PROBLEMS WITH DIMENSION

LINEAR ALGEBRA: $\dim \mathbb{R}^n = n$

THERE IS A BASE CONSISTING OF n VECTORS (THE STANDARD UNIT VECTORS, FOR EXAMPLE).

"TO DESCRIBE A POINT IN $\mathbb{R}^n$ YOU NEED n REAL NUMBERS: ITS COORDINATES."

1878 GEORG CANTOR

"LET $x_1, x_2, \ldots, x_n$ BE INDEPENDENT VARIABLE REAL QUANTITIES THAT KAN ASSUME ALL VALUES ≥ 0 AND ≤ 1 AND LET t BE ANOTHER VARIABLE WITH THE SAME RANGE $(0 \leq t \leq 1)$. THEN IT IS POSSIBLE TO LET THE QUANTITIES t CORRESPOND TO THE SYSTEM OF n QUANTITIES $x_1, x_2, \ldots, x_n$ IN SUCH A WAY THAT TO EACH VALUE OF t CORRESPONDS ONE SYSTEM OF VALUES $x_1, x_2, \ldots, x_n$ AND, CONVERSELY, TO EACH SYSTEM OF VALUES $x_1, x_2, \ldots, x_n$ CORRESPONDS ONE VALUE OF t."

IN MODERN TERMS: THERE IS A BIJECTION BETWEEN THE UNIT INTERVAL $[0,1]$ AND THE n-CUBE $[0,1]^n$.

HOW? IN A FEW STEPS

- $\mathbb{P} = [0,1] \setminus \mathbb{Q}$
- MAKE A BIJECTION BETWEEN $\mathbb{P}$ AND $\mathbb{P}^n$;
- FIRST: BIJECTION BETWEEN $\mathbb{P}$ AND $\mathbb{N}^{\mathbb{N}}$;
- EASY: BIJECTION BETWEEN $\mathbb{N}^{\mathbb{N}}$ AND $(\mathbb{N}^{\mathbb{N}})^n$;
- FINALLY: BIJECTION BETWEEN $\mathbb{P}$ AND $[0,1]$.

① FROM $\mathbb{P}$ TO $\mathbb{N}^{\mathbb{N}}$ AND CONVERSELY

- GIVEN $a \in (0,1)$ APPLY EUCLID'S ALGORITHM TO THE NUMBERS 1 AND a:

$$1 = a_1 \cdot a + r_1$$
$$a = a_2 \cdot r_1 + r_2 \qquad a_1, a_2, \ldots \in \mathbb{N}$$
$$r_1 = a_3 \cdot r_2 + r_3 \qquad a > r_1 > r_2 > \cdots > 0$$
$$r_{i-1} \overset{!}{=} a_{i+1} \cdot r_i + r_{i+1}$$

- THE SEQUENCE IS FINITE IFF $a \in \mathbb{Q}$
- IF $a \neq b$ THEN $(a_1, a_2, \ldots, a_n, \ldots) \neq (b_1, b_2, \ldots, b_n, \ldots)$

- GIVEN $(a_1, a_2, \ldots, a_n, \ldots)$ LET

$$a = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

THIS IS THE LIMIT OF

$$\frac{1}{a_1}, \quad \cfrac{1}{a_1 + \cfrac{1}{a_2}}, \quad \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}}, \quad \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}}, \ldots$$

- THE SEQUENCE CONVERGES

- THESE OPERATIONS ARE EACH OTHER'S INVERSES.

② BIJECTION $\mathbb{N}^{\mathbb{N}} \longleftrightarrow (\mathbb{N}^{\mathbb{N}})^m$

$$(t_1, t_2, t_3, \ldots) \longleftrightarrow \begin{array}{l} (t_1, t_{m+1}, t_{2m+1}, \ldots, t_{km+1}, \ldots) \\ (t_2, t_{m+2}, t_{2m+2}, \ldots, t_{km+2}, \ldots) \\ \vdots \\ (t_m, t_{2m}, t_{3m}, \ldots, t_{(k+1)m}, \ldots) \end{array}$$

(3) BIJECTION $\mathbb{P} \longleftrightarrow \mathbb{P}^n$

$$\mathbb{P} \longleftrightarrow \mathbb{N}^{\mathbb{N}} \longleftrightarrow (\mathbb{N}^{\mathbb{N}})^n \longleftrightarrow \mathbb{P}^n$$

