Dr. Bernard Bolzano's

Paradoxes of the Infinite

edited from the writings of the author

by

Dr. Fr. Příhonský

Je suis tellement pour l'infini actuel, qu'au lieu d'admettre, que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte par-tout, pour mieux marquer les perfections de son Auteur.

— Leibniz, *Opera omnia studio Ludov. Dutens.*, Tom. II, part x, p. 243

Leipzig
C. H. Reclam
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Title page quotation:

I stand for actual infinity so much that instead of admitting that Nature abhors it, as it is commonly said, I hold that [Nature] assumes it everywhere, in order to signal better the perfections of its Author.

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Editor's Preface[a]

The author of this remarkable work on the *Paradoxes of the Infinite* began writing it in 1847 while staying in the company of the editor at the charming villa in Liboch, near Melnik. But he was interrupted by other tasks and did not complete it until the summer months of the following year, the last of his life. The work showed that in spite of his advancing age and the visible decline of his physical powers (he was in his sixty-seventh year at that time) his mental powers had lost nothing of their vigour and alertness. He also showed the learned world his unusual insight into the most abstract depths of mathematics, natural science and metaphysics. Indeed, if Bolzano had written, and left us, nothing else but this work then our firm belief is that on account of this alone he would have to be accounted one of the most distinguished minds of our century. He understands how to solve, with remarkable ease, the most interesting and complex questions which have for long engaged those studying the *a priori* sciences. He can sort them out, in front of the reader with such clarity that anyone who is not a complete stranger to the area, even if he has understood very little before, can follow the author's exposition and grasp at least the majority of his propositions. Furthermore, the experts, provided they give some attention to the work (and surely this is not too much to expect?) are sure to notice how important are the ideas which Bolzano puts forward here, and in other works (especially his *Logic* and *Athanasia*). And they should notice that with these views he aims at nothing less than a complete transformation of all previous scientific presentations.

The editor received this work in manuscript form from the heirs of the author and undertook to have it printed as soon as possible. This obligation was welcome as it coincided with his innermost feelings. Bolzano was his unforgettable teacher and friend. He would gladly have done this earlier if significant obstacles had not been in the way which he was only able to overcome in the course of this year. Now he is at last in the position of being able to improve the manuscript which is not always very readable, and even in places incorrect. He could also facilitate the use of the book with a detailed list of contents and he could find a suitable place for publication. He chose Leipzig because he expects this will offer a greater distribution of the work and also because this famous city of books will be honoured (he is by birth a Bohemian). He is confident that once Bolzano's great genius is generally recognized it will not be the least title to fame for Leipzig to have contributed to the appearance of these *Paradoxes*.

Budissin,
10th July 1850

[a] By Dr Příhonský, who also compiled the following list of *Contents*.

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[b] The German *Gegenständlichkeit* means of a concept there *are* objects associated with that concept. This is in contrast to a concept or idea being 'empty' [*gegenstandlos*]. Thus 'objectivity' is being used here in a specific and unusual sense.

§24. Two sums of quantities that are equal to one another pairwise, may, if their multitude is infinite, not be immediately put equal, but only if both multitudes have equal determining grounds [*Bestimmungsgründe*].

§25. There is an infinite entity in the domain of the actual.

§26. The principle of the 'universal determinateness of all actual existence' does not contradict this assertion.

§27. But those mathematicians surely go wrong who speak of infinitely great intervals of time that are nevertheless bounded on both sides, or as happens more often, they speak of infinitely small parts of time. Similarly they speak of infinitely large and infinitely small distances. Also physicists, and metaphysicians, go wrong if they assume or claim that there are forces in the universe that are infinitely greater or smaller than others.

§28. The chief paradoxes of the infinite in the domain of mathematics. First of all, in the general theory of quantity, and especially in the theory of numbers. How the paradox of a calculation of the infinite can be resolved.

§29. A calculation with the infinitely great does in fact exist.

§30. Similarly a calculation with the infinitely small [exists].

§§31–32. The falsity of some concepts that even mathematicians cherish about the infinitely small and the infinitely great.

§33. The caution to be observed with calculation with the infinite, in order to avoid going astray.

§34. A more precise determination of the concept of zero. Zero may never be used as a divisor in an equation that is to be more than a mere identity.

