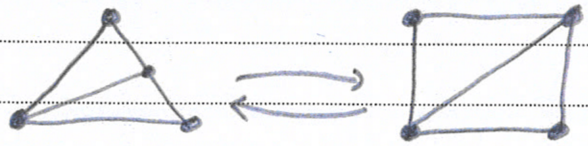
EXERCISE: EVERY 2-DIMENSIONAL SIMPLEX IS HOMEOMORPHIC WITH $[0,1]^2$.

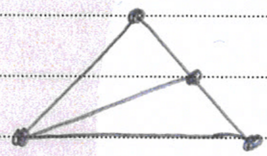
HINT



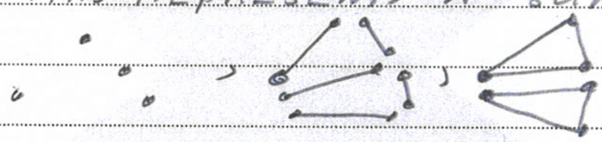
SUBDIVISIONS

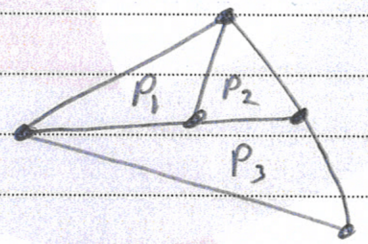
A SIMPLICIAL SUBDIVISION OF A SIMPLEX S IS A FAMILY $\mathcal{P}$ OF SIMPLEXES SUCH THAT

- $\mathcal{P}$ IS FINITE
- $S = \bigcup \mathcal{P}$
- IF $P, Q \in \mathcal{P}$ THEN $P \cap Q = \emptyset$ OR $P \cap Q$ IS A COMMON FACE OF P AND Q
- IF $P \in \mathcal{P}$ THEN ALL FACES OF P ARE IN $\mathcal{P}$



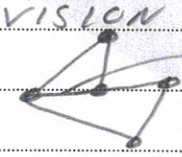
THIS REPRESENTS A SUBDIVISION:





THIS DOES NOT REPRESENT A SUBDIVISION

$P_1 \cap P_3$:



NOT A COMMON FACE, NOT A FACE OF P_3

THE MESH OF A SUBDIVISION IS

$$\max\{\operatorname{diam} P : P \in \mathcal{P}\}$$

WHERE, IN GENERAL, $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$

EXERCISE PROVE

$$\operatorname{diam}[a_0, \dots, a_n] = \operatorname{diam}\{a_0, \dots, a_n\}$$

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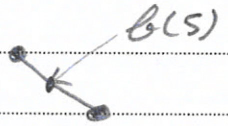
(2)

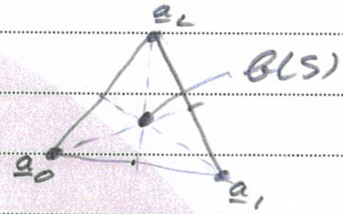
THE BARYCENTER $b(S)$ OF A SIMPLEX $S = [a_0, \dots, a_k]$ IS THE POINT

$$\frac{1}{k+1} a_0 + \frac{1}{k+1} a_1 + \dots + \frac{1}{k+1} a_k$$

$k = 1$ MIDPOINT OF $[a_0, a_1]$



$k = 2$ CENTROID OF $[a_0, a_1, a_2]$



BARYCENTRIC SUBDIVISIONS

LET $S = [a_0, \dots, a_k]$ BE A SIMPLEX

LET $P_0 \supset P_1 \supset \dots \supset P_\ell$ BE A DECREASING SEQUENCE OF FACES OF S.

THEN THE BARYCENTERS $b(P_0), b(P_1), \dots, b(P_\ell)$ ARE AFFINELY INDEPENDENT.

- EACH P_i CORRESPONDS TO A SUBSET F_i OF $\{a_0, \dots, a_k\}$ AND $F_0 \supset F_1 \supset \dots \supset F_\ell$; EXPAND THIS TO A SEQUENCE $G_0 \supset G_1 \supset \dots \supset G_k$ OF SUBSETS SUCH THAT $G_0 = \{a_{i_0}, \dots, a_{i_k}\}$, $G_1 = \{a_{i_1}, \dots, a_{i_k}\}, \dots,$ $G_k = \{a_{i_k}\}$ WHERE $j \mapsto i_j$ IS A PERMUTATION OF $\{0, 1, \dots, k\}$
- SO WLOG $\ell = k$. AND

$$b(P_0) = \frac{1}{k+1} a_{i_0} + \frac{1}{k+1} a_{i_1} + \dots + \frac{1}{k+1} a_{i_k}$$

$$b(P_1) = \frac{1}{k} a_{i_1} + \dots + \frac{1}{k} a_{i_k}$$

$$\vdots$$

$$b(P_k) = a_{i_k}$$

ASSUME $\mu_0 b(P_0) + \dots + \mu_k b(P_k) = 0$

AND $\mu_0 + \dots + \mu_k = 0$

WE GET

$$\mu_0 \frac{1}{k+1} a_{i_0} + \left(\mu_0 \frac{1}{k+1} + \mu_1 \frac{1}{k}\right) a_{i_1} + \left(\mu_0 \frac{1}{k+1} + \dots + \mu_k\right) a_{i_k} = 0.$$

