ . Since  $e = (\lambda - z)f + c$ , we have  $(\lambda - A)z = cx + (\lambda - A)f(A)x$ , whence  $cx = (\lambda - A)(z - f(A)x)$ . Now,  $c \neq 0$  by assumption, and hence  $x \in \mathcal{R}(\lambda - A) = \mathcal{D}((\lambda - A)^{-1})$ .  $\square$

**LEMMA 4.3.** *Let  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]_A$ . Given  $0 \neq \lambda \in \overline{S_\omega}$ , there is a regularizer  $e$  for  $f$  satisfying  $e(\lambda) \neq 0$ .*

*Proof.* Let  $g$  be any regularizer for  $f$ , that is,  $g, fg \in \mathcal{E}[S_\omega]$  and  $g(A)$  is injective. Define  $e := g/(\lambda - z)^n$ , where  $n \in \mathbb{N}$  is the order of the zero  $\lambda$  of  $g$ . (This order  $n$  may be zero.) Then  $e(\lambda) \neq 0$  and  $e \in \mathcal{E}[S_\omega]$  by Lemma 2.4. Furthermore,  $g(A) = (\lambda - A)^n e(A)$ , whence  $e(A)$  is injective. Again by Lemma 2.4, also  $ef = fg/(\lambda - z)^n \in \mathcal{E}[S_\omega]$ . Hence  $e$  is a regularizer for  $f$  with  $e(\lambda) \neq 0$ .  $\square$

We now come to the main part.

**PROPOSITION 4.4.** *Let  $A \in \text{Sect}(\omega)$ ,  $f \in \mathcal{M}[S_\omega]_A$  and  $0 \neq \lambda \in \overline{S_\omega}$ . If  $f(\lambda) = 0$  and  $f(A)$  is injective/invertible, then  $\lambda - A$  is injective/invertible.*

*Proof.* Assume first that  $f(A)$  is injective. By virtue of Lemma 4.3, choose a regularizer  $e \in \mathcal{E}[S_\omega]$  for  $f$  with  $c := e(\lambda) \neq 0$ . Then  $(ef) \in \mathcal{E}[S_\omega]$  and  $(ef)(\lambda) = 0$ . This implies that also  $h := ef/(\lambda - z) \in \mathcal{E}[S_\omega]$ , whence

$$h(A)(\lambda - A) \subset (ef)(A) = f(A)e(A).$$

Because  $e(A)$  and  $f(A)$  are both injective, this shows that  $\lambda - A$  must be injective. Also,  $e$  is a regularizer for  $f/(z - \lambda)$ .

By Lemma 2.4,  $g := (e - c)/(\lambda - z) \in \mathcal{E}[S_\omega]$ . Lemma 4.2 implies that  $e(A)(\lambda - A)^{-1} = (\lambda - A)^{-1}e(A)$ . Inverting both sides of this equation gives us

$(\lambda - A)e(A)^{-1} = e(A)^{-1}(\lambda - A)$ . Now, we compute

$$\begin{aligned} f(A) &= \left( (\lambda - z) \frac{f}{\lambda - z} \right) (A) = e(A)^{-1} \left( (\lambda - z) \frac{ef}{\lambda - z} \right) (A) \\ &= e(A)^{-1}(\lambda - A) \left( \frac{ef}{\lambda - z} \right) (A) = (\lambda - A)e(A)^{-1} \left( \frac{ef}{\lambda - z} \right) (A) \\ &= (\lambda - A) \left( \frac{f}{\lambda - z} \right) (A). \end{aligned}$$

If  $f(A)$  is invertible, then  $\lambda - A$  must be surjective, whence we arrive at  $\lambda \in \varrho(A)$ .  $\square$

*Proof of Theorem 4.1.* We can now prove Theorem 4.1. Let  $0 \neq \lambda \in \sigma(A)$  and define  $\mu := f(\lambda)$ . If  $\mu \neq \infty$ , then Proposition 4.4 applied to the function  $\mu - f$  shows that  $\mu - f(A)$  cannot be invertible. Hence  $\mu \in \sigma(f(A))$ .

Assume now that  $\mu = \infty \notin \tilde{\sigma}(f(A))$ . Then  $f(A) \in \mathcal{L}(X)$  and there is  $\lambda_0 \in \mathbb{C}$  such that  $f(A) - \lambda_0$  is invertible. Hence  $g := 1/(f - \lambda_0) \in \mathcal{M}[S_\omega]_A$  with  $g(A)$  being invertible and  $g(\lambda) = 0$ . Another application of Proposition 4.4 yields  $\lambda \in \varrho(A)$ , which contradicts the assumption made on  $\lambda$ .  $\square$

Since  $\tilde{\sigma}(f(A))$  is a compact subset of the Riemann sphere, we immediately obtain the following corollary.

COROLLARY 4.5. *Let  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]_A$ . Then*

$$\overline{f(\sigma(A) \setminus \{0\})}^{\mathbb{C}^\infty} \subset \tilde{\sigma}(f(A)). \quad (6)$$

## 5. Almost logarithmic limits

As has already been mentioned, the difficulty in the ‘full’ spectral inclusion theorem is that  $f(0)$  and  $f(\infty)$  in general are meaningless. Of course one could think of restriction to functions  $f$  which have limits at 0 and  $\infty$ , but even this does not seem to work. However, we can prove a positive result in the case when the limit behaviour of  $f$  at these points is ‘almost logarithmic’ in the sense of the following definition. (Compare also with Remark 5.4.)

