# Moduli of configurations of points and lines in the plane

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(after Erik Reuvers)

# Overview

- Cross ratio and invariants

- Geometric invariant theory

- Examples

- Gale duality

Work over a field $k$.

Cross ratio for four points on a line:

$$\mathrm{cr}(A,B,C,D) = \frac{CA/CB}{DA/DB}.$$

$$(A,B,C,D) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

$$A = [x_1, y_1],\ \ B = [x_2, y_2],\ \ C = [x_3, y_3],\ \ D = [x_4, y_4]$$

$$\mathrm{cr}(A,B,C,D) = \frac{(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2)}{(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2)}$$

Basic invariant of point sequences on the line (for the action of $\mathsf{SL}_2$):

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} =: [ij]$$

Need multihomogeneous expressions!

Four points:

$$[12][34], \ [13][24], \ [14][23]$$

Relation:

$$[12][34] - [13][24] + [14][23] = 0$$

Cross ratio:

$$\frac{[13][24]}{[14][23]}$$

Note: when all invariants are zero then at least three of the four points coincide.

Such sequences of points are called *unstable*.

Stable sequence: all four points different.

Semistable: each point occurs at most twice.

Three minimal semistable orbits:

$$(P, P, Q, Q),\ (P, Q, P, Q),\ (P, Q, Q, P)$$

Geometric invariant theory:

Quotient map

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)_{\mathrm{ss}} \to \mathbb{P}^2$$

via three basic invariants

$$X_0 = [12][34],\ X_1 = [13][24],\ X_2 = [14][23]$$

image is the line $X_0 - X_1 + X_2 = 0$ with three special points

$$[0,1,1],\ [1,0,-1],\ [1,1,0].$$

Fibres of the quotient map: stable orbits (are closed in semistable locus) or union of semistable orbits (minimal semistable orbit plus all orbits containing it in their closure).

## Invariants for points and hyperplanes

Consider the action of $\mathrm{SL}_{n+1}$ on

$$X := (\mathbb{P}^n)^{N_1} \times ((\mathbb{P}^n)^*)^{N_2}$$

Here $(\mathbb{P}^n)^*$ is the dual projective space: elements are hyperplanes. Equivalently: let $V = k^{n+1}$ and consider sequences

$$(x_1, \ldots, x_{N_1}, x_1^*, \ldots, n_{N_2}^*)$$

where $x_i \in V$, $x_j^* \in V^*$, all nonzero.

Let

$$[i_0 i_1 \ldots i_n] := \det(x_{i_0}, \ldots, x_{i_n})$$

$$[j_0 j_1 \ldots j_n]' := \det(x_{j_0}^*, \ldots, x_{j_n}^*)$$

$$(ij) := \langle x_j^*, x_i \rangle$$

First fundamental theorem of invariant theory:

The algebra of invariant polynomials on $V^{N_1} \times (V^*)^{N_2}$ is generated by all

$$[i_0 i_1 \dots i_n],\ [j_0 j_1 \dots j_n]',\ (ij)$$

For geometric invariants for $X$ need multihomogeneous polynomials.

General situation: sequences

$$(L_1, \ldots, L_N)$$

where

$$L_j \subset \mathbb{P}^{n-1}$$

projective subspace of dimension $d_i - 1$.

Equivalently: let $V$ be a $k$-vector space of dimension $n$. Consider sequences

$$(V_1, \ldots, V_N)$$

where

$$V_j \subset V$$

linear subspace of dimension $d_i$.

Group acting: $\mathrm{SL}_n(k)$ or $\mathrm{SL}(V)$.

Configuration space: product of Grassmannians.

$\mathsf{G}(d, V)$: set of linear subspaces of $V$ of dimension $d$.

Embedded in $\mathbb{P}(\bigwedge^d V)$ by Plücker coordinates.

Consider action of $\mathsf{SL}_n(V)$ on multi-homogeneous expressions in Plücker coordinates.

