Nevanlinna-Pick interpolation and relaxed commutant lifting

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NMC, 27-03-2006

Nevanlinna-Pick interpolation on $\mathbf{H}^\infty$

Data: Distinct points $z_1 \ldots, z_n$ in $\mathbb{D}$, and complex numbers $\beta_1, \ldots, \beta_n$.

Problem: Find $f : \mathbb{D} \to \mathbb{C}$, analytic, such that

(NP1) $f(z_j) = \beta_j$ for $j = 1, \ldots, n$, and

(NP2) $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \leq 1$.

Pick (1916): problem is solvable if and only if the "Pick matrix"

$$\left[\frac{1-\beta_i \overline{\beta}_j}{1-z_i \overline{z}_j} \right]_{i,j=1}^n \text{ is positive.}$$

Nevanlinna (1919): linear fractional description of all solutions.

Nevanlinna-Pick interpolation on $\mathbf{H}^2$

Data: Distinct points $z_1 \dots, z_n$ in $\mathbb{D}$, and complex numbers $\beta_1, \dots, \beta_n$.

Problem: Find $f : \mathbb{D} \to \mathbb{C}$, analytic, such that

(NP1') $f(z_j) = \beta_j$ for $j = 1, \dots, n$, and

(NP2') $\|f\|_2 = (\lim_{r \uparrow 1} \int_0^{2\pi} |f(re^{i\omega})|^2 d\omega)^{\frac{1}{2}} \leq 1.$

Alpay-Bolotnikov-Peretz (1995): problem is solvable if and only if the "Pick matrix"

$$\left[\frac{1}{1 - z_i \overline{z}_j} - \beta_i \overline{\beta}_j \right]_{i,j=1}^{n} \text{ is positive.}$$

Also a linear fractional description of all solutions is given.

The $\mathbf{H}^\infty$ norm

With an $f \in \mathbf{H}^\infty$ we associate a multiplication operator on $\mathbf{H}^2$:

$$\text{If } g \in \mathbf{H}^2 \text{ then } z \mapsto f(z)g(z) \in \mathbf{H}^2.$$

Moreover, for $f \in \mathbf{H}^\infty$:

$$\|f\|_\infty^2 = \lim_{r \uparrow 1} \sup_{g \in \mathbf{H}^2, \|g\|_2 \leq 1} \int_0^{2\pi} |f(re^{i\omega})g(re^{i\omega})|^2 d\omega.$$

Compare this with

$$\|g\|_2^2 = \lim_{r \uparrow 1} \int_0^{2\pi} |g(re^{i\omega})|^2 d\omega$$

for $g \in \mathbf{H}^2$.

Introduction of $\mathbf{H}^2_m$

Put

$\mathcal{P}_m :=$ {polynomials of degree $\leq m-1\} \subset \mathbf{H}^2$.

For $g \in \mathbf{H}^2$ we define

$$\|g\|^2_{2,m} := \lim_{r \uparrow 1} \sup_{h \in \mathcal{P}_m, \|h\|_2 \leq 1} \int_0^{2\pi} |h(re^{i\omega})g(re^{i\omega})|^2 d\omega.$$

Then $\|\cdot\|_{2,m}$ defines an alternative norm for $\mathbf{H}^2$. This space is denoted by $\mathbf{H}^2_m$.

Note:

$$m = 1: \qquad \|g\|_{2,1} = \|g\|_2,$$

and

$$\text{if } f \in \mathbf{H}^\infty: \qquad \lim_{m \to \infty} \|f\|_{2,m} = \|f\|_\infty.$$

Nevanlinna-Pick interpolation on $\mathbf{H}^2_m$

Data: Distinct points $z_1 \ldots, z_n$ in $\mathbb{D}$, and complex numbers $\beta_1, \ldots, \beta_n$.

Problem: Find $f : \mathbb{D} \to \mathbb{C}$, analytic, such that

(RNP1) $f(z_j) = \beta_j$ for $j = 1, \ldots, n$, and

(RNP2) $\|f\|_{2,m} \leq 1$.

Foias-Frazho-Kaashoek (2002): problem is solvable if and only if the "Pick matrix"

$$\left[\frac{1-\beta_i\overline{\beta}_j(1-z_i^m\overline{z}_j^m)}{1-z_i\overline{z}_j} \right]_{i,j=1}^{n} \text{ is positive.}$$

Method:

The relaxed commutant lifting theorem.

