WHAT ARE WE GOING TO DO?

WE WANT TO SHOW

- $[0,1]^n$ AND $[0,1]^m$ ARE TOPOLOGICALLY DIFFERENT WHEN $n \neq m$
- THE SAME FOR $\mathbb{R}^n$ AND $\mathbb{R}^m$

WHAT DOES 'TOPOLOGICALLY DIFFERENT' OR 'TOPOLOGICALLY THE SAME' MEAN?

DEFINITION

LET (X,d) AND (Y,g) BE METRIC SPACES

A MAP $f: X \to Y$ IS CONTINUOUS IF ...

-- FOR EVERY OPEN $U \subseteq Y$ THE PREIMAGE $f^{-1}[U]$ IS OPEN.

IT IS A HOMEOMORPHISM IF IT IS BIJECTIVE AND <u>BOTH</u> f AND f^{-1} ARE CONTINUOUS.

TWO SPACES ARE <u>HOMEOMORPHIC</u> IF THERE IS A HOMEOMORPHISM BETWEEN THEM.

SO WE WILL PROVE

- IF $n \neq m$ THEN $[0,1]^n$ AND $[0,1]^m$ ARE <u>NOT</u> HOMEOMORPHIC
- IF $n \neq m$ THEN $\mathbb{R}^n$ AND $\mathbb{R}^m$ ARE <u>NOT</u> HOMEOMORPHIC

WE WILL ALSO PROVE ....

--- INVARIANCE OF DOMAIN.
IF U IS OPEN IN $\mathbb{R}^n$ AND $P \subseteq \mathbb{R}^n$ IS HOMEOMORPHIC WITH U THEN P IS ALSO OPEN.

SOME EXAMPLES

- A CONTINUOUS BIJECTION IS, IN GENERAL, NOT A HOMEOMORPHISM
  X : $\mathbb{R}$ WITH DISCRETE METRIC
  Y : $\mathbb{R}$ WITH NORMAL METRIC
  $\mathrm{ID}: X \to Y$ IS A CONTINUOUS BIJECTION
  BUT $\mathrm{ID}: Y \to X$ IS NOT CONTINUOUS

- $t \mapsto \exp(2\pi i t)$ IS A CONTINUOUS BIJECTION BETWEEN
  $[0,1)$ AND THE UNIT CIRCLE
  BUT NOT A HOMEOMORPHISM

- $\mathbb{R}$ AND $(-\pi/2, \pi/2)$
  ARE HOMEOMORPHIC
  $x \mapsto \arctan x$ WITH INVERSE $y \mapsto \tan y$

- LAST TIME: C AND $C \times C$ ARE HOMEOMORPHIC
  C IS THE CANTOR SET
  CANTOR'S DEFINITION:
  C CONSISTS OF ALL REAL NUMBERS IN $[0,1]$ WITH A TERNARY EXPANSION CONTAINING ONLY 0 AND 2.

EXERCISE

DEFINE A METRIC ON $\mathbb{N}^{\mathbb{N}}$ AS FOLLOWS

$$d(s,t) = \begin{cases} 0 & \text{IF } s = t \\ 2^{-n} & \text{IF } s \neq t \text{ AND } n = \min\{i : s_i \neq t_i\} \end{cases}$$

a) PROVE THAT d IS INDEED A METRIC

b) PROVE

$$B(s, 2^{-n}) = \{ t \in \mathbb{N}^{\mathbb{N}} : (\forall i \leq n)(s_i = t_i) \}$$

c) PROVE THAT $\mathbb{N}^{\mathbb{N}}$ AND $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ ARE HOMEOMORPHIC

d) PROVE THAT THE BIJECTION BETWEEN $\mathbb{P}$ AND $\mathbb{N}^{\mathbb{N}}$ IS A HOMEOMORPHISM

e) DEDUCE THAT $\mathbb{P}$ AND $\mathbb{P} \times \mathbb{P}$ ARE HOMEOMORPHIC.

INVARIANCE OF DOMAIN CASE $n = 1$

LET $U \subseteq \mathbb{R}$ BE OPEN AND LET $f : U \to \mathbb{R}$ BE CONTINUOUS AND INJECTIVE

THEN $f[U]$ IS OPEN IN $\mathbb{R}$.