§35. Contradictions that arise from the occasionally encountered assertion that infinitely small quantities, when combined with certain others through addition or subtraction become zero or vanish.

§36. These contradictions are not avoided by the assumption of some mathematicians that infinitely small quantities are mere zeros and infinitely large ones are quotients resulting from the division of a finite quantity by a mere zero.

§37. How the author conceives the method of calculation with the infinite in order to free it from all contradiction.

§38. Paradoxes of the infinite in the applied part of the theory of quantity, and indeed in the theory of time and space. The concept of the continuum or of continuous extension already contains apparent contradictions. How these may be resolved.

§39. Paradoxes in the concept of time.

[c] The German here is *geistiger Wesen* which Steele renders, perfectly soundly, as 'spiritual beings'. Both ideas are in the German phrase.

§56. The great paradox of the connection between mental and material substance is automatically resolved on this view.

§57. The erroneous idea of a construction of the universe from mere forces without substances.

§58. In God's creation there is neither a highest nor a lowest level of being.

§59. Variation in the density of bodies is quite compatible with the continuous filling of infinite space with substances, and it is not necessary to assume that substances interpenetrate each other.

§60. Every substance in the world is continually interacting with every other substance.

§61. Among [substances] there are 'dominant' substances, but none of them possesses forces infinitely greater than those of the 'dominated' ones.

§62. Whether every arbitrary collection of substances must contain one dominating substance.

§63. Together with the dominant substances there is yet another world-stuff [*Weltstoff*], the ether, which fills all the remaining spaces outside distinguished substances, and connects all bodies of the world. An attraction and repulsion take place between the substances, and how the author conceives of them. How it comes about that material entities that differ in their forces, that is, in the degree of their mutual attractions, are nevertheless all equal in weight, or that their weights are proportional to their masses.

§64. How the dominance of certain substances or atoms over others is manifested, and what is the consequence of it.

§65. No distinguished substance experiences such a great change that it would be freed thereby from its immediate environment.

§66. Where one body ends and another begins, that is, the question of the boundaries of a body.

§67. Whether, and if, bodies can be in direct contact with one another.

§68. The possible kinds of movements occurring in the universe.

§69. Whether an atom in the universe at any time describes a perfectly straight or a perfectly curved line. Whether the author's views on the infinitude of the universe allow for any movement of the entire [universe] in a given direction, or even a turning motion of it about a world axis, or about a world centre?

§70. Two paradoxes made famous by Euler.

§ 1

Certainly most of the *paradoxical* assertions which we meet with in the area of mathematics, though not all of them as *Kästner* suggests, are propositions that either contain the concept of *infinity* directly, or depend on it in some way for their attempted proof. It is even more indisputable that precisely those mathematical paradoxes which deserve our greatest attention are of this kind. This is because decisions on very important questions in many another subject, such as metaphysics and physics, depend on a satisfactory resolution of their apparent contradictions.

This is the reason why in the present work I am dealing exclusively with the consideration of the paradoxes of the infinite. But it is self-evident that it would not be possible to recognize the appearance of contradiction which is attached to these mathematical paradoxes for what it is, a mere appearance, if we did not make abundantly clear what concept we actually associate with the infinite. Therefore we do this first.

§ 2

The word already indicates that the *infinite* is contrasted [*entgegensetze*] with everything that is merely *finite*. And the fact that the *name* of the former is derived from that of the latter, shows that we think of the *concept* of the infinite as one which arises from that of the finite only through the addition of a new component (such indeed is the mere concept of *negation*). Finally, that both concepts are applied to *multitudes* [*Mengen*], or more specifically to *pluralities* [*Vielheiten*] (i.e. to multitudes of units),[d] therefore also to *quantities*, cannot be denied because it is precisely *mathematics*, i.e. the theory of quantity, where we speak most frequently of the infinite. Since here finite as well as infinite pluralities, and besides *finite* quantities not only *infinitely large* but even *infinitely small* quantities arise as objects of our consideration—and even calculation. Without assuming that both those concepts (namely of the finite and of the infinite) can always only be applied to objects to which *magnitude* and *plurality* can be referred in some respect, we may hope that a more precise investigation of the question of the circumstances in which we define a multitude as finite or as infinite, will also give us information about the *infinite in general*.