AND SO $\mu_0 \frac{1}{k+1} = 0$, $\mu_0 \frac{1}{k+1} + \mu_1 \frac{1}{k} = 0$, $\dots$,

$\mu_0 \frac{1}{k+1} + \mu_1 \frac{1}{k} + \dots + \mu_k = 0$

OR

$$\begin{pmatrix} 1/k+1 & 0 & 0 & \cdots & 0 \\ 1/k+1 & 1/k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ & & & & 0 \\ 1/k+1 & 1/k & \cdots & & 1 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

AND SO $\mu_0 = \mu_1 = \dots = \mu_k = 0$

- THE SIMPLEX $[b(P_0) \dots b(P_k)]$ IS EQUAL TO THE SET

$$\{ x \in S : \lambda_{i_0}(x) \le \lambda_{i_1}(x) \le \dots \le \lambda_{i_k}(x) \}$$

NOTE IN THE PREVIOUS POINT THE μ_i AND λ_i IN

$$\mu_0 b(P_0) + \mu_1 b(P_1) + \dots + \mu_k (P_k) = \lambda_0 a_0 + \lambda_1 a_1 + \dots + \lambda_k a_k$$

SATISFY

$$\begin{pmatrix} \frac{1}{k+1} & 0 & 0 & \cdots & 0 \\ \frac{1}{k+1} & \frac{1}{k} & & \ddots & \vdots \\ \vdots & \vdots & & & 0 \\ \frac{1}{k+1} & \frac{1}{k} & \cdots & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \lambda_{i_0} \\ \lambda_{i_1} \\ \vdots \\ \lambda_{i_k} \end{pmatrix}$$

OR FROM THE BOTTOM TO THE TOP

$1 \cdot \mu_k = \lambda_{i_k} - \lambda_{i_{k-1}}$, $\frac{1}{2} \mu_{k-1} = \lambda_{i_{k-1}} - \lambda_{i_{k-2}}$, $\dots$,

$\dots$, $\frac{1}{k} \mu_1 = \lambda_{i_1} - \lambda_{i_0}$, $\frac{1}{k+1} \mu_0 = \lambda_{i_0}$

So $\lambda_{i_k} - \lambda_{i_{k-1}} \ge 0$, $\lambda_{i_{k-1}} - \lambda_{i_{k-2}} \ge 0$, $\dots$, $\lambda_{i_1} - \lambda_{i_0} \ge 0$, $\lambda_{i_0} \ge 0$

- THE FAMILY OF ALL SIMPLEXES OF THE FORM $[b(P_0), \dots, b(P_\ell)]$ IS A SIMPLICIAL SUBDIVISION OF $[a_0, a_1, \dots, a_k]$
  - THE UNION IS ALL OF S: IF $x \in S$ TAKE A PERMUTATION OF $\{0, \dots, k\}$ WITH $\lambda_{i_k}(x) \ge \dots \ge \lambda_{i_0}(x)$
  - EVERY SIMPLEX IS OF THE FORM $[b(P_{j_0}), \dots, b(P_{j_\ell})]$ WHERE $i_0, \dots, i_k$ IS A PERMUTATION OF $\{0, \dots, k\}$ THAT GIVES A FULL SEQUENCE $P_0 \supset P_1 \supset \dots \supset P_k$ AND OUR SIMPLEX IS THE FACE OF $[b(P_0) \dots b(P_k)]$ GIVEN BY $\mu_j = 0$ IF $j \notin \{j_0, \dots, j_\ell\}$

BUT $\mu_j = 0$ MEANS $\lambda_{i_j} - \lambda_{i_{j-1}} = 0$ OR $\lambda_{i_j} = \lambda_{i_{j-1}}$

SO $[b(P_{j_0}) \ldots, b(P_{j_e})]$ IS GIVEN BY A SET OF EQUALITIES OF THE FORM $\lambda_i = \lambda_j$, AND $\lambda_0 = 0$

IF WE HAVE TWO SUCH SIMPLEXES P AND Q AND $P \cap Q \neq \emptyset$ THEN WE TAKE THE EQUALITIES TOGETHER; EVERY $x \in P \cap Q$ SATISFIES THOSE INEQUALITIES AND ALSO THE SAME INEQUALITIES

WE CAN TAKE A PERMUTATION OF $\{0, 1, \ldots, k\}$ WITH $\lambda_{i_k}(x) \geq \lambda_{i_{k-1}}(x) \geq \ldots \geq \lambda_{i_0}(x)$

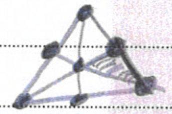
AND $\lambda_{i_j}(x) = \lambda_{i_{j-1}}(x)$ FOR SUITABLE j.