DEFINITION 5.1. Let  $0 < \varphi \leq \pi$ ,  $f \in \mathcal{M}(S_\varphi)$  and  $c \in \mathbb{C}$ . We say that  $f(z) \rightarrow c$  almost logarithmically (a.l.) as  $z \rightarrow 0$  if there is  $\alpha > 0$  such that  $f(z) - c = O(|\log |z||^{-(1+\alpha)})$  as  $z \rightarrow 0$ . We say that  $f(z) \rightarrow \infty$  almost logarithmically as  $z \rightarrow 0$  if  $(1/f)(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow 0$ .

Similarly, we say that  $f(z) \rightarrow d \in \mathbb{C}_\infty$  almost logarithmically as  $z \rightarrow \infty$  if  $f(z^{-1})(0) \rightarrow d$  almost logarithmically as  $z \rightarrow 0$ .

Note that a function  $f$  having limit  $\infty$  almost logarithmically at 0 (at  $\infty$ ) need not be polynomially bounded at 0 (at  $\infty$ ). Obviously, if  $f \in H^\infty(S_\varphi)$  and  $f(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow 0$  and as  $z \rightarrow \infty$ , then  $f \in \mathcal{E}_0(S_\varphi)$ .

With this terminology we can now state the missing information in the spectral inclusion Theorem 4.1.

**THEOREM 5.2.** *Let  $A \in \text{Sect}(\omega)$  and  $\lambda_0 \in \{0, \infty\} \cap \tilde{\sigma}(A)$ . Let  $f \in \mathcal{M}[S_\omega]_A$  and  $\mu \in \mathbb{C}_\infty$  with the property that  $f(z) \rightarrow \mu$  almost logarithmically as  $z \rightarrow \lambda_0$ . Then  $\mu \in \tilde{\sigma}(f(A))$ .*

For the proof we need another lemma.

**LEMMA 5.3.** *Let  $A \in \text{Sect}(\omega)$  with  $\tilde{\sigma}(A) = \{\infty\}$ . If  $f \in \mathcal{M}[S_\omega]_A$  and  $f(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow \infty$ , then  $f(A) \in \mathcal{L}(X)$ .*

*Proof.* Let  $e$  be a regularizer for  $f$ . Since  $A$  is invertible, one can assume that  $e, ef \in \mathcal{E}_0[S_\omega]$ . In the Cauchy integral (5) defining  $e(A), (ef)(A)$ , one can use a contour  $\Gamma$  which lies close to  $\infty$ . More precisely, let  $R > 0$  be large enough for  $f$  to be holomorphic on  $S_\varphi \setminus \overline{B_R(0)}$ . Then  $\Gamma$  can be chosen to be the positively oriented boundary of  $S_{\omega'} \setminus \overline{B_{R'}(0)}$  for some  $R' > R$  and  $\omega < \omega' < \varphi$ . The resolvent identity yields  $(ef)(A) = e(A)T$  for  $T := (1/2\pi i) \int_\Gamma f(z)R(z, A) dz$ . This implies that  $f(A) = e(A)^{-1}(ef)(A) = T \in \mathcal{L}(X)$ .  $\square$

*Proof of Theorem 5.2.* We can now prove Theorem 5.2. For  $\mu \in \mathbb{C}$ , we can consider  $f - \mu$  instead of  $f$  and in this way reduce the problem to the cases  $\mu = 0, \infty$ . Let us start with the case  $\mu = 0, \lambda_0 = \infty$ . We assume that  $f(A)$  is invertible and  $f(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow \lambda_0 = \infty$ . If

$$\infty \in \overline{\sigma(A) \setminus \{0\}}^{\mathbb{C}_\infty},$$

then the assertion follows from (6). Otherwise, there is  $R > 0$  such that  $\sigma(A) \subset \overline{B_R(0)}$ . Define  $Q := (1/2\pi i) \int_\Gamma R(z, A) dz$ , where  $\Gamma = \partial B_\delta(0)$  for (any)  $R < \delta$ . It is easy to see that  $Q$  is a bounded projection. Let  $P := I - Q$  be the complementary projection and let  $Y := \mathcal{R}(P)$  be its range space. Since  $P$  commutes with  $A$ , one has  $Ay \in Y$  for every  $y \in Y \cap \mathcal{D}(A)$ . It is easy to see that  $B := A|_Y$  is a sectorial operator of angle  $\omega$  on  $Y$  with  $\varrho(A) \subset \varrho(B)$  and  $R(\lambda, B) = R(\lambda, A)|_Y$  for all  $\lambda \in \varrho(A)$ . Moreover,  $\tilde{\sigma}(B) \subset \{\infty\}$ . In fact, for each  $\mu \in \sigma(A)$ ,  $R(\mu, B) = (1/2\pi i) \int_\Gamma (1/(\mu - z))R(z, A) dz$  on  $Y$ , as is easily verified. Thus there are only two alternatives:  $Y = 0$  or  $\tilde{\sigma}(B) = \{\infty\}$ .

