Combinatorics: Labelings and Sperner's Lemma

Let S be a k-dimensional simplex $[a_0, a_1, \dots, a_k]$
Let P be the l-th barycentric subdivision of S, and let V be the set of all vertices of simplexes in P.

A good labeling of V is a function

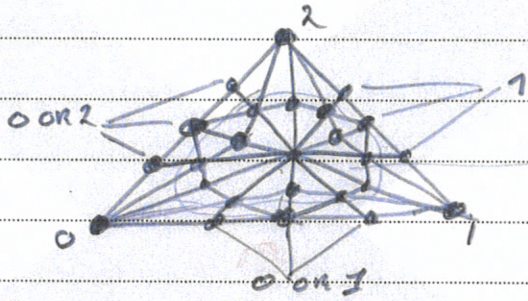
$$h : V \longrightarrow \{0, 1, \dots, k\}$$

such that: if $x \in [a_{i_0} \dots a_{i_l}]$
then $h(x) \in \{i_0, \dots, i_l\}$

So, $h(a_i) = i$ always

- $k = 1$:

- $k = 2$



The rest: 0, 1, or 2

Sperner's Lemma

If h is a good labeling then the number of simplexes on which h is surjective (full simplexes) is odd.

Proof: Induction on the dimension k

$k = 1$: The l-th subdivision of a segment $[a_0, a_1]$ consists of $1 + 2 + \dots + 2^l$ points plus a_0 and a_1, enumerate the points left to right: $a_0 = x_0 < x_1 < \dots < x_{2^l} = a_0$

Now $1 = h(a_1) - h(a_0)$

$$= \sum_{i=1}^{2^l} h(x_i) - h(x_{i-1})$$

NOW $h(x_i) - h(x_{i-1})$ CAN BE 0, 1 OR -1

1 MEANS $h(x_{i-1}) = 0$ $h(x_i) = 1$: $[x_{i-1}, x_i]$ IS FULL

-1 MEANS $h(x_{i-1}) = 1$ $h(x_i) = 0$: $[x_{i-1}, x_i]$ IS FULL

0 MEANS $h(x_{i-1}) = h(x_i)$: $[x_{i-1}, x_i]$ IS NOT FULL

THE SUM IS EQUAL TO 1 SO WE HAVE ONE MORE 1S THAN -1S

SO THE NUMBER OF FULL SIMPLEXES IS OF THE FORM $2n+1$, HENCE ODD.

$k \to k+1$

LET $[a_0, \dots, a_k, a_{k+1}]$ BE A k-DIMENSIONAL SIMPLEX AND LET V BE THE SET OF VERTICES IN THE ℓ^{TH} BARYCENTRIC SUBDIVISION.

LET $h : V \longrightarrow \{0, \dots, k, k+1\}$ BE A GOOD LABELING.

LET $W = V \cap [a_0, \dots, a_k]$ THE VERTICES THAT ARE IN THE FACE $[a_0, \dots, a_k]$

BECAUSE h IS GOOD THE RESTRICTION g OF h TO W IS A GOOD LABELING $g : W \to \{0, \dots, k\}$

WE DEFINE FOUR SETS

- L_1 : THE FULL SIMPLEXES FOR g
  THESE ARE IN THE FACE $[a_0, \dots, a_k]$
- L_2 : THE k-DIMENSIONAL SIMPLEXES P
  THAT ARE NOT IN $[a_0, \dots, a_k]$
  AND SATISFY $h[P] = \{0, \dots, k\}$
- D_1 : THE FULL SIMPLEXES FOR h.
- D_2 : THE $k+1$-DIMENSIONAL SIMPLEXES
  WITH $h[P] = \{0, \dots, k\}$

CONSIDER THE CARTESIAN PRODUCT

$(L_1 \cup L_2) \times (D_1 \times D_2)$

	D_1	D_2
L_1		*
L_2	*	*

LET I BE THE SET OF PAIRS (P, Q)
WHERE P IS A FACE OF Q
WE COUNT I TWICE

- ROW BY ROW
  EACH $P \in L_1$ IS FACE OF EXACTLY ONE $k+1$-DIMENSIONAL SIMPLEX IN D_1 OR D_2
  EACH $P \in L_2$ IS FACE OF EXACTLY TWO $k+1$-DIMENSIONAL SIMPLEXES, BOTH IN D_1 OR D_2
  WE FIND $|I| = |L_1| + 2|L_2|$
- COLUMN BY COLUMN
  EACH Q IN D_1 HAS EXACTLY ONE FACE IN $L_1 \cup L_2$: h IS A BIJECTION ON ITS SET OF VERTICES
  EACH Q IN D_2 HAS EXACTLY TWO FACES IN $L_1 \cup L_2$: Q HAS $k+2$ VERTICES AND THESE HAVE $k+1$ VALUES UNDER h
  EXACTLY ONE VALUE OCCURS TWICE, LEADING TO TWO FACES IN $L_1 \cup L_2$.
  WE FIND $|I| = |D_1| + 2|D_2|$

