ALSO IF $x \in F_i$ THEN $f_i(x) = 0$
SO $f(x) \in A_i$; THIS IMPLIES
THAT $F_i \subseteq f^{-1}[U_i]$
AND LIKEWISE $G_i \subseteq f^{-1}[V_i]$

- SO, IF WE HAVE PARTITIONS $P_1, \ldots, P_n$ BETWEEN A_1 AND B_1, ..., A_n AND B_n THEN FOR ALL i THE PREIMAGE $f^{-1}[P_i]$ IS A PARTITION BETWEEN F_i AND G_i
BUT THEN $\bigcap_{i=1}^{n} f^{-1}[P_i] \neq \emptyset$
BUT $\bigcap_{i=1}^{n} f^{-1}[P_i] = f^{-1}[\bigcap_{i=1}^{n} P_i]$
SO $\bigcap_{i=1}^{n} P_i \neq \emptyset$.

So ⊛ IS ESSENTIALLY THE ONLY WAY TO SHOW THAT $\dim [0,1]^n \geq n$.

How DO WE PROVE ⊛?

WE LOOK FOR SOME EQUIVALENT STATEMENTS AND HOPE FOR ONE THAT WE CAN PROVE.

① IF $f: [0,1]^n \to [0,1]^n$ IS CONTINUOUS THEN THERE IS AN $\underline{x} \in [0,1]^n$ SUCH THAT $f(\underline{x}) = \underline{x}$
[BROUWER'S FIXED-POINT THEOREM]

② THERE IS NO CONTINUOUS MAP $f: B^n \to B^n$ SUCH THAT $f[B_n] \subseteq S^{n-1}$ AND $f(x) = x$ FOR ALL $x \in S^{n-1}$ [NO-RETRACTION THEOREM]

- $B^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$
- $S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$.

EXERCISE PROVE THAT $[0,1]^n$ AND B^n ARE HOMEOMORPHIC USING A HOMEOMORPHISM THAT MAPS $[0,1]^n \setminus (0,1)^n$ ONTO S^{n-1}

$\circledast \Rightarrow$ ② ASSUME $f: [0,1]^n \to [0,1]^n$ IS CONTINUOUS AND SUCH THAT

- $f[[0,1]^n] \subseteq S^{n-1}$ $\quad (= [0,1]^n \setminus (0,1)^n)$
- $f(x) = x$ IF $x \in S^{n-1}$

NOTE THAT THE SETS A_i AND B_i ARE SUBSETS OF S^{n-1}.

ALSO: $Q_i = \{x \in S^{n-1} : x_i = 1/2\}$ IS A PARTITION BETWEEN A_i AND B_i IN S^{n-1}

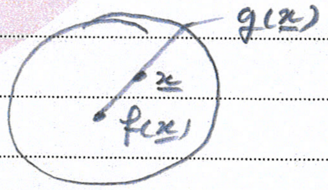
THEN $P_i = f^{-1}[Q_i]$ IS A PARTITION BETWEEN A_i AND B_i IN $[0,1]^n$.
(EXERCISE: CHECK THIS.)

BUT $\bigcap_{i=1}^n Q_i = \emptyset$ AND SO $\bigcap_{i=1}^n P_i = \emptyset$

THIS CONTRADICTS $\circledast$.
SO THERE IS NO SUCH f.

② $\Rightarrow$ ① ASSUME $f: B^n \to B^n$ IS CONTINUOUS AND THAT $f(x) \neq x$ FOR ALL x

DEFINE $g: B^n \to S^{n-1}$ AS IN THE PICTURE $\longrightarrow$



THIS DEFINES A CONTINUOUS MAP SUCH THAT
- $g[B^n] \subseteq S^{n-1}$ AND - $g(x) = x$ IF $x \in S^{n-1}$

CONTRADICTION

EXERCISE: SHOW THAT g IS CONTINUOUS

① $\Rightarrow$ ② IF THERE IS A CONTINUOUS $f: B^n \to S^{n-1}$ SUCH THAT $f(x)=x$ FOR ALL $x \in S^{n-1}$ THEN $-f: B^n \to S^{n-1} \subseteq B^n$ IS ALSO CONTINUOUS AND SATISFIES $-f(x) \neq x$ FOR ALL $x \in B^n$

① $\Rightarrow$ ⊛ ASSUME WE HAVE PARTITIONS $P_1, \dots, P_n$ BETWEEN A_1 AND B_1, ..., A_n AND B_n WITH $\bigcap_{i=1}^{n} P_i = \emptyset$.

