Some Rotations of $\mathbf{R}^3$

6102 [1976, 572]. *Proposed by Barbara Osofsky, Rutgers University*

Let A and B be nontrivial rotations of $\mathbf{R}^3$ about l_1 and l_2, respectively, which are axes through $(0,0,0)$ such that $A^2 = B^3 = \mathrm{Id}$. Hausdorff has shown that if $\cos 2\theta$ is transcendental, where θ is the angle between l_1 and l_2, then all relations between A and B are generated by $A^2 = \mathrm{Id}$ and $B^3 = \mathrm{Id}$. Show that the same is true for $\theta = \frac{1}{4}\pi$.

Solution by Scot Adams, Cornell University. We prove the following theorem:

THEOREM. *Let Ψ and Φ be two lines passing through a given point making an angle of $\pi/4$ with one another. Define ψ to be a rotation of $2\pi/3$ about Ψ and define φ to be a rotation of π about Φ. Then all relations between ψ and φ are generated by $\psi^3 = 1$ and $\varphi^2 = 1$.*

Proof. Form a coordinate system with origin at the intersection of Ψ and Φ, with z-axis Ψ, and such that Φ is contained in the xz-plane. Then, identifying a rotation with its matrix, we have

$$\psi^{\pm 1} = \begin{bmatrix} -\frac{1}{2} & \frac{\mp\sqrt{3}}{2} & 0 \\ \frac{\pm\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \varphi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and so

$$\psi^{\pm 1}\varphi = \frac{1}{2}\begin{bmatrix} 0 & \pm\sqrt{3} & -1 \\ 0 & 1 & \pm\sqrt{3} \\ 2 & 0 & 0 \end{bmatrix}.$$

Suppose n is a positive integer, and η is a rotation of the form $\eta = \psi^{k_1}\varphi\psi^{k_2}\varphi\ldots\psi^{k_n}\varphi$, where $k_1,\ldots,k_n \in \{1,-1\}$. Let

$$\eta = \frac{1}{2^n}\begin{bmatrix} m_{11} & m_{12}\sqrt{3} & m_{13} \\ m_{21}\sqrt{3} & m_{22} & m_{23}\sqrt{3} \\ m_{31} & m_{32}\sqrt{3} & m_{33} \end{bmatrix}$$

for some even integers $m_{11}, m_{21}, m_{31}, m_{32}, m_{33}$ and some odd integers $m_{12}, m_{22}, m_{13}, m_{23}$. The theorem follows by seeing that $\eta \neq 1$. Now

$$2\psi^{\pm 1} \equiv \alpha \equiv \begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(modulo 2)}$$

while

$$\varphi \equiv \beta \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{whence } \; y = \alpha\beta = \begin{bmatrix} 0 & \sqrt{3} & 1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

and $y^2 \equiv y$ (modulo 2). Thus $2^n\eta \equiv y^n \equiv y$, and it follows that $\eta \neq 1$.

Also solved by Roger Lyndon, D. L. Wang & P. Y. Wang, and the proposer.

Editor's note. Lyndon proves a stronger result, that the group generated by three rotations of order 3 about three orthogonal axes in 3-space is the free product of the cyclic groups of order 3 generated by the three rotations separately. By a classical result [Cayley: Philos. Mag. 26 (1845) 141–145] this follows from the following theorem.

THEOREM. *Let $A = (1+\sqrt{3}\,i)/2$, $B = (1+\sqrt{3}\,j)/2$, $C = (1+\sqrt{3}\,k)/2$, quaternions, and let $W = X_1\cdots X_t$, $t \geq 1$, where each X_i is one of $A^{\pm 1}$, $B^{\pm 1}$, $C^{\pm 1}$ and no $X_{i+1} = X_i^{\pm 1}$. Then $W \neq \pm 1$.*

Sketch of proof. Write $W_i = X_1 \cdots X_i = w_i/2^{n_i}$ where $w_i \in Z[\sqrt{3}, i, j, k]$ with n_i maximal. It suffices to show that $n_{i+1} \geqslant n_i$ for all i, $1 \leqslant i < t$. In fact it is shown that $n_{i+1} = n_i$ or $n_{i+1} = n_i + 1$, with at most three successive n_i equal. The proof is by induction, assuming w_{i-1} given of the required form and calculating $w_i, \ldots, w_{i+3}$ modulo successive powers of 2 up through $2^3 = 8$. The argument is complicated by the necessity of considering the various admissible possibilities for $X_i, \ldots, X_{i+3}$.