§ 3

For this purpose we must nevertheless go back to one of the simplest concepts of our understanding so as to agree first of all on the word we wish to use for its designation. It is the concept which underlies the conjunction '*and*' which I believe, if it is to stand out as clearly as required in countless cases for the purpose of mathematics as well as philosophy, can be expressed most suitably by

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[d] On the translation of *Menge* and *Vielheit* see §4 and the *Note on the Translations*.

the words: *a collection* [*Inbegriff*] *of certain things* or *a whole* [*Ganze*] *consisting of certain parts*. That is, if it is agreed that we wish to interpret these words in such a wide sense that they may be asserted in all propositions where the conjunction '*and*' is usually used, e.g. in the following: '*The sun, the earth and the moon* have a mutual effect on one another', '*the rose and the concept of a rose* are a pair of very different things', 'the names *Socrates and son of Sophroniskus* designate one and the same person'—the object which is spoken about in these propositions is *a certain collection of things*, a *whole* consisting of certain *parts*. Namely, in the first one, it is that whole which the sun, earth and moon form together of which it is stated that it is a whole whose parts have mutual effect on one another. In the second proposition it is the collection which the two objects 'the rose and the concept of a rose' jointly make up of which it is judged that they are two very different things etc. These few examples should already be enough for agreement about the concept spoken of here, at least if we add that any arbitrary object A can be combined with all the other arbitrary objects $B, C, D, \ldots$ into a collection or (to speak more correctly) already forms a collection in itself. About this collection several more or less important truths can be stated provided each of the ideas $A, B, C, D, \ldots$ does in fact represent *another* object, or provided none of the propositions 'A is the same as B', 'A is the same as E', 'B is the same as C' etc. is true. For if, for example, A is the same thing as B, then it is of course absurd to speak of a collection of the things A and B.

§ 4

There are collections, which, although containing the same parts $A, B, C, D, \ldots$, nevertheless present themselves as *different* (we call it essentially different) according to the viewpoint (concept) under which we interpret them. For example, a complete glass and a glass broken into pieces considered as a drinking vessel. We call the basis [*Grund*] for this difference in such collections, the *mode of combination* or *arrangement* of its parts. A collection which we put under a concept so that the arrangement of its parts is unimportant (in which therefore nothing essential changes for us if we merely change this arrangement) I call a *multitude* [*Menge*]. And a multitude whose parts are all considered as *units* of a certain kind A, i.e. as objects which come under the concept A, is called a *plurality* [*Vielheit*] of A.

§ 5

It is well known that there are also collections whose parts themselves are compound, i.e. are again collections. Among these are also such as we consider from a viewpoint for which nothing essential changes in them if we conceive the parts of the parts as parts of the whole itself. I call them, with a word borrowed from mathematicians, *sums* [*Summen*]. For it is just the concept of a sum that it must be that $A + (B + C) = A + B + C$.

§ 6

If we consider an object as belonging to a kind [*Gattung*] of thing of which every two, M and N, can have no other relationship to one another than that they are either equal to one another, or that one of them presents itself as a sum which includes a part equal to the other one. That is, that either $M = N$ or $M = N + \nu$ or $N = M + \mu$, where the same must again hold of the parts ν and μ, namely that they are either equal to one another, or one is to be viewed as a part contained in the other, then we consider this object as a *quantity* [*Größe*].

§ 7

If a given collection of things $\ldots, A, B, C, D, E, F, \ldots, L, M, N, \ldots$ has the property that for every part M some one, and only one, other part N can be identified of a kind that we can *determine* by the *same rule* for all parts of the collection either N by its relationship to M, or M by its relationship to N, then I call this collection a *series* [*Reihe*] and its parts the *terms* of this series. I call that rule by which either N is determinable through its relationship to M, or M is determinable through its relationship to N, the *rule of formation* [*Bildungsgesetz*] of the series. One of these terms, whichever one wants, I call (without wishing to designate by this name the concept of an actual sequence in time or space) the *previous* or *preceding* term, the other the *following* or *succeeding* term. Every term M which has a previous term, as well as a following term, i.e. which is not only itself derivable from another but from which also again another term is derivable according to the rule of formation holding for the series, I call an *interior* term of the series, from which it is self-evident which terms, if they exist, I call *exterior*, the *first* or the *last* term.*

§ 8

Let us imagine a *series* of which the *first* term is a *unit* of the kind A, but every succeeding term is derived from its predecessor by our taking an object equal to it and combining it with a new unity of kind A into a sum. Then obviously all the terms appearing in this series—with the exception of the first which is a *mere unit* of the kind A—are *pluralities of the kind A* and in fact these are such as I call *finite* or *countable pluralities*, indeed I call them straightforwardly (and even including the first term) *numbers* [*Zahlen*], and more definitely, *whole* numbers.