(4) BIJECTION $[0,1] \longleftrightarrow \mathbb{P}$

ENUMERATE $\mathbb{Q} \cap [0,1]$: $(q_n)_n$

DEFINE $\varepsilon_n = ((n\pi)^{-1})_n$; $E = \{\frac{1}{n\pi} : n \in \mathbb{N}\}$

NOW $f : [0,1] \longrightarrow \mathbb{P}$

$$q_n \longmapsto \varepsilon_{2n}$$

$$\varepsilon_n \longmapsto \varepsilon_{2n-1}$$

$$x \longmapsto x \qquad (x \notin \mathbb{Q} \cup E)$$

(5) BIJECTION $[0,1] \longleftrightarrow [0,1]^n$

$$[0,1] \longleftrightarrow \mathbb{P} \longleftrightarrow \mathbb{P}^n \longleftrightarrow [0,1]^n$$

" JE LE VOIS, MAIS JE NE LE CROIS PAS "

SO, EVERYTHING IS ONE-DIMENSIONAL?

DEDEKIND: NOT SO FAST; THE BIJECTIONS ARE NOT CONTINUOUS.

EXERCISE: MAKE A BIJECTION BETWEEN $[0,1]$ AND $[0,1]^{\mathbb{N}}$.

1890 G. PEANO

"IN THIS NOTE I DETERMINE TWO CONTINUOUS FUNCTIONS x AND y OF A REAL VARIABLE t THAT VARIES THROUGH THE INTERVAL $[0,1]$ THAT ASSUME ALL PAIRS OF VALUES x AND y WITH $0 \leq x \leq 1$ AND $0 \leq y \leq 1$."

WE WORK IN BASE 3, SO OUR DIGITS ARE 0, 1, 2.
GIVEN A SEQUENCE $t_1, t_2, t_3, \ldots$ OF DIGITS WE WRITE $T = 0.t_1 t_2 t_3 \ldots$
(FOR NOW T IS JUST A SEQUENCE OF DIGITS)

THE COMPLEMENT OF A DIGIT a IS THE DIGIT $2-a$; WE WRITE $k(a)$

SO $k(0) = 2$, $k(1) = 1$, $k(2) = 0$.

NOTE
- $k(a) \equiv a \pmod 2$
- $k^n(a) = k(a)$ (n ODD)
- $k^n(a) = a$ (n EVEN)

TO A SEQUENCE $T = 0.t_1 t_2 t_3 \ldots$ WE ASSOCIATE TWO SEQUENCES

$X = 0.x_1 x_2 x_3 \ldots$ AND $Y = 0.y_1 y_2 y_3 \ldots$

AS FOLLOWS

$x_1 = t_1$ $\qquad$ $y_1 = k^{t_1}(t_2)$

$x_2 = k^{a_2}(t_3)$ $\qquad$ $y_2 = k^{t_1+t_3}(t_4)$

$x_3 = k^{a_2+a_4}(t_5)$ $\qquad$ $y_3 = k^{t_1+t_3+t_5}(t_6)$

$\vdots$

$x_n = k^{t_2+\cdots+t_{2n-2}}(t_{2n-1})$ $\qquad$ $y_n = k^{t_1+\cdots+t_{2n-1}}(t_{2n})$

$\vdots$

WE CAN RECONSTRUCT T FROM X AND Y:

$$t_1 = x_1 \qquad t_2 = k^{x_1}(y_1)$$

$$t_3 = k^{y_1}(x_2) \qquad t_4 = k^{x_1+x_2}(y_2)$$

$$(y_1 \equiv t_2 \pmod 2) \qquad (t_1 = x_1 \;;\; x_2 \equiv t_3 \pmod 2)$$

$$t_{2n-1} = k^{y_1+\cdots+y_{n-1}}(x_n) \qquad t_{2n} = k^{x_1+\cdots+x_n}(y_n)$$

THE VALUE, VAL(T), OF A SEQUENCE T IS DEFINED AS

$$t' = \mathrm{VAL}(T) = \frac{t_1}{3} + \frac{t_2}{3^2} + \cdots + \frac{t_n}{3^n} + \cdots$$

$$= \sum_{n=1}^{\infty} t_n \cdot 3^{-n}$$

EXAMPLES

$$1 = \mathrm{VAL}(0,222\ldots) = \sum_{n=1}^{\infty} 2\cdot 3^{-n}$$

$$\frac{1}{4} = \mathrm{VAL}(0,0202\ldots) = \sum_{n=1}^{\infty} 2\cdot 3^{-2n}$$

EVERY NUMBER IN $[0,1]$ IS THE VALUE OF SOME SEQUENCE (EXERCISE)

WE HAVE TWO TYPES OF NUMBERS

(α) THOSE IN $(0,1)$ THAT, WHEN MULTIPLIED BY SOME POWER OF 3 YIELD AN INTEGER

(β) THE OTHER NUMBERS.