The relaxed commutant lifting problem

Lifting data set (LDS): $\{A, T', U', R, Q\}$ such that

$$A : \mathcal{H} \to \mathcal{H}', \quad T' \text{ on } \mathcal{H}' \text{ a contraction,}$$

$$R, Q : \mathcal{H}_0 \to \mathcal{H}, \quad R^*R \leq Q^*Q,$$

$$T'AR = AQ,$$

$$U' \text{ on } \mathcal{K}' = \mathcal{H}' \oplus \mathcal{M} \text{ an isometric lifting of } T',$$

$$\text{i.e., } U' = \begin{bmatrix} T' & 0 \\ * & * \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}, \ U' \text{ isometry}$$

Problem (RCLP): Find $B : \mathcal{H} \to \mathcal{K}'$ such that

1) $B = \begin{bmatrix} A \\ * \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}$, a contraction,

2) $U'BR = BQ$.

RCLT (FFK'02): Solvable iff $\|A\| \leq 1$.

Frazho-H-Kaashoek '06:

A description of all solutions.

Connection to operator theory

The Hilbert space ℓ^2:

$$\ell^2 = \{\left[\begin{array}{ccc} x_0 & x_1 & \dots \end{array}\right]^T | x_k \in \mathbb{C}, \sum_{k=0}^{\infty} |x_k|^2 < \infty\}.$$

Let $f : \mathbb{D} \to \mathbb{C}$ analytic, $f(z) = \sum_{k=0}^{\infty} f_k z^k$.

For $\mathbf{H}^2$:

$$\begin{aligned} f \in \mathbf{H}^2 &\iff T_{f,1} := \left[\begin{array}{ccc} f_0 & f_1 & \dots \end{array}\right]^T \in \ell^2 \\ &\iff T_{f,1} : \mathbb{C} \to \ell^2 \text{ bounded.} \end{aligned}$$

And $\|f\|_2 = \|T_{f,1}\|$.

For $\mathbf{H}^\infty$: $f \in \mathbf{H}^\infty$ iff

$$T_f := \left[\begin{array}{cccc} f_0 & 0 & 0 & \dots \\ f_1 & f_0 & 0 & \dots \\ f_2 & f_1 & f_0 & \\ \vdots & & & \ddots \end{array}\right] \text{ on } \ell^2 \text{ bounded.}$$

And $\|f\|_\infty = \|T_f\|$.

Condition RNP2 in operator form

Now for $\mathbf{H}^2_m$:

Let f be a function from $\mathbf{H}^2$. Define

$$T_{f,m} := \begin{bmatrix} f_0 & & 0 \\ \vdots & \ddots & \\ f_{m-1} & \cdots & f_0 \\ f_m & \cdots & f_1 \\ \vdots & \vdots & \vdots \end{bmatrix} : \mathbb{C}^m \to \ell^2 \text{ bounded.}$$

Then $\|f\|_{2,m} = \|T_{f,m}\|$.

Thus the condition (RNP2) is equivalent to

(RNP2') $T_{f,m}$ is a contraction.

Identifying $T_{f,m}$ in $\mathbf{L}(\mathbb{C}^m, \ell^2)$

Define

$$S = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \text{ on } \ell^2 \text{ (forward shift)},$$

$$R = \begin{bmatrix} I_{m-1} \\ 0 \end{bmatrix}, Q = \begin{bmatrix} 0 \\ I_{m-1} \end{bmatrix} : \mathbb{C}^{m-1} \to \mathbb{C}^m.$$

Then for $f \in \mathbf{H}^2_m$

$$ST_{f,m}R = T_{f,m}Q.$$

In fact:

LEMMA. *Let $B : \mathbb{C}^m \to \ell^2$ be a (bounded) operator. Then*

$$SBR = BQ \iff B = T_{f,m} \text{ for some } f \in \mathbf{H}^2_m.$$

Some operators

Put $Z = \text{diag}\,(z_1, \ldots, z_n)$ on $\mathbb{C}^n$, and

$\alpha = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T, \beta = \begin{bmatrix} \beta_1 & \ldots & \beta_n \end{bmatrix}^T : \mathbb{C} \to \mathbb{C}^n.$

Define

$$\begin{aligned} W &= \begin{bmatrix} \alpha & Z\alpha & Z^2\alpha & \ldots \end{bmatrix} \\ &= \begin{bmatrix} 1 & z_1 & z_1^2 & \ldots \\ \vdots & \vdots & \vdots & \\ 1 & z_n & z_n^2 & \ldots \end{bmatrix} : \ell^2 \to \mathbb{C}^n. \end{aligned}$$

$$\begin{aligned} W_m &= \begin{bmatrix} \beta & Z\beta & \ldots & Z^{m-1}\beta \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 & z_1\beta_1 & \ldots & z_1^{m-1}\beta_1 \\ \vdots & \vdots & & \vdots \\ \beta_n & z_n\beta_n & \ldots & z_n^{m-1}\beta_n \end{bmatrix} : \mathbb{C}^m \to \mathbb{C}^n. \end{aligned}$$

$$P = WW^* \text{ on } \mathbb{C}^n, \quad P_m = W_m W_m^* \text{ on } \mathbb{C}^n.$$

Then

$$z_i \neq z_j \text{ if } i \neq j \;\Rightarrow\; P \text{ is invertible.}$$

And

$WS = ZW, \quad W_m R = W_{m-1}, \quad W_m Q = ZW_{m-1},$

where W_{m-1} is defined similar as W_m.