* More precise discussions about this, as also about some of the concepts put forward in the previous paragraphs, are found in the *Wissenschaftslehre*.[e]

[e] Explanations of the terms for collections are given in *WL* §§ 82–86.

§ 9

According to the different nature of the concept designated here by *A* there may sometimes be a greater and sometimes a smaller multitude of objects which it comprehends, i.e. the units of the kind *A*. And therefore there is sometimes a greater and sometimes a smaller multitude of terms in the series being discussed. In particular there can even be so many of them that this series, to the extent that it is to exhaust *all* these units (taken in themselves), may have absolutely *no last term*. We shall prove this in more detail in what follows. Therefore assuming this for the time being I shall call a plurality which is greater than every finite one, i.e. a plurality which has the property that every finite multitude represents only a part of it, an *infinite plurality*.

§ 10

I hope it will be granted that the definition put forward here of both the concepts of a *finite* and of an *infinite* plurality truly determine the difference between them as intended by those who have used these expressions in a strict sense. It will also be granted that there is no hidden circularity in these definitions. Therefore it only a question of whether through a mere definition of what is called an infinite *plurality* we are in a position to determine what is [the nature of] the *infinite* in general. This would be the case if it should prove that, strictly speaking, there is nothing other than pluralities to which the concept of infinity may be applied in its true meaning, i.e. if it should prove that infinity is really only a property of a plurality or that everything which we have defined as *infinite* is only called so because, and in so far as, we discover a property in it which can be regarded as an infinite plurality. Now it seems to me that is really the case. The mathematician obviously never uses this word in any other sense. For generally it is nearly always quantities with whose determination he is occupied and for which he makes use of the assumption of one of those of the same kind for the *unit*, and then of the concept of a number. If he finds a quantity greater than every number of the unit taken, then he calls it *infinitely large*; if he finds one so small that every multiple of it is smaller than the unit, then he calls it *infinitely small*. Outside these two classes of infinities and the kinds further derived from them of infinitely greater and infinitely smaller *quantities of higher order*, which all proceed from the same concepts, there is no other infinity for him.

§ 11

Now some philosophers, particularly of more recent times, like *Hegel* and his followers, are not satisfied with this infinity so well known to mathematicians. They call it contemptuously 'the bad infinity' and claim to know a much higher one, the true, the *qualitative infinity* which they find especially in *God* and generally only in the *absolute*. If they, like *Hegel*, *Erdmann* and others, imagine the mathematical infinity only as a quantity which is *variable* and has no limit to its growth (which is, of course, as we shall soon see, what some mathematicians have put forward as the definition of their concept), then I would agree with them in their criticism of this concept of a quantity itself never *reaching* but only *growing* into infinity. A *truly infinite* quantity, e.g. the length of the whole straight line unbounded in both directions (i.e. the magnitude of that spatial thing which contains all points which are determined by their merely conceptual relationship to two given points), needs precisely not to be variable, as it is in fact not in the example mentioned. A quantity which can always be taken greater than it has already been taken, and may become greater than every given (finite) quantity can nevertheless always remain a merely finite quantity, as holds in particular of every number quantity [*Zahlgröße*] 1, 2, 3, 4, ... What I do not concede is merely that the philosopher may know an object on which he is justified in conferring the predicate of being infinite without first having identified in some respect an infinite magnitude or plurality in this object. If I can prove that even in God as that being which we consider as the most perfect unity, viewpoints can be identified from which we see in him an infinite plurality, and that it is only from these viewpoints that we attribute infinity to him, then it will hardly be necessary to demonstrate further that similar considerations underlie all other cases where the concept of infinity is well justified. Now I say we call God infinite because we concede to him powers of more than one kind that have an infinite magnitude. Thus we must attribute to him a power of knowledge that is true omniscience, that therefore comprehends an infinite multitude of truths because all truths in general etc. And what would be the concept that anyone would want to press upon us in place of the concept of true infinity put forward here? It should be the universe [*das All*], which comprehends every possible thing, the absolute universe, apart from which there is nothing. According to this statement there would be an infinity which included, according to our definition, infinitely many things. It would be a collection of not only all actual things, but also all those things which have no reality, the propositions and truths in themselves. And thus even with all the other errors in mind which are mixed up in this theory of the universe there should be no basis for abandoning our concept of the infinite so as to adopt that other one.