Assume the latter. Clearly we have  $e(B) = e(A)|_Y$  for each  $e \in \mathcal{E}[S_\omega]$ . From this it follows easily that  $\mathcal{D}(f(B)) = \mathcal{D}(f(A)) \cap Y$  and  $f(B)x = f(A)x \in Y$  for all  $x \in \mathcal{D}(f(A)) \cap Y$ . Since  $f(A)$  is invertible by assumption,  $f(B)$  is also with  $f(B)^{-1} = f(A)^{-1}|_Y$ . Since  $f$  decays fast enough at  $\infty$ , we can apply Lemma 5.3 to conclude that  $f(B) \in \mathcal{L}(X)$ . Moreover, with  $n \in \mathbb{N}$  large enough, the function  $g := \log(z)f^n$  still has nice decay at  $\infty$ . Another application of Lemma 5.3 yields  $g(B) = \log(B)f(B)^n \in \mathcal{L}(X)$ . However,  $f(B)$  is invertible, whence  $\log(B)$  must be bounded. This implies that  $B$  is bounded (see [14] and Section 7). However, this contradicts the fact that  $\tilde{\sigma}(B) = \{\infty\}$ . Hence we arrive at  $Y = 0$  and thus  $A \in \mathcal{L}(X)$ , that is,  $\infty \notin \tilde{\sigma}(A)$ .

Consider now the case  $\mu = 0, \lambda_0 = 0$ . Assume that  $f(A)$  is invertible and  $f(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow 0$ . By Theorem 3.3, we can assume without restriction that  $A$  is injective. Let  $B := A^{-1}$  and  $g(z) := f(z^{-1})$ . Now we can apply the composition rule [16, Corollary 7.5] to obtain  $g \in \mathcal{M}[S_\omega]_B$  and  $g(B) = f(A)$ . The latter is invertible and  $g(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow \infty$ . By what we have shown above,  $B$  is bounded, that is,  $A$  is invertible.

Finally, we deal with the case  $\mu = \infty$ . Assume that  $\infty = \mu \notin \tilde{\sigma}(f(A))$ . By definition,  $f(A)$  must be bounded. Hence we can find  $\lambda \in \mathbb{C}$  such that  $\lambda - f(A)$  is invertible. By Theorem 2.2,  $g := 1/(\lambda - f) \in \mathcal{M}[S_\omega]_A$ ,  $g(A)$  is invertible and  $g(z) \rightarrow 0$  almost logarithmically as  $z \rightarrow \lambda_0$ . By the results achieved so far, we can conclude that  $\lambda_0 \notin \tilde{\sigma}(A)$ . □

REMARK 5.4. Theorem 5.2 is certainly not the best one can achieve. In fact, one could weaken the hypothesis on the decay of  $f$  to, for example,

$$f(z) - c = O\left(\frac{1}{|\log |z|| |\log \log |z||^{1+\alpha}}\right),$$

or to even weaker decay. We do not know if just integrability with respect to  $|dz|/|z|$  is sufficient.

### 6. The spectral mapping theorem

In this section we want to prove our main result, namely the identity

$$f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A))$$

for functions  $f \in \mathcal{M}[S_\omega]_A$  having limits almost logarithmically at  $\{0, \infty\} \cap \tilde{\sigma}(A)$ . Note that Theorem 4.1 and Theorem 5.2 together show the inclusion  $f(\tilde{\sigma}(A)) \subset \tilde{\sigma}(f(A))$ . The proof of the other inclusion needs some preparation. In fact, we need extensions of our functional calculus from Section 2 in case the operator  $A$  is bounded and/or invertible.

Consider an operator  $A \in \text{Sect}(\omega)$  which is invertible. Fix  $\omega < \varphi < \pi$ . There is  $\varepsilon > 0$  such that the ball  $\overline{B_\varepsilon(0)}$  is entirely contained in  $\varrho(A)$ . Consider the algebra

$$\mathcal{E}(S_\varphi, \varepsilon) := \left\{ f \in H^\infty(S_\varphi \setminus \overline{B_\varepsilon(0)}) \mid \lim_{z \rightarrow \infty} f(z) = 0 \text{ and } \int_{\partial S_{\omega', |z| > 2\varepsilon}} |f(z)| \frac{|dz|}{|z|} < \infty \text{ for all } 0 \leq \omega' < \varphi \right\}.$$

One can extend our elementary functional calculus  $\Phi$  constructed in Section 2 from the algebra  $\mathcal{E}(S_\varphi)$  to the algebra  $\mathcal{E}(S_\varphi, \varepsilon)$  simply by using in (5) a contour  $\Gamma$  which avoids 0 at a distance of  $\varepsilon$ . This yields a new abstract functional calculus  $(\mathcal{E}(S_\varphi, \varepsilon), \mathcal{M}(S_\varphi), \Phi)$  (see [16]) which is an extension of the former. Since the new algebra  $\mathcal{E}(S_\varphi, \varepsilon)$  is larger than  $\mathcal{E}(S_\varphi)$ , more functions  $f \in \mathcal{M}(S_\varphi)$  become regularizable. If  $f$  is regularizable by  $\mathcal{E}(S_\varphi, \varepsilon)$ , then we say that ‘ $f(A)$  is defined by the natural functional calculus for invertible sectorial operators’. Clearly, Theorem 2.2 remains valid when it is adapted appropriately.

A similar extension is possible when  $A \in \text{Sect}(\omega)$  is bounded. Again take  $\omega < \varphi < \pi$  and let  $r > r(A)$ , where  $r(A)$  denotes the spectral radius of  $A$ . Then one considers the algebra

$$\left\{ f \in H^\infty(S_\varphi \cap B_r(0)) \mid \lim_{z \rightarrow 0} f(z) = 0, \int_{\partial S_{\omega', |z| < r/2}} |f(z)| \frac{|dz|}{|z|} < \infty \text{ for all } 0 \leq \omega' < \varphi \right\} \quad (7)$$

and adds the space  $\mathbb{C}(1/(1+z))$ . The elementary calculus on this algebra is constructed along the lines of Section 2: one uses a Cauchy integral for those

functions  $e$  which have limit 0 at 0 and then extends this calculus by adding multiples of  $(1 + A)^{-1}$ . Again one ends up with a nice functional calculus for which Theorem 2.2 holds *mutatis mutandis*. If a function  $f$  is regularizable with respect to the algebra (7), then we say that ‘ $f(A)$  is defined by the natural functional calculus for bounded sectorial operators’.

