TODAY: A WEAK FORM OF INVARIANCE OF DOMAIN.

THEOREM:
LET $A \subseteq \mathbb{R}^n$.
THEN $\dim A \leq n-1$ IF AND ONLY IF $\operatorname{Int} A = \emptyset$.

ONE DIRECTION IS EASY:
IF $\operatorname{Int} A \neq \emptyset$ THEN A CONTAINS A COPY OF $[0,1]^n$ AND IT FOLLOWS THAT $\dim A \geq n$ THANKS TO THE PAIRS OF OPPOSITE SIDES OF THE CUBE.

THE OTHER IS MORE WORK.

STEP 1
IF X IS A METRIC SPACE AND $A \subseteq X$ THEN $\dim A \leq \dim X$.

PROOF
- ALREADY KNOWN FOR CLOSED SUBSPACES
- ALSO FOR OPEN SUBSPACES, BY THE COUNTABLE-CLOSED-SUM THEOREM
- ASSUME $\dim X = n$ AND LET $(A_0, B_0), \ldots, (A_n, B_n)$ BE $n+1$ PAIRS OF DISJOINT CLOSED SETS IN A.
  NOW $\overline{A_i} \cap \overline{B_i}$ NEED NOT BE EMPTY, BUT WE DO KNOW THAT $\overline{A_i} \cap \overline{B_i} \cap A = \emptyset$.
  LET $F = \bigcup_{i=0}^{n} (\overline{A_i} \cap \overline{B_i})$ AND $O = X \setminus F$.
  THEN F IS CLOSED IN X, SO O IS OPEN IN X.

So: $\dim O \leq n$.
WE HAVE PAIRS OF DISJOINT CLOSED SETS IN O ($n+1$ OF THEM):
$(\overline{A}_0 \cap O, \overline{B}_0 \cap O), \ldots, (\overline{A}_n \cap O, \overline{B}_n \cap O)$
IN O WE HAVE PARTITIONS
$P_0, \ldots, P_n$ BETWEEN THESE
WITH $\bigcap_{i=0}^{n} P_n = \emptyset$.
BUT THEN $P_0 \cap A, \ldots, P_n \cap A$ ARE PARTITIONS IN A BETWEEN A_0 AND $B_0, \ldots, A_n$ AND B_n, WITH EMPTY INTERSECTION.

STEP 2 $\dim(\mathbb{R}^n \setminus \mathbb{Q}^n) = n-1$

- $\dim(\mathbb{R}^n \setminus \mathbb{Q}^n) \geq n-1$ IS NOW EASY
  $\mathbb{R}^n \setminus \mathbb{Q}^n$ CONTAINS $\{\pi\} \times [0,1]^{n-1}$
  AND $\dim [0,1]^{n-1} = n-1$
- LET $(A_1, B_1), \ldots, (A_n, B_n)$ BE A SET OF n PAIRS OF DISJOINT CLOSED SETS IN $\mathbb{R}^n \setminus \mathbb{Q}^n$

WE CAN FIND INFINITELY MANY SUBSETS OF $\mathbb{Q}$ THAT ARE DENSE IN $\mathbb{R}$ AND PAIRWISE DISJOINT

EXERCISE: LET p BE A PRIME NUMBER
LET $\mathbb{Q}_p = \{ k \cdot p^{-m} : k \in \mathbb{Z}, m \in \mathbb{N}, k \not\equiv 0 \bmod p \}$
PROVE
- $\mathbb{Q}_p$ IS DENSE
- IF $p \neq q$ THEN $\mathbb{Q}_p \cap \mathbb{Q}_q = \emptyset$.

TAKE n OF THOSE SETS, $D_1 = \mathbb{Q}_2, D_2 = \mathbb{Q}_3, \ldots, D_n = \mathbb{Q}_{p_n}$
AS IN THE PREVIOUS PROOF LET
$F = (\overline{A}_1 \cap \overline{B}_1) \cup \ldots \cup (\overline{A}_n \cap \overline{B}_n)$ AND $O = \mathbb{R}^n \setminus F$.
WE NEED COMPACT SETS; DEFINE
$K_m = \{ x \in \mathbb{R}^n : \|x\| \leq m \wedge d(x,F) \geq 2^{-m} \}$

- K_m IS CLOSED AND BOUNDED HENCE COMPACT

- FOR ALL m WE HAVE $K_m \subseteq \operatorname{Int} K_{m+1}$,

  IF $x \in K_m$ THEN

  - $\|x\| \leq m$ HENCE $\|y\| < m+1$ IF $\|x-y\| < 1$
  - $d(x, F) \geq 2^{-m}$ HENCE $d(y, F) > 2^{-(m+1)}$ IF $\|x-y\| < 2^{-(m+1)}$

  WE FIND THAT $B(x, 2^{-(m+1)}) \subseteq K_{m+1}$

NOW FIX i.