Condition RNP1 in operator form

Define $\mathcal{H}' = \mathrm{Im}\, W^* \subset \ell^2$ and

$$A = W^* P^{-1} W_m : \mathbb{C}^m \to \mathcal{H}'.$$

Then for $f \in \mathbf{H}^2_m$:

$$\begin{aligned} f(z_i) = \beta_i \ \forall i \iff WT_{f,m} &= W_m \\ \iff WT_{f,m} &= WW^* P^{-1} W_m \\ &= WA \\ \iff T_{f,m} &= \begin{bmatrix} A \\ * \end{bmatrix}. \end{aligned}$$

Thus the condition (RNP1) is equivalent to

(RNP1') $T_{f,m} = \begin{bmatrix} A \\ * \end{bmatrix}$.

NP-interpolation on $\mathbf{H}^2_m$ in operator language:

Let $f : \mathbb{D} \to \mathbb{C}$ be analytic. Then f is a NP-interpolant in $\mathbf{H}^2_m$ if and only if

(RNP1') $T_{f,m} = \begin{bmatrix} A \\ * \end{bmatrix}$, and

(RNP2') $T_{f,m}$ is a contraction.

Connection with relaxed commutant lifting

Put

$$T' = \Pi_{\mathcal{H}'} S|\mathcal{H}' \text{ on } \mathcal{H}'.$$

Then $\{A, T', S, R, Q\}$ is a LDS if

(1) S is an isometric lifting of T' and

(2) $T'AR = AQ$.

Proof of (1): S is an isometry and

$$S^*W^* = W^*Z^* \Rightarrow S^*\mathcal{H}' \subset \mathcal{H}'.$$

Proof of (2):

$$\begin{aligned} WT'AR &= WSW^*P^{-1}W_mR = ZWW^*P^{-1}W_{m-1} \\ &= ZW_{m-1} = W_mQ = WW^*P^{-1}W_mQ \\ &= WAQ. \end{aligned}$$

Thus

$$\operatorname{Im} W^* = \mathcal{H}' \;\Rightarrow\; T'AR = AQ.$$

Conclusion 1

THM. *Let* $f : \mathbb{D} \to \mathbb{C}$ *be analytic and let* $\Omega := \{A, T', S, R, Q\}$ *be the LDS as constructed above. Then* f *is a NP-interpolant in* $\mathbf{H}^2_m$ *for our given data if and only if there exists a solution* B *to the RCLP for LDS* Ω *such that* $B = T_{f,m}$.

RCLT: There exists a solution to the RCLP for LDS Ω iff $\|A\| \leq 1$.

$$\begin{aligned} \|A\| \leq 1 &\iff I - A^*A \geq 0 \\ &\iff I - W_m^* P^{-1} W_m \geq 0 \\ &\iff P - W_m W_m^* \geq 0 \\ &\iff P - P_m \geq 0 \end{aligned}$$

Conclusion 2: The Pick matrix

Note

$$\begin{aligned} P &= \sum_{k=0}^{\infty} Z^k \alpha\alpha^* (Z^*)^k = \left[\, \textstyle\sum_{k=0}^{\infty} z_i^k \bar{z}_j^k \, \right]_{i,j=1}^{n} \\ &= \left[\, \tfrac{1}{1-z_i\bar{z}_j} \, \right]_{i,j=1}^{n} \end{aligned}$$

and

$$\begin{aligned} P_m &= \sum_{k=0}^{m-1} Z^k \beta\beta^* (Z^*)^k = \left[\, \textstyle\sum_{k=0}^{m-1} z_i^k \beta_i \bar{\beta}_j \bar{z}_j^k \, \right]_{i,j=1}^{n} \\ &= \left[\, \frac{\beta_i\bar{\beta}_j(1-z_i^m\bar{z}_j^m)}{1-z_i\bar{z}_j} \, \right]_{i,j=1}^{n}. \end{aligned}$$

Therefore

$$P - P_m = \left[\, \frac{1-\beta_i\bar{\beta}_j(1-z_i^m\bar{z}_j^m)}{1-z_i\bar{z}_j} \, \right]_{i,j=1}^{n}.$$