Similar remarks apply in the case when  $A \in \text{Sect}(\omega)$  is bounded and invertible.

DEFINITION 6.1. Let  $f \in \mathcal{M}(S_\varphi)$  with  $0 < \varphi < \pi$ . We say that  $f$  is regular at 0 if  $\lim_{z \rightarrow 0} f(z) =: c \in \mathbb{C}$  exists and

$$\int_{\partial S_{\omega'}, |z| < \varepsilon} |f(z) - c| \frac{|dz|}{|z|} < \infty$$

for small  $\varepsilon > 0$ . We say that  $f$  is regular at  $\infty$  if  $f(z^{-1})$  is regular at 0. We say that  $f$  is quasi-regular at 0 ( $\infty$ ) if  $f$  or  $1/f$  is regular at 0 ( $\infty$ ). Finally, if  $M \subset \{0, \infty\}$ , then we say that  $f$  is (quasi-)regular at  $M$  if  $f$  is (quasi-)regular at each point of  $M$ .

Note that if  $f$  is regular at 0 with  $f(0) \neq 0$ , then  $1/f$  is also regular at 0. If  $f$  is quasi-regular at 0, then  $\mu - f$  and  $1/f$  are also quasi-regular at 0 (for each  $\mu \in \mathbb{C}$ ). A function  $f$  which is quasi-regular at 0 has a well defined limit  $\lim_{z \rightarrow 0} f(z) \in \mathbb{C}_\infty$ .

LEMMA 6.2. Let  $A \in \text{Sect}(\omega)$  and  $f \in \mathcal{M}[S_\omega]$ . Assume that  $f$  is regular at  $\tilde{\sigma}(A) \cap \{0, \infty\}$  and that all poles of  $f$  are contained in  $\mathbb{C} \setminus P\sigma(A)$ . Then the following assertions are true.

- (1) In the case  $\{0, \infty\} \subset \tilde{\sigma}(A)$ ,  $f(A)$  is defined by the natural functional calculus for sectorial operators.
- (2) In the case when  $A$  is invertible but not bounded,  $f(A)$  is defined by the natural functional calculus for invertible sectorial operators.
- (3) In the case when  $A$  is bounded but not invertible,  $f(A)$  is defined by the natural functional calculus for bounded sectorial operators.
- (4) In the case when  $A$  is bounded and invertible,  $f(A)$  is defined by the natural functional calculus for bounded and invertible sectorial operators.

Moreover, if the poles of  $f$  are contained even in  $\varrho(A)$ , then  $f(A) \in \mathcal{L}(X)$ .

Proof. Assume that  $f \in \mathcal{M}(S_\varphi)$  is as required. One can assume without loss of generality that the limits at the points  $\tilde{\sigma}(A) \cap \{0, \infty\}$  are 0 (add a function of the form  $c/(1 + z) + d$ ). By making  $\varphi$  smaller, we can assume that the only possible accumulation points of the poles of  $f$  are 0 and  $\infty$ .

Take  $\lambda_0 \in S_\varphi$  to be a pole of  $f$ . Then, for suitably large  $n_0 \in \mathbb{N}$ , the function

$$f_1 := \frac{(\lambda_0 - z)^{n_0}}{(1 + z)^{n_0}} f(z)$$

is also regular at the points  $\tilde{\sigma}(A) \cap \{0, \infty\}$ , but it has one pole less than  $f$ . Moreover, letting  $r_0 := (\lambda_0 - z)^{n_0}/(1 + z)^{n_0}$ , we see that  $r_0(A)$  is bounded and injective.

Now assume that we are in the situation of (1). Then  $f$  can have only finitely many poles. By induction, we find a bounded rational function  $r$  such that  $r(A)$  is bounded and injective and  $rf \in \mathcal{E}(S_\varphi)$ . Hence  $r$  regularizes  $f$  to  $\mathcal{E}$  and  $f(A)$  is defined.

Similar reasoning applies in the other cases. Namely, only finitely many poles of  $f$  lie in the ‘relevant’ part of the domain of  $f$ . For example, if  $A$  is invertible, the poles of  $f$  may accumulate at 0, but for the natural functional calculus for  $A$ , the behaviour of  $f$  near 0 is irrelevant.

Now assume that the poles of  $f$  lie inside  $\varrho(A)$ . Then the operator  $r_0(A)$  (see above) is not only injective but even invertible. Thus  $r(A)$  is also invertible, whence  $f(A) = r(A)^{-1}(rf)(A) \in \mathcal{L}(X)$ .  $\square$

PROPOSITION 6.3 (second inclusion theorem). *Let  $A \in \text{Sect}(\omega)$ , and take  $f \in \mathcal{M}[S_\omega]_A$  to be quasi-regular at  $\{0, \infty\} \cap \tilde{\sigma}(A)$ . Then*

$$\tilde{\sigma}(f(A)) \subset f(\tilde{\sigma}(A)).$$

*Proof.* Note first that by assumption,  $f(\lambda)$  is defined for each  $\lambda \in \tilde{\sigma}(A)$ . Take  $\mu \in \mathbb{C}$  such that  $\mu \notin f(\tilde{\sigma}(A))$ . Then the function  $(\mu - f)^{-1} \in \mathcal{M}[S_\omega]$  is regular in  $\{0, \infty\} \cap \tilde{\sigma}(A)$ , and all of its poles (namely the points  $\lambda \in \overline{S_\omega} \setminus \{0\}$  where  $f(\lambda) = \mu$ ) are contained in  $\varrho(A)$ . Hence one can apply Lemma 6.2 to conclude that  $(\mu - f)^{-1}(A)$  is defined and bounded. However, this implies that  $\mu - f(A)$  is invertible, whence  $\mu \notin \tilde{\sigma}(f(A))$ .