- THERE IS A FAMILY OF PAIRWISE DISJOINT INTERVALS $\mathcal{J}$ SUCH THAT $U = \bigcup \mathcal{J}$

  PROOF: CALL x AND y IN U EQUIVALENT IF $[x, y] \subseteq U$ (OR $[y, x] \subseteq U$)

  EACH EQUIVALENCE CLASS IS AN INTERVAL (POSSIBLY UNBOUNDED)

- $f[U] = f[\bigcup \mathcal{J}] = \bigcup \{ f[I] : I \in \mathcal{J} \}$

  SO IT SUFFICES TO SHOW EACH $f[I]$ IS OPEN.

- IF $I \in J$ THEN f IS MONOTONE:

EITHER $x < y \to f(x) < f(y)$ (ALL x, y)

OR $x < y \to f(x) > f(y)$ (ALL x, y)

PROOF. ASSUME NOT, SO

THERE ARE x, y, u AND v WITH

$x < y$, $u < v$ AND

$f(x) < f(y)$, $f(u) > f(v)$

THERE ARE A FEW CASES TO CONSIDER

- $x = u$ THEN $f(v) < f(x) < f(y)$

  SO THERE IS A w BETWEEN v AND y WITH $f(w) = f(x)$

  CONTRADICTION

- $x < u$ WITH SOME SUBCASES

  – $f(x) < f(u)$ THEN

  (i) $f(x) < f(v) < f(u)$ OR

  (ii) $f(v) < f(x) < f(u)$

  (i): THERE IS $w \in (x, u)$ WITH $f(w) = f(v)$

  (ii) THERE IS $w \in (u, v)$ WITH $f(w) = f(x)$

  – $f(u) < f(x)$

  THEN THERE IS A w BETWEEN u AND y WITH $f(w) = f(x)$

- $u < x$: MUCH LIKE THE PREVIOUS CASE

SO IN ALL CASES THE INTERMEDIATE VALUE THEOREM LEADS TO A CONTRADICTION.

- FOR EACH $I \in J$ THE IMAGE $f[I]$ IS AN OPEN INTERVAL, BY THE I.V.T. AGAIN.

USEFUL RESULT

IF X IS <u>COMPACT</u> AND $f: X \to Y$ IS A CONTINUOUS BIJECTION THEN f IS A HOMEOMORPHISM

PROOF LET $g: Y \to X$ BE THE INVERSE OF f. WE CLAIM g IS CONTINUOUS

LET $U \subseteq X$ BE OPEN, WE MUST SHOW THAT $g^{-1}[U]$ IS OPEN.

LET $F = X \setminus U$. THEN F IS CLOSED AND HENCE COMPACT.

AND SO $f[F]$ IS COMPACT AND <u>HENCE</u> CLOSED.

$$\text{SO } g^{-1}[U] \overset{!}{=} f[U] = f[X \setminus F]$$
$$= f[X] \setminus f[F] \quad \text{(BIJECTION!)}$$

<u>IS OPEN</u>

EXERCISE

LET $f: [0,1]^2 \to [0,1]$ BE CONTINUOUS.

PROVE THAT f IS NOT INJECTIVE

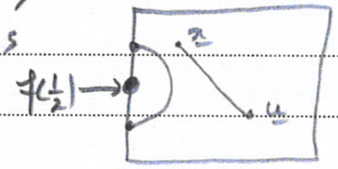
HINT LET $a = f(0,0)$ AND $b = f(1,1)$

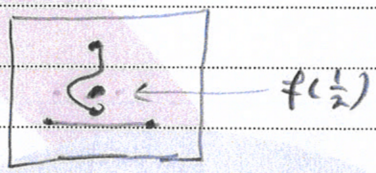
WLOG $a < b$; FIND INFINITELY MANY POINTS (x,y) WITH $f(x,y) = \frac{1}{2}(a+b)$.

SO, $[0,1]^2$ AND $[0,1]$ ARE NOT HOMEOMORPHIC.

CAN WE SEE THAT DIRECTLY?

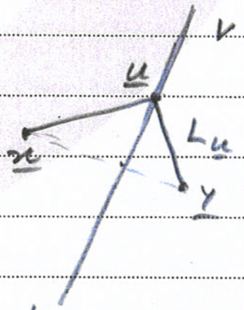
ASSUME $f : [0,1] \to [0,1]^2$
IS A HOMEOMORPHISM.
THEN $f : [0,1] \setminus \{\frac{1}{2}\} \to [0,1]^2 \setminus \{f(\frac{1}{2})\}$
IS ALSO A HOMEOMORPHISM.
BUT ① $[0,1] \setminus \{\frac{1}{2}\}$ IS NOT CONNECTED
② $[0,1]^2 \setminus \{f(\frac{1}{2})\}$ IS CONNECTED
① EASY $[0,\frac{1}{2})$, $(\frac{1}{2},1]$ SPLITS THE SPACE
② ALMOST EASY
TWO CASES





THE COMPLEMENT IS EVEN PATHWISE CONNECTED.