FOR EACH i TAKE A CONTINUOUS FUNCTION $f_i: [0,1]^n \to [0,1]$ SUCH THAT

- $x \in A_i \iff f_i(x) = 1$
- $x \in P_i \iff f_i(x) = \frac{1}{2}$
- $x \in B_i \iff f_i(x) = 0$

DEFINE $f: [0,1]^n \to [0,1]^n$ BY

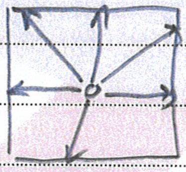
$$f(x) = (f_1(x), \dots, f_n(x))$$

BECAUSE $\bigcap_{i=1}^{n} P_i \neq \emptyset$ WE FIND THAT

$$f(x) \neq (\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2})$$

FOR ALL x

LET g BE THE CENTRAL PROJECTION OF $[0,1]^2 \setminus \{(\frac{1}{2}, \dots, \frac{1}{2})\}$ ONTO S^{n-1}



THEN g IS CONTINUOUS AND

$$(g \circ f)(x) \neq x$$

FOR ALL x

EXERCISE :
- CONSTRUCT THE FUNCTIONS f_i
- SHOW THAT g IS CONTINUOUS
- SHOW THAT $(g \circ f)(x) \neq x$ FOR ALL x

We are going to prove Brouwer's Fixed-point Theorem

If $f : [0,1]^n \to [0,1]^n$ is continuous

then there is an $\underline{x} \in [0,1]^n$ with $f(\underline{x}) = \underline{x}$

Exercise: prove this for $n=1$.

We work with simplexes.

Let $\{\underline{a}_0, \dots, \underline{a}_k\}$ be a set of points in $\mathbb{R}^n$

We say the set is affinely independent (or in general position) if the set

$\{\underline{a}_1 - \underline{a}_0, \dots, \underline{a}_k - \underline{a}_0\}$

is linearly independent.

Lemma $\{\underline{a}_0, \dots, \underline{a}_k\}$ is affinely independent if and only if

$\lambda_0 \underline{a}_0 + \lambda_1 \underline{a}_1 + \dots + \lambda_k \underline{a}_k = \underline{0}$ and $\lambda_0 + \lambda_1 + \dots + \lambda_k = 0$

imply that $\lambda_0 = \lambda_1 = \dots = \lambda_k = 0$.

Proof "$\Rightarrow$" Assume $\lambda_0, \lambda_1, \dots, \lambda_k$ are such that

$\lambda_0 + \lambda_1 + \dots + \lambda_m = 0$ and $\lambda_0 \underline{a}_0 + \lambda_1 \underline{a}_1 + \dots + \lambda_k \underline{a}_k = 0$

Then $\lambda_0 = -\lambda_1 - \dots - \lambda_k$ so we get

$\lambda_1 \underline{a}_1 + \dots + \lambda_k \underline{a}_k - \lambda_1 \underline{a}_0 - \dots - \lambda_k \underline{a}_0 = 0$

or $\lambda_1 (\underline{a}_1 - \underline{a}_0) + \dots + \lambda_k (\underline{a}_k - \underline{a}_0) = 0$

But then $\lambda_1 = \dots = \lambda_k = 0$ [linear independence]

and also $\lambda_0 = 0$.

"$\Leftarrow$" Assume $\mu_1 (\underline{a}_1 - \underline{a}_0) + \dots + \mu_k (\underline{a}_k - \underline{a}_0) = 0$

or $-(\mu_1 + \dots + \mu_k)\underline{a}_0 + \mu_1 \underline{a}_1 + \dots + \mu_k \underline{a}_k = 0$

now $-(\mu_1 + \dots + \mu_k) + \mu_1 + \dots + \mu_k = 0$

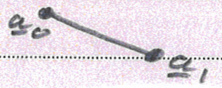
so $\mu_1 = \dots = \mu_k = -(\mu_1 + \dots + \mu_k) = 0$ [affine independence]

LET $\{a_0, \dots, a_k\}$ BE AFFINELY INDEPENDENT IN $\mathbb{R}^n$.
THE k-DIMENSIONAL **SIMPLEX** SPANNED BY THIS SET IS THE SET OF POINTS OF THE FORM

$$\lambda_0 a_0 + \lambda_1 a_1 + \dots + \lambda_k a_k$$

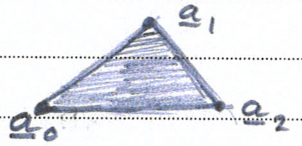
WHERE $\lambda_0 + \lambda_1 + \dots + \lambda_k = 1$ AND $\lambda_0 \geq 0, \lambda_1 \geq 0, \dots, \lambda_k \geq 0$.