§ 12

I cannot also help rejecting as incorrect many other definitions of infinity, which have been proposed even by mathematicians in the opinion that they present only the components of this one and the same concept.

1. In fact, as I have just mentioned earlier, some mathematicians, among them even *Cauchy* (in his *Cours d'Analyse* and many other writings), and the author of the article '*Unendlich*' in *Klügel*'s *Wörterbuch*, have believed infinity to be defined if they describe it as a variable quantity whose value increases *without bound* and which can be proved to become greater than *every given quantity however large*. The *limit* of this unbounded increase is the *infinitely large quantity*. Thus the tangent of a right angle, thought of as a continuous quantity, is unbounded, without end,

and in the *proper sense infinite*. The mistakenness of this definition is clear from the fact that what mathematicians call a *variable quantity* is not really a quantity but is the mere concept, the mere *idea* of a quantity, and in fact such an idea that is concerned not with a single quantity but an infinite multitude of quantities differing from one another in value, i.e. quantities distinguishable by their *magnitude* [*Großheit*]. What is called infinite are indeed not those *different* values which, in the example mentioned here, are represented by the expression tang. ϕ for different values of ϕ, but only that single value which is imagined (although wrongly in this case) that that expression takes for the value $\phi = \frac{\pi}{2}$. Also it is certainly a contradiction to speak of the limit of an unbounded increase, and equally for the definition of the infinitely small, of the limit of an unbounded decrease. And if the infinitely large is defined by the former, then by analogy the infinitely small should be defined by the latter, i.e. the mere *zero* (a nothing). But this is certainly incorrect and neither *Cauchy* nor *Grunert* allow themselves to say it.

2. If the definition just considered was too wide, in contrast that adopted by Spinoza and many other philosophers as well as mathematicians, that *only that is infinite which is capable of no further increase*, or to which nothing more can be attached (added), is much too narrow. The mathematician is allowed to add to every quantity, even infinitely large ones, other quantities, and not only finite ones but even other quantities which are already infinite. Indeed he may even multiply the infinitely large infinitely many times etc. And if some dispute whether this procedure is even a legitimate one, which mathematicians, providing they do not reject everything infinite, will not have to admit that the length of a straight line which is bounded on only one side while on the other side it continues indefinitely, is infinitely large and nevertheless can be increased by additions on the first side?

3. No more satisfactory is the definition of those who adhere precisely to the components of the word and say infinite is *what has no end*. If they think thereby only of an end in time, a cessation, then they could only call things which are in time, finite or infinite. However we also ask about things which are not in time, e.g. lines or quantities in general, whether they are finite or infinite. But if they take the word in a wider sense, roughly equivalent to *limits* in general, then I point out *firstly* that there many objects for which one cannot reasonably show that a limit exists for them without attributing to the word a highly unreliable, confused meaning, and which nevertheless nobody counts as infinite. Thus every simple part of time or space (a point in time or space) has no limit, but is instead usually considered itself only as a limit (of a time interval or line), indeed most of them are directly defined so that this belongs to their nature. But it occurs to nobody (unless they are Hegel) to wish to see an infinity in a mere point. Just as little does the mathematician regard the circumference of a circle and many other lines and surfaces which turn back on themselves as a limit and consider them only as finite things. (It would have to be that he may come to speak of the infinite multitude of the points contained in them, and in that respect he must also recognize in every bounded line something infinite.) *Secondly*, I remark that there are many objects which are undeniably bounded, but are regarded as quantities belonging to the

infinite. It is so not only with the straight line already mentioned earlier, which only extends into infinity on one side, but also with the surface area which a pair of infinite parallel lines encloses between them, or the two indefinitely extended arms of an angle drawn in the plane, and several others. Thus also in rational psychology we shall call an intellect infinitely large if, even without being omniscient, it is just capable of surveying some infinite multitude of truths, e.g. just the complete infinite series of decimal places which the single quantity $\sqrt{2}$ contains.