# Dimension of the moduli space

Naive dimension count: Virtual dimension := dimension of configuration space minus dimension of the group acting

$$\text{vdim}(M) = \sum_i d_i(n - d_i) - n^2 + 1$$

Example: lines in space. Choose $2 \times 4$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$$

whose row space is the given line. For $0 \leq i < j \leq 3$ let

$$p_{ij} = x_i y_j - x_j y_i$$

Then

$$\mathrm{G}(2,4) \hookrightarrow \mathbb{P}^5$$

via

$$\ell \mapsto [p_{01}(\ell), p_{02}(\ell), p_{03}(\ell), p_{12}(\ell), p_{13}(\ell), p_{23}(\ell)]$$

Two lines $\ell$ and $m$ intersect if and only if

$$h(\ell,m) := p_{01}(\ell)p_{23}(m) - p_{02}(\ell)p_{13}(m) + p_{03}(\ell)p_{12}(m)$$

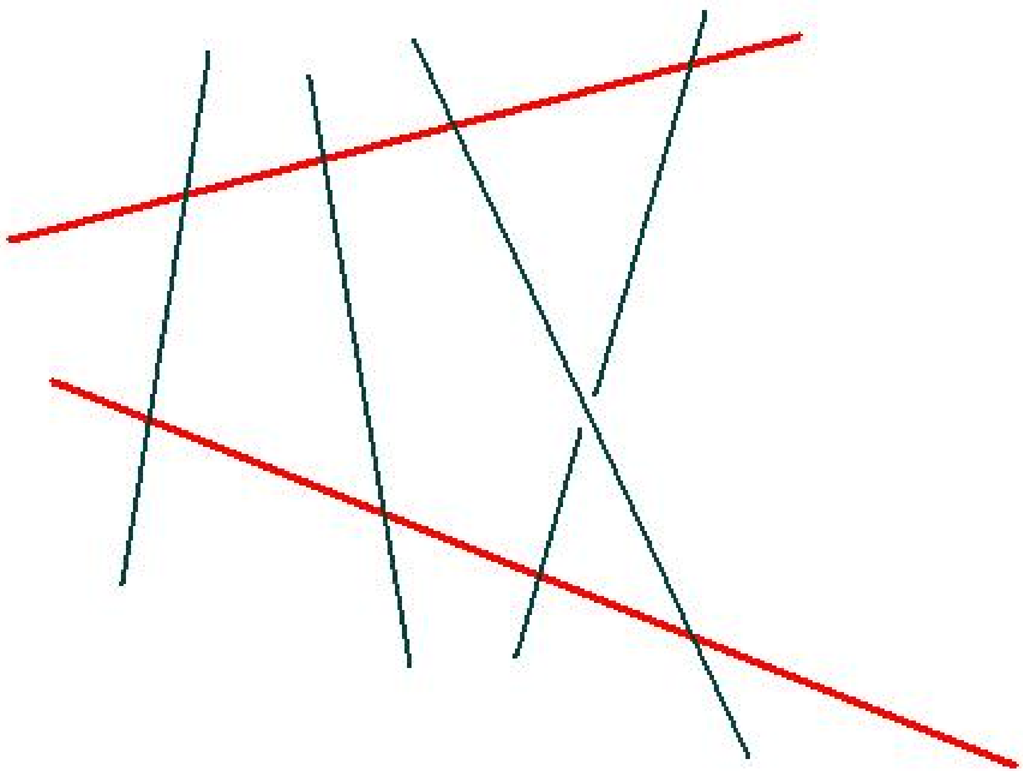
$$+p_{12}(\ell)p_{03}(m) - p_{13}(\ell)p_{02}(m) + p_{23}(\ell)p_{01}(m) = 0$$

Invariants for four lines $(\ell_1,\ell_2,\ell_3,\ell_4)$:

$X_0 := h(\ell_1,\ell_2)h(\ell_3,\ell_4), X_1 := h(\ell_1,\ell_3)h(\ell_2,\ell_4), X_2 := h(\ell_1,\ell_4)h(\ell_2,\ell_3).$

Moduli space = GIT quotient = $\mathbb{P}^2$. Note that

$$\text{vdim}M = 1,\ \dim(M) = 2$$



Geometric explanation

Generalisation of this example:

Four $\mathbb{P}^{d-1}$'s in $\mathbb{P}^{2d-1}$.

Equivalent with $\mathrm{vdim}(M) = 1$.

Moduli space: $\mathbb{P}^d$.

## Quartic surfaces in $\mathbb{P}^3$ with a double conic

Are projection from a general point of intersection of two quadrics in $\mathbb{P}^4$ = Del Pezzo surface of degree four (in thesis of Martijn Grooten: cyclide quartic surface)

Moduli space of these surfaces (suitably marked)? Open subset of $G(2,5)$.

Generic pencil of quadrics in $\mathbb{P}^5$: five singular ones, rank five, get five poles.

Center of projection $\mathbb{P}^5 \to \mathbb{P}^3$: sixth point. Del Pezzo surface contains sixteen lines...