FOR EACH $x \in \bar{A}_i \cap O$ WE TAKE AN n-CUBE $C_x = \prod_{j=1}^n (p_j, q_j)$ SUCH THAT

- $x \in C_x$
- $p_j, q_j \in D_i$
- $\bar{C}_x = \prod_{j=1}^n [p_j, q_j]$ IS DISJOINT FROM $\bar{B}_i$ AND CONTAINED IN O
- IF $x \in K_{m+1} \setminus K_m$ THEN $\operatorname{DIAM} C_x < 2^{-(m+2)}$ AND $\bar{C}_x \cap K_m = \emptyset$.

NOW $\{C_x : x \in \bar{A}_i \cap O\}$ COVERS $\bar{A}_i \cap O$.

- TAKE A FINITE SET $G_1 \subseteq \bar{A}_i \cap K_1$ SUCH THAT $\bar{A}_i \cap K_1 \subseteq U_1 = \bigcup\{C_x : x \in G_1\}$
- TAKE A FINITE SET $G_2 \subseteq (\bar{A}_i \cap K_2) \setminus U_1$ SUCH THAT $(\bar{A}_i \cap K_2) \setminus U_1 \subseteq U_2 = \bigcup\{C_x : x \in G_2\}$

  ⋮

- TAKE A FINITE SET $G_{m+1} \subseteq (\bar{A}_i \cap K_{m+1}) \setminus \bigcup_{j \leq m} U_j$ SUCH THAT $(\bar{A}_i \cap K_{m+1}) \setminus \bigcup_{j \leq m} U_j \subseteq \bigcup\{C_x : x \in G_{m+1}\}$

  ⋮

LET $W_i = \bigcup\{C_x : x \in \bigcup_{m=1}^{\infty} G_m\}$

- W_i IS OPEN
- $\bar{A}_i \cap O \subseteq W_i$

- $\overline{W_i} \cap O = \bigcup\{\overline{C_x} : x \in \bigcup_{m=1}^{\infty} G_m\} \cap O$

  (HENCE $\overline{W_i} \cap O \cap \overline{B_i} = \emptyset$)

  $\supseteq$: IS CLEAR: $C_x \subseteq W_i$ HENCE $\overline{C_x} \subseteq \overline{W_i}$
  FOR ALL x

  $\subseteq$: LET $x \in O \cap \overline{W_i}$ AND TAKE m SUCH
  THAT $x \in K_m$.
  NOW WE KNOW THAT $B(x, 2^{-(m+1)}) \subseteq K_{m+1}$
  HENCE IF $y \notin K_{m+1}$ THEN
  $B(x, 2^{-(m+1)}) \cap \overline{C_y} = \emptyset$
  AND SO $B(x, 2^{-(m+1)}) \cap \bigcup\{C_y : y \in \bigcup_{j \geq m+2} G_j\} = \emptyset$
  IT FOLLOWS THAT
  $x \in \overline{\bigcup\{C_y : y \in \bigcup_{j \leq m+1} G_j\}}$
  $\subseteq \bigcup\{\overline{C_y} : y \in \bigcup_{j \leq m+1} G_j\}$
  BECAUSE THE G_j ARE FINITE.

- IT FOLLOWS THAT THE BOUNDARY P_i OF W_i
  IS CONTAINED IN THE UNION OF
  THE BOUNDARIES OF THE C_x ($x \in \bigcup_m G_m$)
  HENCE IF $x \in P_i$ THEN AT LEAST
  ONE COÖRDINATE OF x IS IN D_i.

- OF COURE P_i IS A PARTITION IN O
  BETWEEN $\overline{A_i} \cap O$ AND $\overline{B_i} \cap O$

- FINAL CONCLUSION:
  $\bigcap_{i=1}^{n} P_i \subseteq \mathbb{Q}^n$
  BECAUSE IF $x \in \bigcap_i P_i$ THEN x HAS
  A COORDINATE IN EACH OF THE D_i
  SO ALL COORDINATES ARE RATIONAL.