4. Most commonly what is called infinitely large is what is greater than every quantity that *could be given* [*angebliche Größe*]. Here we need most of all a more exact determination of what is in mind with the words '*could be given*'. Should it only mean that something is *possible*, i.e. *can* have reality, or only that it is *nothing contradictory*? In the first case, the concept of *finite thing* is limited solely to that kind of thing which has *real existence* [*Wirklichkeit*], either they are real at all times, or have been or will be real at certain times, or at least *could* become real at some time. In fact it is in this sense which *Fries* (*Naturphilosophie*, §47) seems to have taken the infinite when he calls it the *incompletable*. But usage applies the concept of finite and also that of infinite both to objects which have real existence, like God, and also to others which cannot be spoken of as having any existence at all such as the pure propositions and truths in themselves, together with their components the ideas in themselves, since we assume finite as well as infinite multitudes of them. But if by 'what could be given' is understood everything which is just *not contradictory*, then one already puts into the definition of the concept that there may be no infinity, for a quantity which is to be greater than every [quantity] which is not contradictory, would also have to be greater than itself, which is, of course, absurd. However there is still a third meaning in which the words '*could be given*' could be taken, if one understood by them only such a thing as *can only be given to us*, i.e. can become an object of *our experience*. But I ask everyone whether—if a beneficial use is to be made of it in science—he does not in any case take the words '*finite*' and '*infinite*' in a sense, and he must necessarily adopt only such a sense, that they refer to a certain *internal* property of the object which we call thus, but in no way do they refer to a mere relationship of it to our *perception*, even to our *sense awareness* (whether we may be able, or not, to have *experiences* of it). Thus the question of whether something is finite or infinite can certainly not depend upon whether the object in question possesses a quantity which we are able to perceive (for example, to look at, or not).

§ 13

If we have now come to agreement on which concept we shall associate with the word '*infinite*' and if we have also made clear the components from which we compose this concept, then the next question is whether it also has *objectivity*,[f] i.e. whether there are also things to which it can be applied, multitudes, which we

[f] See the footnote on p. 596.

may call infinite in the sense defined? And I venture to *affirm* this categorically. In the *realm of those things which make no claim to reality but only to possibility*, there are indisputably multitudes which are infinite. *The multitude of propositions and truths in themselves* is, as may very easily be seen, infinite. For if we consider some truth, perhaps the proposition that there are actually truths, or otherwise any arbitrary truth, which I shall designate by *A*, then we find that the proposition that the words '*A is true*' express is different from *A* itself, for the latter obviously has a completely different subject from the former. Namely, its subject is the whole proposition *A* itself. However, by the rule by which we derived from the proposition *A*, this different one, which I shall call *B*, we can again derive from *B* a third proposition *C*, and continue in this way without end. The collection of all these propositions in which each successive one stands in the relationship just given to the one immediately before it, in that it makes it its subject and states of it that it is a true proposition, this collection—I say—comprises a multitude of parts (propositions) which are greater than every finite multitude. For without my reminder the reader may notice the similarity between the series of these propositions formed by the rule just given, and the *series of numbers* considered in §8. This is a similarity consisting in this, that to every term of the latter there is a term of the former corresponding to it, that therefore for every number, however large, there is also a number of distinct propositions equal to it, and that we can always form new propositions, or to say it better, that there are such propositions in themselves regardless of whether we form them or not. Whence it follows that the collection of all these propositions has a plurality which is greater than every number, i.e. is infinite.

§ 14

Nevertheless, simple and clear as the proof just given is, there are a considerable number of scholarly and intelligent men who declare the proposition which I believe I have proved here, to be not only paradoxical but downright false. They deny that there *is any infinity*. According to their claim, not only among things which have reality but also among the others, there is no single thing, not even a collection of several things, for which an infinite multitude of parts could in any respect be assumed. We shall consider later the arguments which they raise against infinity in the realm of reality because we shall also bring forward later the reasons for the existence of such an infinity. Therefore let us examine here the arguments through which it is to be proved that there may never be something infinite, not even among the things which make no claim to being real.