Now assume that  $\mu = \infty \notin f(\tilde{\sigma}(A))$ . This implies that  $f$  itself is regular at  $\{0, \infty\} \cap \tilde{\sigma}(A)$  and the poles of  $f$  are contained in  $\varrho(A)$ . Another application of Lemma 6.2 yields that  $f(A)$  is a bounded operator, whence  $\infty \notin \tilde{\sigma}(f(A))$ .  $\square$

THEOREM 6.4 (spectral mapping theorem). *Let  $A \in \text{Sect}(\omega)$  on the Banach space  $X$ , and take  $f \in \mathcal{M}[S_\omega]_A$  with limits (almost logarithmically) at  $\{0, \infty\} \cap \tilde{\sigma}(A)$ . Then*

$$\tilde{\sigma}(f(A)) = f(\tilde{\sigma}(A)).$$

Unfortunately, we cannot prove the analogous statement for the point spectrum. However, we have the following.

PROPOSITION 6.5. *Let  $A \in \text{Sect}(\omega)$  on the Banach space  $X$ , and take  $f \in \mathcal{M}[S_\omega]_A$  to be quasi-regular at  $M := \{0, \infty\} \cap \tilde{\sigma}(A)$ . Then*

$$P\sigma(f(A)) \subset f(P\sigma(A)) \cup f(M).$$

*Proof.* Take  $\mu \in \mathbb{C} \setminus f(P\sigma(A))$  and consider the function  $g := 1/(u - g)$ . Clearly,  $g$  is quasi-regular at  $M$ . If we assume in addition that  $\mu \notin f(M)$ , then we can conclude that  $g$  is even regular at  $M$ , whence we can apply Lemma 6.2. This yields that  $g(A)$  is defined; hence  $\mu - f(A)$  is injective, that is,  $\mu \notin P\sigma(f(A))$ .  $\square$

COROLLARY 6.6 (spectral mapping theorem for the point spectrum). *Let  $A \in \text{Sect}(\omega)$  on the Banach space  $X$ , and take  $f \in \mathcal{M}[S_\omega]_A$  to be quasi-regular at  $M := \{0, \infty\} \cap \tilde{\sigma}(A)$ . Then*

$$f(P\sigma(A)) \subset P\sigma(f(A)) \subset f(P\sigma(A)) \cup f(M).$$

7. Fractional powers, logarithms and holomorphic semigroups

We want to apply the results developed above to fractional powers and the logarithm of a sectorial operator  $A \in \text{Sect}(\omega)$ . For  $\text{Re } \alpha > 0$ , the function  $z^\alpha$  is regularized by  $(1+z)^{-n}$  where  $n > \text{Re } \alpha$ . Hence  $A^\alpha := (z^\alpha)(A)$  is defined. Since  $z^\alpha$  clearly has even ‘polynomial’ limits at 0 and  $\infty$ , Theorem 6.4 and Corollary 6.6 apply. From [18] or [15, Chapter 3], we know that  $A$  is injective if and only if  $A^\alpha$  is injective. Thus we arrive at the following corollary. (Note that  $z^\alpha = \infty$  if and only if  $z = \infty$ .)

COROLLARY 7.1. *Let  $A \in \text{Sect}(\omega)$  on a Banach space  $X$ . Then*

$$\sigma(A^\alpha) = \sigma(A)^\alpha \quad \text{and} \quad P\sigma(A^\alpha) = P\sigma(A)^\alpha$$

for every  $\text{Re } \alpha > 0$ .

Now consider a sectorial operator  $A \in \text{Sect}(\omega)$  with  $\omega < \pi/2$ . Then for  $t > 0$ , the function  $e^{-tA}$  is contained in  $\mathcal{E}[S_\omega]$ , whence  $e^{-tA} := (e^{-tz})(A)$  is defined and yields a bounded operator. The family  $(e^{-tA})_{t>0}$  is a holomorphic semigroup bounded at 0, and its generator (defined via the Laplace transform) is  $-A$ ; see [10] for the case when  $A$  is densely defined and [3] or [15, Chapter 3] for the general case. Theorem 6.4 yields the well known result

$$\sigma(e^{-tA}) = \tilde{\sigma}(e^{-tA}) = e^{-t\tilde{\sigma}(A)} = e^{-t\sigma(A)} \cup \{0\}$$

(in the case when  $A$  is not bounded); cf. [10, Corollary IV.3.12].