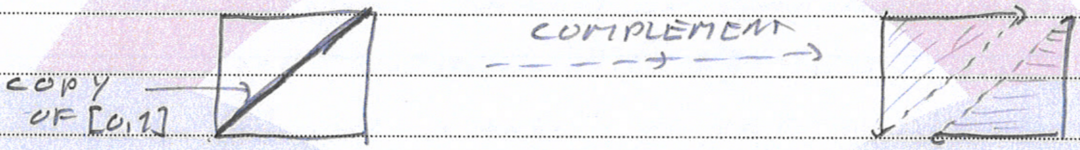
THEOREM [CANTOR]
LET $n \geq 2$, AND LET $A \subseteq \mathbb{R}^n$ BE COUNTABLE
THEN $\mathbb{R}^n \setminus A$ IS (PATHWISE) CONNECTED
PROOF
LET $\underline{x}, \underline{y} \in \mathbb{R}^n \setminus A$
LET V BE THE PERPENDICULAR BISECTING HYPERPLANE: $V = \{\underline{u} : \|\underline{x} - \underline{u}\| = \|\underline{y} - \underline{u}\|\}$
FOR $\underline{u} \in V$ LET $L_{\underline{u}}$ BE THE UNION OF THE LINE SEGMENTS $[\underline{x}, \underline{u}]$ AND $[\underline{u}, \underline{y}]$.



NOTE IF $\underline{u} \neq \underline{v}$ THEN
$L_{\underline{u}} \cap L_{\underline{v}} = \{\underline{x}, \underline{y}\}$
SO $L_{\underline{u}} \cap L_{\underline{v}} \cap A = \emptyset$
CONCLUSION: $\{\underline{u} \in V : L_{\underline{u}} \cap A \neq \emptyset\}$ IS COUNTABLE
SO THERE IS A $\underline{u} \in V$ (ALMOST ALL OF THEM) WITH $L_{\underline{u}} \cap A = \emptyset$; SO $L_{\underline{u}}$ CONNECTS $\underline{x}$ AND $\underline{y}$ THROUGH $\mathbb{R}^n \setminus A$.

WE HAVE AN 'EASY' PROPERTY THAT DISTINGUISHES $[0,1]$ FROM ALL $[0,1]^n$ $(n \geq 2)$:
"THERE IS A POINT x SUCH THAT THE SPACE MINUS THAT POINT IS NOT CONNECTED"

POSSIBLE PROPERTY FOR $[0,1]^2$:
"THERE IS A (HOMEOMORPHIC) COPY OF $[0,1]$ WHOSE COMPLEMENT IS NOT CONNECTED"



ALL WE NEED TO DO NOW IS PROVE
"NO (HOMEOMORPHIC) COPY OF $[0,1]$ IN $[0,1]^n$ $(n \geq 3)$ HAS A DISCONNECTED COMPLEMENT."

THAT IS ACTUALLY QUITE HARD TO PROVE.

WE SHALL WORK TOWARDS SOMETHING MORE GENERAL.

WE DEFINE DIMENSION FOR METRIC SPACES AND WE WILL PROVE THAT
$[0,1]^n$ AND $\mathbb{R}^n$
HAVE DIMENSION EXACTLY n.

THE DEFINITION SHOULD BE IN TERMS OF OPEN AND CLOSED SETS ONLY, SO THAT HOMEOMORPHIC SPACES GET THE SAME DIMENSION.

SOME DEFINITIONS

LET X BE A METRIC SPACE.
LET A AND B BE DISJOINT CLOSED SETS IN X.
A SET P IS A <u>PARTITION</u> BETWEEN A AND B IF $X \setminus P$ IS THE UNION OF TWO DISJOINT OPEN SETS U AND V WITH

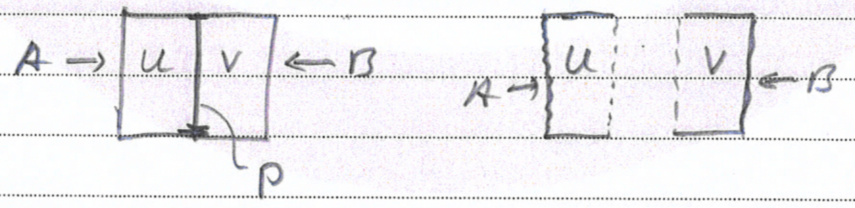
$$A \subseteq U \quad \text{AND} \quad B \subseteq V.$$