1. They say, 'There can never be an infinite multitude, just for this reason because an infinite multitude *can never be united into a whole, can never be gathered together in thought*.' I must immediately call this assertion an error which is produced by the false view that in order to think of a whole consisting of certain objects *a*, *b*, *c*, *d*, ... one would first have to have formed *ideas* which represent each one of

these objects individually (individual ideas of them). It is definitely not so; I can imagine the multitude, or the collection if preferred, the *whole* [*Ganze*] of the inhabitants of Prague or of Beijing without imagining each of these inhabitants individually, i.e. through an idea corresponding exclusively to each one. I am actually doing this just now, since I am speaking of this very multitude, and, for example, make the judgement, that their number in Prague lies between the two numbers 100 000 and 120 000. That is, as soon as we possess an idea A which represents each of the objects a, b, c, d, . . . , but nothing else, it is extremely easy to reach an idea which represents the *collection* which all these objects make up together. In fact nothing extra is needed other than the concept which the word '*collection*' denotes, connected with the idea A in such a way as indicated by the words: *the collection of all A*. By this single remark, whose correctness I believe must be clear to everyone, all difficulty which may be found with the concept of a multitude if it consists of infinitely many parts, is removed. As soon as a category [*Gattungsbegriff*] for each of these parts exists, but which covers nothing else, as is the case with the concept: '*The multitude of all propositions or truths in themselves*', where the required category is already: 'a proposition or truth in itself'. However, I cannot leave uncriticized a *second* error which is revealed in that objection.

It is the opinion, 'that a multitude would not exist unless first somebody were to exist who *conceives* it'. Whoever asserts this, in order to be as consistent as one can actually be with an error, should not only assert that there may be no *infinite* multitude of propositions and truths in themselves, but he should assert that actually there may *not be any* propositions and truths in themselves *at all*. For if we have brought about a clear awareness in ourselves of the concept of propositions and truths in themselves and do not in fact doubt the objectivity of them, then we could hardly make assertions like the one just mentioned, and could certainly not persist with them. In order to show this in a way clear to everybody, I permit myself to put the question whether there do not exist at the poles of the earth fluid as well as solid bodies, air, water, rocks and such like, whether these bodies do not act upon another according to certain laws, e.g. that the speeds which they impart to one another on impact are inversely proportional to their masses and such like, and whether all these things occur even if no person, or any other thinking being, is there to observe it? If one agrees with this (and who would not have to agree?) then there are also propositions and truths in themselves which express all these proceedings without anyone knowing and thinking them. And in these propositions there is frequent reference to wholes and multitudes, for every body is a whole and produces many of its effects only through the multitude of parts of which it consists. Therefore there are multitudes and wholes without the presence of a being which conceives them. And if this were not so, if these multitudes were not there themselves, how could the judgements which we make about them be true? Or rather, what would be the meaning of these judgements if they should only become true if somebody is there who perceives these proceedings? If I say, 'This boulder broke off from that cliff in front of my eyes, cut through the air, and crashed down below,' this would have to have roughly the following meaning: While I thought of certain simple entities together up there, a combination of them arose which I call a boulder, this combination withdrew from certain others, which, while I think of them together, united into a whole which I call a cliff, etc.

2. However, one might say, 'for all this it remains true that it is only *our act* [*Werk*], and in fact a largely very arbitrary act, whether we want to think of certain simple objects together in a collection or not, and only if we do this first do relationships arise between them. The central particle in this button on my coat and the central particle in the top of that tower there have nothing to do with one another and have no connection with one another at all, only through my present thinking of them together does any kind of connection between them originate.' Even this I must contradict. The two particles were, even before the thinking being put together their ideas, in mutual effect on one another, e.g. through the force of attraction and such like, and if, on the other hand, that thinking being does not, by virtue of his thoughts, also adopt actions which produce a change in the relationships between the two particles, then it is absolutely untrue that it is only through that thinking of them together that relationships arise among them, which apart from this would not be there. If I should judge truly that the former [particle] is lower, and the latter is higher, and that therefore the latter may be pulled up by the former by some small amount in height etc, then all this would have to be the case even if I had not thought about it, etc.

3. Other people say, 'It is not the case that for a collection to exist it is necessary for it to have *actually been thought* by a thinking being, but rather it is necessary that it *could* be thought. Now because no being is possible that can imagine each one of an infinite multitude of things individually and then connect these ideas, then also no collection which comprises an infinite multitude of things as parts in itself is possible.'