Let us turn to logarithms. Consider an injective operator  $A \in \text{Sect}(\omega)$ . Then the function  $\log z$  is regularized by  $z/(1+z)^2$ , whence we can define  $\log A := (\log z)(A)$ . The function  $\log z$  has limit  $\infty$  at points 0 and  $\infty$ . However, these limits are not almost logarithmic, so we cannot invoke Theorem 6.4. At least we can apply the inclusion theorem and obtain

$$\log(\sigma(A) \setminus \{0\}) \subset \sigma(\log A). \tag{8}$$

We give a sketch of the proof for the reverse inclusion. By Nollau’s theorem, the operator  $B := \log A$  is a so-called *strip-type* operator, that is, its spectrum is contained in the horizontal strip  $\overline{H_\omega} = \{|\text{Im } z| \leq \omega\}$  and is uniformly bounded outside every larger strip; see [14, Proposition 3.2] or [15]. One can construct a functional calculus for  $B$  in a similar way as for sectorial operators. Call  $\mathcal{E}_0(H_\varphi)$  the functions  $f$  holomorphic on the horizontal strip  $H_\varphi$  with a good decay as  $|\text{Re } z| \rightarrow \infty$ . For such  $f$ , one defines  $f(B) = \Phi(f)$  via a Cauchy integral similar to (5). Then one extends this definition to the algebra  $\mathcal{E}(H_\varphi) := \mathcal{E}_0(H_\varphi) \oplus \mathbb{C}(1/(\pi - z))$ . This yields a proper abstract functional calculus  $(\mathcal{E}(H_\varphi), \mathcal{M}(H_\varphi), \Phi)$  in the sense of [16]. If  $f \in \mathcal{M}(H_\varphi)$  is regularizable by  $\mathcal{E}(H_\varphi)$ , then we say that ‘ $f(B)$  is defined by the natural functional calculus for strip-type operators’.

Next, one has to note that the considerations which lead to the spectral inclusion theorem Theorem 4.1, namely Lemmas 4.2 and 4.3 as well as Proposition 4.4, turn over to the strip-type case almost verbally. Hence one has the following theorem.

THEOREM 7.2 (spectral inclusion theorem for strip-type operators). *Let  $B \in \text{Strip}(\omega)$  be a strip-type operator on a Banach space  $X$ . If  $f \in \mathcal{M}[H_\omega]$  and*

$f(A)$  is defined by the natural functional calculus for strip-type operators, then  $f(\sigma(B)) \subset \tilde{\sigma}(f(B))$ .

Now, as  $\varphi < \pi$ , for all  $\lambda, \mu \notin \overline{S_\varphi}$  the function  $h(z) := e^z / (\lambda - e^z)(\mu - e^z)$  is a member of  $\mathcal{E}(H_\varphi)$ . By the composition rule [14, Corollary 4.1], one has  $h(B) = h(\log(A)) = AR(\lambda, A)R(\mu, A)$ , which is injective. (In fact,  $h(B)$  is injective for every strip-type operator  $B$  if the strip height is less than  $\pi$ ; see [15, Lemma 4.5].) Hence the function  $e^z$  is regularizable by  $h^2 \in \mathcal{E}(H_\varphi)$ . Theorem 7.2 shows that  $e^{\sigma(B)} \subset \tilde{\sigma}(e^B)$ . As  $B = \log A$ ,  $A = e^B$ , and since  $e^z$  has no poles and no zeros, we have in fact  $e^{\sigma(\log A)} \subset \sigma(A) \setminus \{0\}$ . Moreover,  $\log B$  is bounded if and only if  $A$  is bounded and invertible. Thus we have proved the following.

**THEOREM 7.3.** *Let  $A \in \text{Sect}(\omega)$  be injective. Then  $\sigma(\log A) = \log(\sigma(A) \setminus \{0\})$  and  $\tilde{\sigma}(\log A) = \log(\tilde{\sigma}(A))$ .*

Also an adaptation of Theorem 3.3 to the strip-type case yields the identity  $P\sigma(\log A) = \log(P\sigma(A))$ .

## 8. Operators with a bounded $H^\infty$ -calculus

In this section we examine to what extent the theorems of the previous sections can be improved if one supposes the operator to have a bounded  $H^\infty$ -calculus. We will see that a slight improvement is possible (Theorem 8.5), but that substantial improvements are not to be expected in general.

**DEFINITION 8.1.** Let  $A \in \text{Sect}(\omega)$  be injective and let  $\omega < \varphi < \pi$ . We say that the natural  $H^\infty(S_\varphi)$ -calculus is bounded if  $f(A) \in \mathcal{L}(X)$  for all  $f \in H^\infty(S_\varphi)$ . Similar terminology is used for other types of operator and their corresponding functional calculi.

We define

$$\omega_{H^\infty}(A) := \inf\{\omega < \varphi \mid \text{the natural } H^\infty(S_\varphi)\text{-calculus is bounded}\}.$$

Hence we write  $\omega_{H^\infty}(A) < \pi$  for the fact that there is  $\omega < \varphi < \pi$  such that the natural  $H^\infty(S_\varphi)$ -calculus for  $A$  is bounded.

Note that by the closed graph theorem, if the natural  $H^\infty(S_\varphi)$ -calculus for  $A$  is bounded, then the mapping  $(f \mapsto f(A)) : H^\infty(S_\varphi) \rightarrow \mathcal{L}(X)$  is bounded. The next result is almost trivial.

**PROPOSITION 8.2.** *Let  $A \in \text{Sect}(\omega)$  be injective and assume that for some  $\varphi > \omega$ , the natural  $H^\infty(S_\varphi)$ -calculus is bounded. Then*

$$\sigma(f(A)) \subset \overline{f(S_\varphi)}$$

for any  $f \in \mathcal{M}(S_\varphi)_A$ .

*Proof.* Let  $\lambda \notin \overline{f(S_\varphi)}$ ; then  $g(z) := 1/(\lambda - f) \in H^\infty(S_\varphi)$ . Hence, by assumption,  $g(A)$  is bounded. Theorem 2.2(f) yields  $\lambda \in \varrho(f(A))$ .  $\square$

To proceed further, we need an auxiliary result.