SO $P \cap Q$ IS AGAIN OF THE GIVEN FORM.

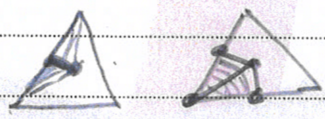
- LET $P = b(P_0) \ldots b(P_{k-1})$ BE A $k-1$-DIMENSIONAL SIMPLEX.

CASE 1 $P_0 \neq S$ THEN P IS CONTAINED IN P_0 AND IT IS A FACE OF JUST ONE k-DIMENSIONAL SIMPLEX IN THIS SUBDIVISION!

$[b(S), b[P_0], \ldots, b(P_{k-1})]$



CASE 2 $P_0 = S$



WE HAVE A PERMUTATION SUCH THAT

$P_0 = [a_{i_0}, \ldots, a_{i_k}]$

$P_1 = [a_{i_1}, \ldots, a_{i_k}]$

$\lambda_{i_j} = \lambda_{i_{j-1}}$ $P_{j-1} = [a_{i_{j-1}}, \ldots, a_{i_k}]$ $P_j = [a_{i_{j+1}}, \ldots, a_{i_k}]$

OR $\lambda_{i_k} = 0$ $P_0 = [a_{i_0}, \ldots, a_{i_k}] \ldots P_{k-1} = [a_{i_{k-1}}, a_{i_k}]$

IN EITHER CASE P IS A FACE OF EXACTLY TWO k-DIMENSIONAL SIMPLEXES:

- INSERT $[a_{i_j}, \ldots, a_{i_k}]$ BETWEEN P_{j-1} AND P_j

OR WRITE $P_{j-1} = [a_{i_j}, a_{i_{j-1}}, a_{i_{j+1}} \ldots a_{i_k}]$

AND INSERT $[a_{i_{j-1}}, a_{i_{j+1}} \ldots a_{i_k}]$

- SECOND CASE. $P_k = [a_{i_{k-1}}]$ OR $P_k = [a_{i_k}]$.

LEMMA THE MESH OF THE BARYCENTRIC SUBDIVISION IS NOT LARGER THAN $\frac{k}{k+1}$ DIAM S.

PROOF FIRST

IF $\underline{x}, \underline{y} \in [a_0, \ldots, a_k]$

THEN $\|\underline{x} - \underline{y}\| \leq \max_{i \leq k} \|\underline{a}_i - \underline{y}\|$

FOR $\underline{x} - \underline{y} = \lambda_0 \underline{a}_0 + \cdots + \lambda_k \underline{a}_k - \underline{y}$

$= \lambda_0(\underline{a}_0 - \underline{y}) + \cdots + \lambda_k(\underline{a}_k - \underline{y})$

SO $\|\underline{x} - \underline{y}\| \leq (\lambda_0 + \cdots + \lambda_k) \max_{i \leq k} \|\underline{a}_i - \underline{y}\|$

$= \max_{i \leq k} \|\underline{a}_i - \underline{y}\|$

SECOND TAKE A PERMUTATION AND $l < m \leq k$ AND TWO BARYCENTERS

$\underline{b}_1 = \frac{1}{l+1}(a_{i_0} + \cdots + a_{i_l})$ AND $\underline{b}_2 = \frac{1}{m+1}(\underline{a}_{i_0} + \cdots + a_{i_m})$

NOW $\|\underline{b}_1 - \underline{b}_2\| \leq \max_{j \leq m} \|a_{i_j} - \underline{b}_2\|$ (WORK IN $[a_{i_0} \ldots a_{i_m}]$)

TAKE SUCH A j:

$\|a_{i_j} - \underline{b}_2\| = \|a_{i_j} - \frac{1}{m+1}(a_{i_0} + \cdots + a_{i_m})\|$

$= \|\frac{1}{m+1}((a_{i_j} - a_{i_0}) + (a_{i_j} - a_{i_1}) + \cdots + (a_{i_j} - a_{i_m}))\|$

$j \leq m$ SO ONE TERM IS ZERO $\leq \frac{1}{m+1} \cdot m \cdot \text{DIAM } S$

COROLLARY:

LET $\varepsilon > 0$. THERE IS AN l SUCH THAT THE MESH OF THE lTH BARYCENTRIC SUBDIVISION IS SMALLER THAN ε.

THE BARYCENTRIC SUBDIVISION IS THE FIRST. TAKING THE BARYCENTRIC SUBDIVISION OF ALL SIMPLEXES IN THE FIRST GIVES US THE SECOND BARYCENTRIC SUBDIVISION, ETC.