Abel Prize 2004:
The Atiyah–Singer index theorem

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$$\mathrm{index}(D) = \int_{T^*M} \mathrm{Ch}([\sigma(D)])\mathrm{Td}(T^*_{\mathbb{C}}M)$$

Why is the index theorem famous?

- Relates algebra, analysis, geometry and topology:

  l.h.s. $\sim$ functional analysis (linear algebra in $d = \infty$)

  r.h.s. $\sim$ differential geometry and algebraic topology

- Synthesis and extension of many known deep results:

  Gauß–Bonnet, Riemann–Roch, de Rham,...

- Very hard to prove (yet has many different proofs)
- Created its own research field: index theory ($\rightarrow$ NCG)
- Unexpected applications to quantum physics:

  Anomalies in quantum field theory $\rightarrow$ baryogenesis

  Quantization theory (Raoul Bott, 1923–2005)

The index in linear algebra

A linear map $L : V \to W$ **defines an equation** $Lv = w$

Are there any solutions? (take inner product on W**)**

Yes, iff $w \in \mathrm{ran}(L) = \mathrm{ran}(L)^{\perp\perp} \subset W$ **or:** $w \in \mathrm{coker}(L)^{\perp}$
with $\mathrm{coker}(L) := \mathrm{ran}(L)^{\perp}$**; w.r.t. a basis** (e_i) **of** $\mathrm{coker}(L)$**,**
this gives $\dim(\mathrm{coker}(L))$ **conditions** $(w, e_i) = 0$ **on** w

What is the dimension of the solution space S**?**

$\dim(S) = \dim(\ker(L))$**:** $L(v + v') = w$ **if** $Lv = w$ **and** $Lv' = 0$

Discontinuous dependence on L**: if** $V = W$**, then**

$$\dim(\ker(\varepsilon \cdot 1)) = \dim(\mathrm{coker}(\varepsilon \cdot 1)) = 0 \text{ if } \varepsilon \neq 0 \text{ but } \dim(V) \text{ if } \varepsilon = 0$$

Stable quantity: $\mathrm{index}(L) := \dim(\ker(L)) - \dim(\mathrm{coker}(L))$

The baby index theorem $(\dim(\cdot) < \infty)$

$$\mathrm{index}(L) = \dim(V) - \dim(W)$$

1. $\mathrm{index}(L)$ **is independent of** L
2. **depends on** V **and** W **only through topological invariants** $\dim(V)$ **and** $\dim(W)$ **(L.E.J. Brouwer)**
3. **The index of** $L : V \to V$ **is 0**

The index of an infinite-dimensional linear operator L**:**

- **may depend on** L **itself as well (again topologically)**
- **may be nonzero even if** $V = W$
- **may even be infinite; hence require** $\dim(\ker(L)) < \infty$ **and** $\dim(\mathrm{coker}(L)) < \infty$ **i.e.** L **is 'almost invertible'**

Elliptic differential operators

Example: $D_1 = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ **on** $V = W = L^2(\mathbb{R}^2)$

$f \in \ker(D_1) \Rightarrow f = f(x+iy) \Rightarrow \dim(\ker(D_1)) = 0$ **(Liouville)**

Also: $\dim(\text{coker}(D_1)) = 0 \Rightarrow \text{index}(D_1) = 0 - 0 = 0$ **(R–R)**

Not an example: $D_2 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ **on** $L^2(\mathbb{R}^2)$ **(has no index)**

Better example: $D_3 = \frac{d}{dx} + x$ **on** $L^2(\mathbb{R})$

$f \in \ker(D_3) \Rightarrow f \sim \exp(-x^2/2) \Rightarrow \dim(\ker(D_3)) = 1$

$f \in \text{coker}(D_3) \Rightarrow f \sim \exp(+x^2/2) \Rightarrow \dim(\ker(D_3)) = 0$

Hence $\text{index}(D_3) = 1 - 0 = 1$

Atiyah–Singer is about elliptic (pseudo) differential operators (like D_1 and D_3) on compact spaces

Dirac operators

Two ways to probe a space with elliptic operators:

- **structureless particle** $\rightarrow$ **Riemannian geometry & Laplace operator** Δ **(e.g.** $\Delta = \partial/\partial x^2 + \partial/\partial y^2$ **on** $L^2(\mathbb{R}^2)$**)**
- **spinning particle with internal structure** E **(vector bundle)** $\rightarrow$ **'spin geometry' & Dirac operator** $\not{D}_E$

e.g. on $\mathbb{R}^2$ **with** $z = x + iy$**:** $\not{D} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}$ **on** $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$

"The geometrical significance of spinors is still very mysterious. (...) When Singer and I were investigating these questions we 'rediscovered' for ourselves the Dirac operator. Had we been better educated in physics, or had there been the kind of dialogue with physicists that is now common, we would have got there much sooner"

(Michael Atiyah, 1998)

Index theorem for Dirac operators

$$\mathrm{index}(\not{D}_E) = \int_M \hat{A}(M)\mathrm{ch}(E)$$

1. $\mathrm{index}(\not{D}_E)$ **is independent of** $\not{D}_E$
2. **depends on space** M **and internal structure** E **only through topological invariants** $\hat{A}(M)$ **and** $\mathrm{ch}(E)$
3. **The index of** $\not{D}_E$ **may be nonzero!**

Two ways to read the index formula:

- **'topological' r.h.s. computes 'analytical' l.h.s.**
- **l.h.s. explains integrality of topological r.h.s.**

e.g. Gauß–Bonnet: $(\#\Delta) - (\#\partial) + (\#V) = (1/2\pi)\int_\Sigma K$

Application 1: baryogenesis

"The index theorem created matter in the early Universe" (Ed Witten, 2004)

Baryons B (protons, neutrons) have antiparticles $\overline{B}$

Today: $(n_B - n_{\overline{B}})/(n_B + n_{\overline{B}}) \simeq 1$ **but** $< 10^{-9}$ **after Big Bang**

Naive conservation law: $\frac{d}{dt}(n_B - n_{\overline{B}}) = 0 \Rightarrow$

Problem: how did baryons wipe out antibaryons?

Solution due to Sakharov (1967), 't Hooft (1976), . . . ,:

$$\frac{d}{dt}(n_B - n_{\overline{B}}) \sim \mathrm{index}(\not{D}_B) \neq 0$$

"However, now a consensus has been reached that the effect is far too small. (. . .) However, all of the above may be wrong (. . .)" (A. Dolgov, 1997)

Viability depends on future experimental data!

Application 2: what is quantization?

Traditionally (Heisenberg, Dirac, von Neumann, ∼1930):

(i) phase space (symplectic manifold) → Hilbert space

(ii) functions on phase space → linear operators

Abstraction 1 (Guillemin & Sternberg, 1982):

Quantization is equivalence class of unitary G-modules

Abstraction 2 (Bott, ∼ 1990): Quantization is

'G-index' of Dirac operator $\not{D}_L$ on phase space

'Dutch school': functoriality of Bott's definition ⇒

Abstraction 3 (2005): Quantization is a functor

But: mathematical precision ∼ 1/physical relevance!