PROPOSITION 8.3. Let  $A \in \text{Sect}(\omega)$  with  $\omega_{H^\infty}(A) < \pi$ .

(a) If  $A \in \mathcal{L}(X)$ , then, for each  $\omega_{H^\infty}(A) < \varphi \leq \pi$  and each  $r > r(A)$ , the natural  $H^\infty(S_\varphi \cap B_r(0))$ -calculus is bounded.

(b) If  $0 \in \varrho(A)$ , then, for each  $\omega_{H^\infty}(A) < \varphi \leq \pi$  and each  $r > r(A^{-1})$ , the natural  $H^\infty(S_\varphi \setminus \overline{B_{1/r}(0)})$ -calculus is bounded.

*Proof.* Assume that  $A$  is bounded and choose  $\varphi, r$  as in (a). Let  $\omega_{H^\infty}(A) < \omega' < \varphi$  and  $r(A) < r' < r$ . Let  $\Gamma_1 := -[0, r']e^{i\omega'} \oplus [0, r']e^{-i\omega'}$  and  $\Gamma_2 := r'e^{i[-\omega', \omega']}$ . For  $f \in H^\infty(S_\varphi \cap B_r(0))$ , define

$$f_1(z) := \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} dw \quad \text{and} \quad f_2(z) := \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z} dw.$$

Then  $f(z) = f_1(z) + f_2(z)$  for  $z \in S_{\omega'} \cap B_{r'}(0)$ . Now,  $f_2$  is holomorphic at 0, whence  $f_2(A) \in \mathcal{L}(X)$ . In particular,  $f_2$  is bounded at 0, whence  $f_1$  is also bounded at 0. This shows that  $f_1 \in H^\infty(S_{\omega'})$ . By hypothesis,  $f_1(A) \in \mathcal{L}(X)$ . This yields  $f(A) \in \mathcal{L}(X)$ .

To prove (b), apply (a) to  $B := A^{-1}$  and use the composition rule. □

COROLLARY 8.4. Let  $A \in \text{Sect}(\omega)$  be injective with  $\omega_{H^\infty}(A) < \pi$ . Take  $\omega_{H^\infty}(A) < \varphi < \pi$  and  $f \in \mathcal{M}(S_\varphi)$ . Assume that  $f$  has finite limits at  $\{0, \infty\} \cap \tilde{\sigma}(A)$  and  $\{\lambda \in \overline{S_\omega} \mid f(\lambda) = \infty\} \subset \varrho(A)$ . Then Lemma 6.2(1)–(4) hold and  $f(A) \in \mathcal{L}(X)$ .

*Proof.* As in the proof of Lemma 6.2, one first removes the poles of  $f$  in the ‘relevant’ part of the sector by multiplication with certain rational functions. Then one applies Proposition 8.3. □

THEOREM 8.5. Let  $A \in \text{Sect}(\omega)$  be injective with  $\omega_{H^\infty}(A) < \pi$  and take  $f \in \mathcal{M}(S_\varphi)_A$ . Assume that  $f$  has (not necessarily finite) limits at  $\{0, \infty\} \cap \tilde{\sigma}(A)$ . Then the spectral mapping theorem

$$f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A))$$

holds.

*Proof.* We first prove the inclusion  $\subset$ . By virtue of Theorem 4.1, we only have to deal with the points 0 and  $\infty$ . Assume that 0 is a spectral point of  $A$ . We have to prove that  $f(0) \in \tilde{\sigma}(f(A))$ . By Corollary 4.5, we can assume that 0 is isolated, whence, by the same arguments as in the proof of Lemma 5.3, we can reduce the situation to  $A \in \mathcal{L}(X) \cap \text{Sect}(\omega)$ , injective with  $\sigma(A) = \{0\}$ . Assume first that  $\mu := f(0) \in \mathbb{C}$ . Define  $f_n(z) := f(z + 1/n)$ . Making  $\varphi$  a bit smaller if necessary and  $r > 0$  sufficiently small, we can assume that  $f_n \rightarrow f$  uniformly on  $S_\varphi \cap B_r(0)$ . Moreover, by Proposition 8.3, the natural  $H^\infty(S_\varphi \cap B_r(0))$  for  $A$  is bounded. This yields  $f_n(0) - f_n(A) \rightarrow \mu - f(A)$  in norm. However,  $f_n(0) - f_n(A) = f(1/n) - f(A + 1/n)$  by the composition rule, and this operator cannot be invertible, by the spectral inclusion theorem Theorem 4.1. Hence  $\mu - f(A)$  cannot be invertible, whence  $\mu \in \sigma(f(A))$ .

In the case  $\mu = \infty$ , apply the foregoing to  $f^{-1}$ .

The spectral point  $\infty$  is dealt with similarly. A second possibility is to apply the results already obtained to the function  $f(z^{-1})$  and the operator  $B := A^{-1}$  in view of the composition rule.

Let us turn to the reverse inclusion  $\supset$ . Take  $\mu \in \mathbb{C} \setminus f(\tilde{\sigma}(A))$ . The function  $g := 1/(\mu - f)$  satisfies the hypothesis of Corollary 8.4. Hence  $(1/(\mu - f))(A)$  is bounded, whence  $\mu - f(A)$  is invertible.