SO $|L_1| + 2|L_2| = |D_1| + 2|D_2|$
AS $|L_1|$ IS ODD (INDUCTIVE ASSUMPTION)
HENCE SO IS $|D_1|$.

THEOREM

LET S BE A k-DIMENSIONAL SIMPLEX

LET $f: S \to S$ BE CONTINUOUS

THEN THERE IS AN $x \in S$ SUCH THAT $f(x) = x$

PROOF

- FOR $i = 0, \dots, k$ LET

$$F_i = \{ x \in S : \lambda_i(x) \geq \lambda_i(f(x)) \}$$

- $\lambda_i(a_i) = 1 \geq \lambda_i(f(a_i))$ SO $a_i \in F_i$.
- IF $x \in [a_i, a_j]$ THEN $x = \lambda_i(x) \cdot a_i + \lambda_j(x) \cdot a_j$

SO $\lambda_i(x) + \lambda_j(x) = 1 \geq \lambda_i(f(x)) + \lambda_j(f(x))$

WE FIND THAT $x \in F_i$ OR $x \in F_j$.

- IF $x \in [a_{i_0} \dots a_{i_\ell}]$ THEN

$$x = \sum_{j=0}^{\ell} \lambda_{i_j}(x) \cdot a_{i_j}$$

HENCE $\sum_{j=0}^{\ell} \lambda_{i_j}(x) = 1 \geq \sum_{j=0}^{\ell} \lambda_{i_j}(f(x))$

AND SO $x \in \bigcup_{j=0}^{\ell} F_{i_j}$

WE SHOW $\bigcap_{i=0}^{k} F_i \neq \emptyset$

FOR IF $\lambda_i(x) \geq \lambda_i(f(x))$ FOR ALL i

THEN $\lambda_i(x) = \lambda_i(f(x))$ FOR ALL i

BECAUSE $\lambda_0(x) + \dots + \lambda_k(x) = \lambda_0(f(x)) + \dots + \lambda_k(f(x))$

FOR EVERY ℓ DEFINE A GOOD LABELING

$$h_\ell : V_\ell \to \{0, \dots, k\},$$

WHERE V_ℓ IS THE SET OF VERTICES OF THE ℓTH BARYCENTRIC SUBDIVISION OF S.

FOR EVERY $v \in V_\ell$ LET $h_\ell(v)$ BE SUCH THAT $v \in F_{h_\ell(v)}$

AND IN SUCH A WAY THAT

IF $v \in [a_{i_0}, \dots, a_{i_\ell}]$ THEN $h_\ell(v) \in \{i_0, \dots, i_\ell\}$

THIS IS POSSIBLE BY THE ABOVE OBSERVATIONS.

FOR EVERY ℓ THERE IS A FULL SIMPLEX P_ℓ FOR h_ℓ

SO $P_\ell \cap F_i \neq \emptyset$ FOR ALL i.

For each ℓ let x_ℓ be a vertex of P_ℓ.
By the Bolzano-Weierstrass theorem
a subsequence of $(x_\ell : \ell \in \mathbb{N})$ converges
to a point x.
Let $\varepsilon > 0$. Take ℓ so large that

- $\|x - x_\ell\| < \varepsilon/3$
- $\left(\frac{k}{k+1}\right)^\ell < \varepsilon/3$

Then $P_\ell \subseteq B(x, \varepsilon)$ by the triangle
inequality and $\operatorname{diam} P_\ell \leq \left(\frac{k}{k+1}\right)^\ell$.
But then $B(x,\varepsilon) \cap F_i \neq \emptyset$ for all i.
We see that $x \in \bigcap_{i=0}^{k} \overline{F_i}$.
But the F_i are closed so $x \in \bigcap_{i=0}^{k} F_i$.

Exercise.
Show that a k-dimensional simplex
is homeomorphic to $[0,1]^k$.

So, now we know $\dim [0,1]^n = n$ for all n
and so $[0,1]^n$ and $[0,1]^m$ are
homeomorphic if and only if $n = m$.
And likewise for the $\mathbb{R}^n$.