$k = 1$:



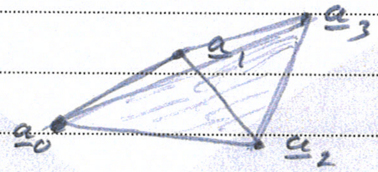
LINE SEGMENT

$k = 2$:



TRIANGLE

$k = 3$:



TETRAHEDRON

NOTATION $[a_0, a_1, \dots, a_k]$

IF $\{a_{i_0}, \dots, a_{i_\ell}\} \subseteq \{a_0, \dots, a_k\}$ (ALL i_j DIFFERENT)
THEN $\{a_{i_0}, \dots, a_{i_\ell}\}$ IS AFFINELY INDEPENDENT
IT SPANS A SIMPLEX $[a_{i_0}, \dots, a_{i_\ell}]$ OF COURSE
IT IS CALLED A FACE OF $[a_0, \dots, a_k]$
AN ℓ-DIMENSIONAL FACE

NOTE $[a_{i_0}, \dots, a_{i_\ell}] = \{\lambda_0 a_0 + \lambda_1 a_1 + \dots + \lambda_k a_k : \lambda_i = 0,\ i \notin \{i_0, \dots, i_\ell\}\}$

LEMMA: IF $x \in [a_0, \dots, a_k]$
AND $x = \lambda_0 a_0 + \dots + \lambda_k a_k = \mu_0 a_0 + \dots + \mu_k a_k$
WITH $\lambda_0 + \dots + \lambda_k = 1$ AND $\mu_0 + \dots + \mu_k = 1$
THEN $\lambda_0 = \mu_0, \lambda_1 = \mu_1, \dots, \lambda_k = \mu_k$

PROOF $(\lambda_0 - \mu_0) a_0 + \dots + (\lambda_k - \mu_k) a_k = 0$
AND $(\lambda_0 - \mu_0) + \dots + (\lambda_k - \mu_k) = 0$
AND SO $\lambda_0 - \mu_0 = \dots = \lambda_k - \mu_k = 0$

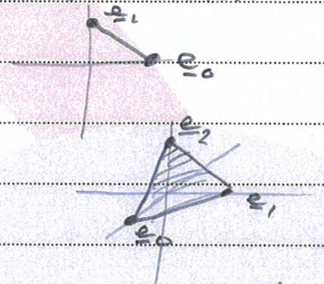
THE NUMBERS $\lambda_0, \lambda_1, \dots, \lambda_k$ IN $\underline{x} = \lambda_0 \underline{a}_0 + \dots + \lambda_k \underline{a}_k$ ARE THE <u>BARYCENTRIC</u> <u>COORDINATES</u> OF $\underline{x}$

WE WRITE $\lambda_0(\underline{x}), \dots, \lambda_k(\underline{x})$

THEOREM $[\underline{a}_0, \dots, \underline{a}_k]$ IS COMPACT AND THE COORDINATE FUNCTIONS $\lambda_0, \lambda_1, \dots, \lambda_k$ ARE CONTINUOUS.

PROOF: IN $\mathbb{R}^{k+1}$ LET $\underline{e}_0, \underline{e}_1, \dots, \underline{e}_k$ BE THE STANDARD UNIT VECTORS.

THESE ARE AFFINELY INDEPENDENT.



IF $\underline{x} \in [\underline{e}_0, \dots, \underline{e}_k]$

THEN $\underline{x} = x_0 \underline{e}_0 + \dots + x_k \underline{e}_k$ AND $x_0 + x_1 + \dots + x_k = 1$

SO FOR THIS (STANDARD) SIMPLEX WE HAVE $\lambda_i(\underline{x}) = x_i$.

THE SET $[\underline{e}_0, \dots, \underline{e}_k]$ IS CLOSED AND BOUNDED HENCE COMPACT.

THE MAP
$$\Lambda : [\underline{e}_0, \dots, \underline{e}_k] \longrightarrow [\underline{a}_0, \dots, \underline{a}_k]$$
$$\underline{x} \longmapsto x_0 \underline{a}_0 + \dots + x_k \underline{a}_k$$
IS CONTINUOUS AND INJECTIVE.

THE MAP IS ALSO ONTO BECAUSE $\Lambda(\underline{e}_i) = \underline{a}_i$ FOR ALL i.

[SIMPLE LINEAR ALGEBRA!]

SO $[\underline{a}_0, \dots, \underline{a}_k]$ IS COMPACT AS WELL.

BY COMPACTNESS $\Lambda^{-1} : [\underline{a}_0, \dots, \underline{a}_k] \longrightarrow [\underline{e}_0, \dots, \underline{e}_k]$ IS ALSO CONTINUOUS.

BUT $\Lambda^{-1}(\underline{x}) = (\lambda_0(\underline{x}), \lambda_1(\underline{x}), \dots, \lambda_k(\underline{x}))$

SO $\lambda_0, \lambda_1, \dots, \lambda_k$ ARE ALL CONTINUOUS.

EXERCISE: FIND FORMULAS FOR THE FUNCTIONS λ_i.

ANY TWO k-DIMENSIONAL SIMPLEXES ARE HOMEOMORPHIC