If  $\mu = \infty \notin f(\tilde{\sigma}(A))$ , then  $f$  itself satisfies the hypothesis of Corollary 8.4, whence  $f(A)$  is bounded. This is by definition the same as  $\infty \notin \tilde{\sigma}(f(A))$ .  $\square$

## 9. Final remarks

### 9.1. Other types of holomorphic functional calculi

Most proofs which we have presented in this text are generic. That is to say that, if we have a construction of an elementary functional calculus analogous to that in Section 2 with canonical extension to meromorphic functions as described in [16], and if we have the analogue of Lemma 2.4, then we obtain analogues of Theorem 3.3, Theorem 4.1, Corollary 4.5, Proposition 6.3 and Corollary 6.6. The appropriate definition of ‘regularity’ depends on the integrability condition which is part of the definition of the elementary functional calculus. For example, for strip-type operators, this means that

$$\int_{\text{Im } z = \omega', |\text{Re } z| \geq R} |f(z)| |dz| < \infty \quad \text{for } R > 0 \text{ large and } \omega' < \varphi.$$

More or less the only result which is *not* generic is Theorem 5.2, which involves the notion of almost logarithmical limit behaviour. In the proof of Lemma 5.3, we used special knowledge about sectorial operators and their logarithms. However, if we impose a stronger condition on the limit behaviour, the result and the proof become generic. This condition is a *regular polynomial limit*, by which we mean polynomial decay to a limit of such an order that the regularity condition is met. (In the proof of Lemma 5.3, the only fact that is actually needed is that  $zf(z)^n$  is still regular at  $\infty$  for some power of  $n$ .) For a strip-type operator, for example, the decay at  $\infty$  to a limit  $f(\infty)$  has to be of order  $|z|^{-(1+\varepsilon)}$  for some  $\varepsilon > 0$ . We then obtain the following theorem.

**THEOREM 9.1** (spectral mapping theorem for strip-type operators). *Let  $B \in \text{Strip}(\omega)$  and  $f \in \mathcal{M}[H_\omega]_B$ . Assume that  $f$  has a (regular polynomial) limit at  $\infty$ . Then*

$$f(\tilde{\sigma}(B)) = \tilde{\sigma}(f(B)).$$

### 9.2. An example

Let us consider an operator  $A$  with spectrum within  $\overline{S_\omega}$ ,  $0 \in \varrho(A)$  and  $|\lambda|^\gamma \|R(\lambda, A)\|$  bounded for  $\lambda \notin S_\varphi$  ( $\varphi > \omega$ ) and a fixed  $0 < \gamma < 1$ . For the construction of an elementary functional calculus, we use contours  $\Gamma$  which avoid 0 and eventually bound a sector. (Such operators and the corresponding functional calculus were considered in [20] and before in [7].) The regularity condition amounts to

$$\int_{\partial S_\omega} |f(z)| \frac{|dz|}{|z|^\gamma} < \infty.$$

A ‘regular polynomial’ decay at  $\infty$  would be  $O(|z|^\alpha)$  for  $\alpha < \gamma - 1$ . Correspondingly, one would expect that a spectral mapping theorem failed for the functions  $f(z) = z^\alpha$  with  $0 < \alpha < 1 - \gamma$ . By [7, Example 3.16], this is indeed the case. On the other hand, our (generic) approach leads to a spectral mapping theorem for the fractional powers  $A^\alpha$  with  $1 - \gamma < \alpha$ .

### 9.3. Operators with a bounded $H^\infty$ -calculus

In Section 8, we saw that in the presence of a bounded  $H^\infty$  calculus, the spectral mapping theorem is true for functions  $f$  with definite limits at 0 and  $\infty$ . One can ask if Proposition 8.2 can be improved. That this seems improbable is shown by the following example.

[10, IV.3.4] gives the construction of a certain operator  $B$  on a Hilbert space  $H$ . This operator  $B$  has compact resolvent and it generates a  $C_0$ -group  $T$  with group type equal to 1. Moreover,  $0 = s(B) < \omega_0(T) = 1$ , where  $s(B) := \sup\{\lambda \in \mathbb{C} \mid \lambda \in \sigma(B)\}$  is the spectral bound and  $\omega_0(T)$  is the growth bound of the semigroup  $(T(t))_{t \geq 0}$ . Now,  $B$  has a bounded  $H^\infty$ -calculus on strips  $\{-1 - \varepsilon < \operatorname{Re} z < 1 + \varepsilon\}$  for each  $\varepsilon > 0$ . This follows from [13]. Define  $A := e^{-iB}$ . Then  $A$  is an injective sectorial operator of spectral angle 1, with  $T$  being its group of imaginary powers. Of course,  $A$  has a bounded  $H^\infty(S_{1+\varepsilon})$ -calculus and  $\sigma(A) \subset \overline{S_1} \cap \{\operatorname{Im} z \geq 0\}$  by Theorem 7.3. For  $f(z) = z^i = e^{i \log z}$ , we find that

$$\overline{f(\sigma(A) \setminus \{0\})} \subset \{|z| \leq 1\}$$

but  $\sigma(A^i) \cap \{|z| = e\} \neq \emptyset$ , since  $A^i = T(1)$  and  $r(T(1)) = e^{\omega_0(T)} = e$ .

### 9.4. Spectral mapping theorems for non-holomorphic functional calculi

There are several functional calculi known in the literature which are not based on the Cauchy integral formula but on real integrals. We just mention the Hirsch calculus [18] and the Phillips calculus [17]. In disguised form, the latter also appears in the Mellin transform functional calculus for generators of  $C_0$ -groups; see [18, 21]. For these calculi, certain (sometimes weak) spectral mapping theorems are known, the most recent of which was given by Fašangová in [11].

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