- $\{1/2\}$ IS A PARTITION BETWEEN $\{0\}$ AND $\{1\}$ IN THE SPACE $[0,1]$: $U = [0, 1/2)$, $V = (1/2, 1]$
  - $[1/3, 2/3]$ IS ALSO A PARTITION BETWEEN $\{0\}$ AND $\{1\}$: $U = [0, 1/3)$; $V = (2/3, 1]$.

- $\emptyset$ IS A PARTITION BETWEEN $\{e\}$ AND $\{\pi\}$ IN $\mathbb{P}$: $U = (-\infty, 3) \cap \mathbb{P}$; $V = (3, \infty) \cap \mathbb{P}$.

- THE y-AXIS IS A PARTITION BETWEEN THE LINES "$x = \pi$" AND "$x = -\sqrt{2}$" IN $\mathbb{R}^2$: $U = \{(x,y) : x > 0\}$, $V = \{(x,y) : x < 0\}$.

- $\{(\frac{1}{2}, y) : 0 \le y \le 1\}$ IS A PARTITION BETWEEN THE LEFT AND RIGHT SIDE OF $[0,1]^2$:
  $U = \{(x,y) : x < 1/2\}$; $V = \{(x,y) : x > 1/2\}$



2019-09-06

Definition of dimension

For metric spaces X and natural numbers n (including 0) we say

$\dim X \leq n$ if

for every family of $n+1$ pairs of disjoint closed sets $(A_0, B_0), (A_1, B_1), \ldots, (A_n, B_n)$ there are partitions

P_0 between A_0 and B_0,

P_1 between A_1 and B_1, ...

P_n between A_n and B_n.

Such that $P_0 \cap P_1 \cap \ldots \cap P_n = \emptyset$.

We say $\dim X = n$ if $\dim X \leq n$ and <u>not</u> $\dim X \leq n-1$

$\dim X < 0$ would mean $X = \emptyset$

so for us $\dim X \leq 0$ and $\dim X = 0$ mean the same: for every pair (A, B) of disjoint closed sets the empty set is a partition between A and B

In that case we have disjoint open sets U and V with

$A \subseteq U$, $B \subseteq V$ and $X = U \cup V$

so U and V are closed as well, they are <u>cl</u>osed-and-<u>open</u>, in short <u>clopen</u>

Exercise prove $\dim \mathbb{Q} = 0$

and $\dim \mathbb{P} = 0$

- DIM $[0,1] \geq 1$ BECAUSE $[0,1]$ IS CONNECTED.
- DIM $[0,1] \leq 1$.

LET (A_0, B_0) AND (A_1, B_1) BE TWO PAIRS OF DISJOINT CLOSED SETS.
FOR EVERY POINT x IN A_0 TAKE TWO RATIONAL NUMBERS p_x AND q_x SUCH THAT $x \in (p_x, q_x)$ AND

$$[p_x, q_x] \cap B_0 = \emptyset.$$

BY COMPACTNESS WE CAN TAKE A FINITE SET $F_0 \subseteq A_0$ SUCH THAT $A_0 \subseteq \bigcup_{x \in F_0} (p_x, q_x)$
LET $U_0 = \bigcup_{x \in F_0} (p_x, q_x)$
AND $V_0 = [0,1] \setminus \bigcup_{x \in F_0} [p_x, q_x]$
THEN $U_0 \cap V_0 = \emptyset$, $A_0 \subseteq U_0$ AND $B \subseteq V_0$
SO $P_0 = [0,1] \setminus (U_0 \cup V_0)$ IS A PARTITION BETWEEN A_0 AND B_0.
NOTE THAT $P_0 \subseteq G_0 = \bigcup_{x \in F_0} \{p_x, q_x\}$
DO THE SAME THING FOR A_1 AND B_1 BUT CHOOSE THE p_x AND q_x FOR $x \in A_1$ IN $\mathbb{Q} \setminus G$.
WE GET U_1, V_1 AND $P_1 \subseteq G_1$
BY CONSTRUCTION

$$P_0 \cap P_1 \subseteq G_0 \cap G_1 = \emptyset.$$