Example: three points and three lines in the projective plane

$$\begin{aligned} W &:= [012][012]' \\ X_\sigma &:= (0\sigma(0))(1\sigma(1))(2\sigma(2)) \end{aligned}$$

where $\sigma$ runs over the permutations of $\{0, 1, 2\}$.

Moreover:

$$W = \sum_\sigma \epsilon(\sigma) X_\sigma$$

and

$$\prod_\sigma X_\sigma^{\epsilon(\sigma)} = 1$$

The hypersurface of $\mathbb{P}^5$ with equation

$$X_0 X_1 X_2 = X_3 X_4 X_5$$

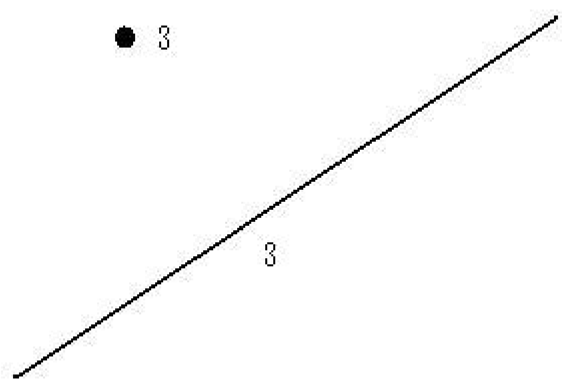
contains as a dense open subset the torus

$$\{[1, s, t, u, v, stu^{-1}v^{-1}] \mid s, t, u, v \neq 0\}$$

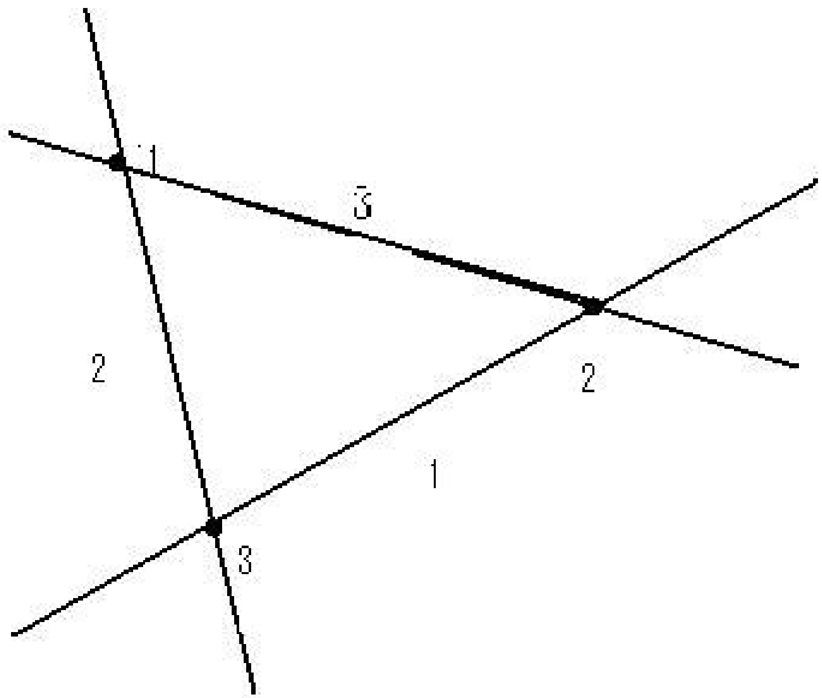
and is a toric variety!

Has a birational map to moduli space of six points in the plane!

Some minimal semistable orbits:




$$[1, 1, 1, 1, 1, 1]$$




$[1, 0, 0, 0, 0, 0]$

Duality:

$$X_\sigma \mapsto X_{\sigma^{-1}}$$

## Gale duality

Consider configuration as projection of generic configuration. $D := \sum d_i$, then generic configuration in $\mathbb{P}^{D-1}$. Project to $\mathbb{P}^{n-1}$ from center $\simeq \mathbb{P}^{D-n-1}$. Corresponds to exact sequence

$$0 \to W \to \bigoplus V_i \to V \to 0$$

Dualise to

$$0 \to V^* \to \bigoplus V_i^* \to W^* \to 0$$

Example: five points on $\mathbb{P}^1$ give five points on $\mathbb{P}^2$.

Theorem: Gale duality induces an isomorphism of moduli spaces.

Example: six points in plane are Gale self-dual if they lie on a conic.