Principal Ideal Domains

6116 [1976, 748]. *Proposed by S. H. Cox, Jr., Universidad de Puerto Rico*

Let A be an integral domain satisfying the condition: For every nonzero ideal I of A there is an epimorphism $A \to A'$ of rings such that I and A' are isomorphic A-modules. For example, a principal ideal domain satisfies the condition with $A \to A'$ the identity $A = A'$. Show that each domain satisfying the condition is a principal ideal domain. (Warning: ring epimorphisms need not be surjective.)

Solution by Leonard Scott and Douglas Costa, University of Virginia. The assertion is false. To see this, let R be an integral domain with quotient field K. Let $A = R + XK[X] \subseteq K[X]$. If I is an ideal of A, it extends to a principal ideal of $K[X]$ generated by some $f(X) \in K[X]$. Set $L = \{a \in K \mid af(X) \in I\}$. It is easy to check that $I = f(X)LA$. Now let us add the assumption that R is a discrete valuation ring. Since L is an R-submodule of K, either $L = K$ or L is cyclic. If L is cyclic, I is principal and $I \cong A$. Otherwise, $I = f(X)KA = f(X)K[X] \cong K[X]$. Now $K[X]$ is the localization of A with respect to the nonzero elements of R and is therefore an epimorphic extension of A. Thus the hypotheses of the assertion are satisfied. The ideal $XK[X]$ however is not even finitely generated in A.

The assertion is true if A is assumed to be either noetherian or one-dimensional. By exercise 10, p. 8, of Kaplansky, *Commutative Rings*, A will be a PID if all of its prime ideals are principal. In either of the two present cases it is therefore sufficient, by the *Hauptidealsatz*, to show that every maximal ideal is principal.

Let M be a maximal ideal of A and let $A \to A'$ be a ring epimorphism with $A' \cong M$ as A-modules. Since A' is torsion-free of rank one as an A-module, it is isomorphic to a subring of the quotient field of A containing A. Replace A' by this ring. Then the A-module isomorphism $A' \cong M$ may be extended to an A'-module isomorphism, whence $M = A'x$ for some $x \in M$. Tensoring the epimorphism $A \to A'$ by A/M gives an epimorphism $A/M \to A'/M$, which must be surjective since A/M is a field. Thus $A' = A$ and $M = Ax$ is principal.

Linear Compositions of Two Entire Functions

6117 [1976, 748]. *Proposed by M. J. Pelling, Watford, Hertfordshire, England*

A well-known theorem asserts that given entire functions $f(z), g(z)$ with no common zero, then there exist entire functions $a(z), b(z)$ such that $af + bg = 1$ identically.

(i) Show that it is always possible to choose $a(z)$ zero-free.

(ii) Is it always possible to choose both $a(z)$ and $b(z)$ to be zero-free?

Solution by Lee A. Rubel, University of Illinois. (i) We want $af + bg = 1$ with $a \neq 0$ supposing that f, g, have no common zeros. By Helmer's theorem, there are entire functions A, B so that $Af + Bg = 1$ and hence for any entire function λ, $(A + \lambda g)f + (B - \lambda f)g = 1$, so we may choose $a = A + \lambda g$ and want $A + \lambda g = e^h$. At a zero z_n of g we need $e^{h(z_n)} = A(z_n)$ (with appropriate multiplicity) and by the interpolation theorem for entire functions, this is certainly possible since $A(z_n) \neq 0$ since $A(z_n)f(z_n) = 1$.

(ii) We prove the following theorem.

THEOREM. *Let f and g be nonconstant polynomials. Then there do not exist nonconstant entire functions a and b, having no zeros, such that $af + bg = 1$.*