(α) THE NUMBERS OF THIS TYPE ARE REPRESENTED BY TWO SEQUENCES, E.G.,

$$\frac{1}{9} = \mathrm{VAL}(0,0100\ldots) \quad \text{AND}$$

$$\frac{1}{9} = \mathrm{VAL}(0,0022\ldots)$$

If n is minimal with $3^n t \in \mathbb{N}$ then $3^n t$ is not divisible by 3 (otherwise $3^{n-1} t \in \mathbb{N}$), and $3^n t < 3^n$, so

$$3^n t = \sum_{k=0}^{n-1} a_k 3^k$$

with $a_0 = 1$ or $a_0 = 2$

$$\text{So} \quad t = \sum_{k=0}^{n-1} a_k 3^{k-n} = \sum_{k=1}^{n} a_{n-k} \cdot 3^{-k}$$

We see

$$t = \mathrm{VAL}(0, a_{n-1} \ldots a_0 0000 \ldots)$$

But since $\sum_{m=n+1}^{\infty} 2 \cdot 3^{-m} = 2 \cdot 3^{-(n+1)} \cdot \sum_{m=0}^{\infty} 3^{-m} = 2 \cdot 3^{-(n+1)} \cdot \frac{3}{2} = 3^{-n}$

We also have

$$t = \mathrm{VAL}(0, a_{n-1} \ldots a_1 (a_0 - 1) 2222 \ldots)$$

In this case we write

$$t = \mathrm{VAL}(T) \qquad T = 0, t_1 \ldots t_{m-1} t_m 2222 \ldots$$
$$= \mathrm{VAL}(T') \qquad T' = 0, t_1, \ldots, t_{m-1} t'_m 0000 \ldots$$

with $t_m < 2$ and $t'_m = t_m + 1$.

(β) Numbers of type (β) correspond to exactly one sequence.

Claim if $\mathrm{VAL}(T) = \mathrm{VAL}(T')$ as above then $\mathrm{VAL}(X) = \mathrm{VAL}(X')$ and $\mathrm{VAL}(Y) = \mathrm{VAL}(Y')$

Proof Assume $T = 0, t_1 t_2 \ldots t_{2n-1} t_{2n} 2222 \ldots$ where $a_{2n-1} < 2$ or $a_{2n} < 2$

(every number of type (α) is equal to $\mathrm{VAL}(T)$ for such a sequence T)

Write $T' = 0, s_1 s_2 \ldots s_{2n-1} s_{2n} 0000$

NOW $X = 0.x_1 x_2 \ldots x_{n-1} x_n x_{n+1} \ldots$

AND $X' = 0.z_1 z_2 \ldots z_{n-1} z_n z_{n+1} \ldots$

FOR $i < n$ WE HAVE $x_i = z_i$

BECAUSE FOR $j < 2n-1$ WE HAVE $t_j = s_j$

E.G. $x_1 = t_1 = s_1 = z_1$ ETC

FOR n: $x_n = k^a(t_{2n-1})$ $\quad a = t_2 + \ldots + t_{2n-2}$

$z_n = k^a(s_{2n-1})$ $\quad = s_2 + \ldots + s_{2n-2}$

FOR $n+1$: $x_{n+1} = k^{a+t_{2n}}(t_{2n+1}) = k^{a+t_{2n}}(2)$

$z_{n+1} = k^{a+s_{2n}}(s_{2n+1}) = k^{a+s_{2n}}(0)$

FOR $m > n+1$ $\quad x_m = k^{a+t_{2n}+2+\ldots+2}(t_{2m-1}) = k^{a+t_{2n}}(2)$

$z_m = k^{a+t_{2n}+0+\ldots+0}(s_{2m-1}) = k^{a+s_{2n}}(0)$

CASE 1 $\quad t_{2n} < 2$; SO $s_{2n-1} = t_{2n-1}$

$s_{2n} = t_{2n} + 1$

WE SEE – $x_n = z_n$

– $a + s_{2n} = a + t_{2n} + 1$

– FOR $m > n+1$:

$x_m = k^{a+t_{2n}}(2) = k^{a+t_{2n}+1}(0)$

$= k^{a+s_{2n}}(0)$

$= z_m$.