- SO $P_i \setminus \mathbb{Q}^n$ IS A PARTITION BETWEEN
  A_i AND B_i IN $\mathbb{R}^n \setminus \mathbb{Q}^n$
  AND $\bigcap_{i=1}^{n} (P_i \setminus \mathbb{Q}^n) = \emptyset$.

STEP 3 IF $\operatorname{Int} A = \emptyset$ THEN THERE IS A HOMEOMORPHISM $h : \mathbb{R}^n \to \mathbb{R}^n$ SUCH THAT $h[A] \subseteq \mathbb{R}^n \setminus \mathbb{Q}^n$.
(AND THAT ENSURES $\dim A \leq n-1$.)

1. $\mathbb{R}^n \setminus A$ IS DENSE IN $\mathbb{R}^n$ (DEFINITION OF $\operatorname{Int} A$)
2. THERE IS A COUNTABLE SET $D \subseteq \mathbb{R}^n \setminus A$ THAT IS DENSE IN $\mathbb{R}^n$.
   - CHOOSE, FOR EVERY OPEN BLOCK $\prod_{i=1}^n (p_i, q_i)$ WITH $p_i, q_i \in \mathbb{Q}$ A POINT $x(\underline{p}, \underline{q})$ IN $\prod_{i=1}^n (p_i, q_i) \setminus A$. THEN $D = \{ x(\underline{p}, \underline{q}) : (\forall i)(p_i < q_i) \}$ IS A COUNTABLE DENSE SUBSET OF $\mathbb{R}^n$
3. THERE IS A HOMEOMORPHISM $h : \mathbb{R}^n \to \mathbb{R}^n$ SUCH THAT $h[D] = \mathbb{Q}^n$
   (AND SO $h[A] \subseteq \mathbb{R}^n \setminus \mathbb{Q}^n$).

EXERCISE

a) DO THIS FIRST FOR $n = 1$

b) $n = 2$. LET $D \subseteq \mathbb{R}^2$ BE COUNTABLE AND DENSE
   PROVE THAT THERE IS A ROTATION R SUCH THAT FOR $\underline{d}, \underline{e} \in R[D]$ IF $\underline{d} \neq \underline{e}$ THEN $d_1 \neq e_1$ AND $d_2 \neq e_2$
   - LET D_1 AND D_2 BE COUNTABLE AND DENSE IN $\mathbb{R}^n$ AND ASSUME BOTH SATISFY THE PROPERTY IN THE PREVIOUS BIT: IF $\underline{d} \neq \underline{e}$ IN D_1 (OR D_2) THEN $d_1 \neq d_2$ AND $e_1 \neq e_2$

   CONSTRUCT A BIJECTION $b : D_1 \to D_2$ SUCH THAT FOR ALL $\underline{d}, \underline{e} \in D_1$ WITH $\underline{d} \neq \underline{e}$ WE HAVE
   - $d_1 - e_1$ AND $b(d)_1 - b(e)_1$ HAVE THE SAME SIGN
   - $d_2 - e_2$ AND $b(d)_2 - b(e)_2$ HAVE THE SAME SIGN

- Use the bijection b to define $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ such that $h = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$
  $(x,y) \mapsto (f_1(x), f_2(y))$
  is a homeomorphism such that $h[D_1] = D_2$
- Now show how to define, given a countable and dense $D \subseteq \mathbb{R}^2$, a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h[D] = \mathbb{Q}^2$.

c) n arbitrary

- Let $D \subseteq \mathbb{R}^n$ be countable and dense. Show that there is an orthogonal matrix U such that $U[D]$ satisfies the following property: if $\underline{d}, \underline{e} \in U[D]$ and $\underline{d} \neq \underline{e}$ then
  $d_i \neq e_i$ for all $i \leq n$.
- Let D_1 and D_2 be countable and dense in $\mathbb{R}^n$ that both satisfy the property above. Construct a bijection $b : D_1 \to D_2$ such that for all distinct $\underline{d}$ and $\underline{e}$ in D_1 and for all $i \leq n$
  $d_i - e_i$ and $b(\underline{d})_i - b(\underline{e})_i$
  have the same sign.
- Use b to construct homeomorphisms $f_i : \mathbb{R} \to \mathbb{R}$ $(i \leq n)$ such that $h = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$
  $\underline{x} \mapsto (f_1(x_1), \dots, f_n(x_n))$
  is a homeomorphism and maps D_1 onto D_2
- Let $D \subseteq \mathbb{R}^n$ be countable and dense. Construct a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h[D] = \mathbb{Q}^n$.