SO, IN FACT, $X = X'$ AND, CERTAINLY, VAL(X) = VAL(X')

CASE 2 $\quad t_{2n-1} < 2$ AND $t_{2n} = 2$

SO THEN $s_{2n} = 0$; $s_{2n-1} = t_{2n-1} + 1$

WRITE $a = t_2 + \ldots + t_{2n-2} = s_2 + \ldots + s_{2n-2}$ (AGAIN)

NOW $x_n = k^a(t_{2n-1})$ $\quad z_n = k^a(s_{2n-1})$

BECAUSE $t_{2m} = 2$ AND $s_{2m} = 0$ $(m \geq n)$

WE FIND, FOR $m \geq n+1$

$x_m = k^a(t_{2m-1}) = k^a(2)$

$z_m = k^a(s_{2m-1}) = k^a(0)$

IF a IS ODD THEN

$$X = 0, x_1 x_2 \dots x_{n-1} (k(t_{2n-1})) 00 \dots$$

$$X' = 0, x_1 x_2 \dots x_{n-1} (k(s_{2n-1})) 22 \dots$$

BUT $s_{2n-1} = t_{2n-1} + 1$

SO $k(t_{2n-1}) = k(s_{2n-1}) + 1$

WE FIND THAT $\mathrm{VAL}(X) = \mathrm{VAL}(X')$

IF a IS EVEN THEN

$$X = 0, x_1 x_2 \dots x_{n-1}\, t_{2n-1}\, 22 \dots$$

$$X' = 0, x_1 x_2 \dots x_{n-1}\, s_{2n-1}\, 00 \dots$$

SO AGAIN $\mathrm{VAL}(X) = \mathrm{VAL}(X')$

EXERCISE: VERIFY THAT $\mathrm{VAL}(Y) = \mathrm{VAL}(Y')$ ALSO

CONCLUSION $\mathrm{VAL}(T) \longmapsto (\mathrm{VAL}(X), \mathrm{VAL}(Y))$
UNAMBIGUOUSLY DEFINES A FUNCTION
FROM $[0,1]$ TO $[0,1]^2$.

- THE FUNCTION IS SURJECTIVE BECAUSE
  $T \longmapsto (X, Y)$
  IS SURJECTIVE (BIJECTIVE EVEN)

- THE FUNCTION IS CONTINUOUS
  LET $n \in \mathbb{N}$ AND ASSUME
  $|t - s| < 3^{-n}$
  WHERE $t = \mathrm{VAL}(T)$ AND $s = \mathrm{VAL}(S)$
  CLAIM IF THERE IS A $j \leq n$ SUCH THAT $t_j \neq s_j$
  THEN (WLOG) $t_j - s_j = 1$ AND
  $s_i = 2$ AND $t_i = 0$ WHEN $j < i \leq n$.
  IF $t_j - s_j = 2$ THEN
  $$t - s \geq 2 \cdot 3^{-j} - \sum_{i > j} 2 \cdot 3^{-i} = 2 \cdot 3^{-j} - 3^{-j} = 3^{-j} \geq 3^{-n} \quad \times$$

So $t_j - s_j = 1$

Now assume $t_k - s_k \geq -1$ for some k with $j < k \leq n$.

Then
$$t - s = 3^{-j} + \sum_{i>j} (t_i - s_i)\, 3^{-i}$$
$$\geq 3^{-j} + \sum_{i>j} (-2)\, 3^{-i} + 3^{-k}$$
$$= 3^{-j} - 3^{-j} + 3^{-k} \geq 3^{-n} \quad \times$$

• Now assume $s = val(S)$, $t = val(T)$, $S \mapsto (U,V)$ and $T \mapsto (X,Y)$

Also assume $|s - t| < 9^{-n} = 3^{-2n}$

Case 1 $\quad s_i = t_i$ for $i \leq 2n$

Then, according to the construction, we find $x_i = u_i$ and $y_i = v_i$ for $i \leq n$

and so $|u - x| \leq 3^{-n}$ and $|v - y| \leq 3^{-n}$.

Case 2 $\quad$ There is a $j \leq 2n$ such that

$s_i = t_i \quad (i < j)$

$t_j - s_j = 1$

$s_i = 2,\ t_i = 0 \quad j < i \leq 2n$

Then we get (as before)

• $u_i = x_i$ for $i \leq n$

or
$$\begin{cases} u_i = x_i & \text{for } i < k \\ |u_k - x_k| = 1 & \\ |u_i - x_i| = 2 & k < i \leq n \end{cases}$$

First case $\quad |u - x| \leq 2\sum_{i>n} 3^{-i} = 3^{-n}$

Second case $\quad |u - x| \leq \left[3^{-k} - 2\sum_{i=k+1}^{n} 3^{-i}\right] + \sum_{i>n} 2 \cdot 3^{-i}$
$$= 3^{-n} + 3^{-n}$$
$$= 2 \cdot 3^{-n}$$

• Similarly for y and v

$|y - v| \leq 2 \cdot 3^{-n}$.

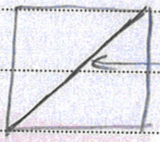
THE FUNCTION IS (UNIFORMLY) CONTINUOUS:
IF $|s-t| < 9^{-n}$ THEN $|u-x|, |v-y| \leq 2 \cdot 3^{-n}$

EXERCISE LET C BE THE CANTOR SET
SHOW THAT C IS MAPPED ONTO $C \times C$
AND THAT THE FUNCTION IS INJECTIVE ON C
DEDUCE THAT C AND $C \times C$ ARE HOMEOMORPHIC

EXERCISE LET $f : \mathbb{R} \to \mathbb{R}$ BE CONTINUOUS
SUCH THAT
- $f[\mathbb{R}] \subseteq [0,1]$
- $f(t+2) = f(t)$ FOR ALL t
- $f(t) = 0 \quad 0 \leq t \leq 1/3$
- $f(t) = 1 \quad 2/3 \leq t \leq 1$

DEFINE
$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t)$$
$$y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n} t)$$
PROVE THAT
$$t \mapsto (x(t), y(t))$$
MAPS $[0,1]$ ONTO $[0,1]^2$ AND IS CONTINUOUS
[EVEN THE CANTOR SET IS MAPPED ONTO $[0,1]^2$]

EXERCISE LET $f_0(t) = (x_0(t), y_0(t))$
WHERE $x_0(t) = y_0(t) = t$


CURVE f_0

GIVEN x_n AND y_n DEFINE x_{n+1} AND y_{n+1} AS FOLLOWS

$$
\begin{aligned}
x_{n+1}(t) = {} & \tfrac{1}{3} x_n(9t) && (0 \leq t \leq 1/9); \\
& \tfrac{1}{3} - \tfrac{1}{3} x_n(9t-1) && (1/9 \leq t \leq 2/9) \\
& 1/3\, x_n(9t-2) && (2/9 \leq t \leq 3/9) \\
& 1/3 + 1/3\, x_n(9t-3) && (3/9 \leq t \leq 4/9) \\
& 2/3 - 1/3\, x_n(9t-4) && (4/9 \leq t \leq 5/9) \\
& 1/3 + 1/3\, x_n(9t-5) && (5/9 \leq t \leq 6/9) \\
& 2/3 + 1/3\, x_n(9t-6) && (6/9 \leq t \leq 7/9) \\
& 1 - 1/3\, x_n(9t-7) && (7/9 \leq t \leq 8/9) \\
& 2/3 + 1/3\, x_n(9t-8) && (8/9 \leq t \leq 1)
\end{aligned}
$$

$$y_{n+1}(t) = \begin{cases} y_n(3t) & (0 \le t \le 1/3) \\ 1 - y_n(3t-1) & (1/3 \le t \le 2/3) \\ y_n(3t-2) & (2/3 \le t \le 1) \end{cases}$$

$\longleftarrow f_1(t) = (x_1(t), y_1(t))$

IN GENERAL $f_n(t) = (x_1(t), y_1(t))$

a) SHOW THAT $\| f_1(t) - f_0(t) \| \le \sqrt{2}$ FOR ALL t

b) SHOW THAT $\| f_2(t) - f_1(t) \| \le \sqrt{2} \cdot 3^{-1}$ FOR ALL t

c) SHOW THAT $\| f_{n+1}(t) - f_n(t) \| \le \sqrt{2}\, 3^{-n}$ FOR ALL t

d) SHOW THAT $(f_n)_n$ IS A CAUCHY-SEQUENCE OF CONTINUOUS FUNCTIONS (UNIFORM METRIC)

e) SHOW THAT $(f_n)_n$ CONVERGES UNIFORMLY TO A CONTINUOUS FUNCTION $f: [0,1] \to [0,1]^2$

f) SHOW THAT f IS SURJECTIVE

g) SHOW THAT f IS ACTUALLY PEANO'S FUNCTION