We have already seen in no.1 how much in error is the assumption that is repeated here, that for the thinking of a collection the thinking of all its parts individually, i.e. the thinking of each individual part by means of a single idea representing it, is required. Also we do not need to refer at the outset to the omniscient being as such a being for which the conception of an infinite multitude of things, each one individually, causes no trouble. However, we may not even grant the first assumption, namely that the existence of a collection of things rests on the condition that such a collection *can be thought of*. For the '*capacity of a thing to be thought of*' can never include the basis of its possibility, instead it is exactly conversely that the possibility of a thing is firstly the basis on which a reasonable being, providing it is not mistaken, and the thing is *possible*, *can think* it, or as we say (but improperly) finds it *thinkable*. One will be even more convinced of the complete correctness of this remark and the fact that the admittedly very widespread view which I am attacking here is completely untenable, if one tries to clarify the components of which the highly important concept of possibility consists. That one calls that which is *possible*, what *can* be, is obviously not an analysis of this concept, for the concept of possibility is altogether involved in the word '*can*'. But it would be still more incorrect to wish to set up the definition that that is possible *which can be thought*. We can even think, in the true sense of the word where it concerns mere *representation*, of the impossible, and we actually think it whenever we judge about it and e.g. explain something as impossible, as when we say that there is, and can be, no quantity which represented by 0 or $\sqrt{-1}$. But even if one understands by thinking here not a mere representing but an actual *asserting* [*Fürwahrhalten*], it is false that everything is possible which we can assert as true. By mistake we sometimes even hold the impossible, e.g. that we had found the square of the circle, as true. Therefore it would have to be said (as I already adopted in modified form above) that is possible about which a thinking being, if it judged the truth appropriately, expresses the judgement that it can be, i.e. that it is possible. A definition which contains an obvious circularity! We are therefore required to drop completely the reference to a thinking being for the definition of the possible and look for another characteristic. One sometimes hears people say that 'possible' is '*what does not contradict itself*'. Of course, everything which already contains a contradiction within itself, e.g. that a sphere is not a sphere, is impossible. But not everything impossible is of such a kind that the contradiction which is already in the components from which we have composed the idea of it, is found. It is impossible that a solid which is enclosed by seven plane polygonal surfaces, may be enclosed by equal polygonal surfaces. But the contradictory nature does not lie open to view in the words which are connected here. We must therefore extend our definition further. But if we want to say that the impossible is what stands in contradiction to some truth, then we would by this be defining everything which is not, as impossible, because the proposition that it is, would contradict the truth, that it is not. We would therefore admit no difference between the possible and the actual, and even the necessary, which nevertheless we all do distinguish. Accordingly we see the domain of truths which the impossible contradicts must be limited only to a certain class, and now we can hardly fail to notice which class of truths this is. They are the pure conceptual truths [*Begriffswahrheiten*]. Whatever some pure conceptual truth contradicts is called the *impossible*. Therefore the possible is what stands in contradiction with no pure conceptual truth. Whoever has once realized that this is the correct concept of possibility, to them it can hardly occur to make the assertion that something is only possible if it is thought, i.e. is viewed as possible by a thinking being which does not err in its judgement. For this is to say: 'A proposition contradicts no pure conceptual truth if it contradicts no pure conceptual truth that there is a thinking being which judges of this proposition the truth that it contradicts no pure conceptual truth.' Who does not see how irrelevant here is this addition of a thinking being? But if it is decided that the *thinking* does not make the possibility, where is there some reason for concluding from the supposed circumstance that an infinite multitude of things cannot be *thought together* that such multitudes cannot exist?

§ 15

I consider it now as sufficiently proved and defended that there are infinite multitudes, at least among the things which have no reality, in particular that the multitude of all *truths in themselves* is an infinite multitude. It will also be admitted, that in a similar way as it is derived in §13, the multitude of *all numbers* (the so-called natural, or whole numbers, whose concept we defined in §8) is infinite. But this proposition does sound *paradoxical* and we might actually consider it as the *first paradox* appearing in the area of mathematics, because the one considered before properly belongs to a more general science than the theory of quantity.

It might be said, 'If every number, as a concept, is a merely finite multitude, how can the multitude of *all* numbers be an infinite multitude? If we consider the series of natural numbers:

$$1, 2, 3, 4, 5, 6, \ldots$$

then we notice that the multitude of numbers which this series contains, starting from the first (the unit) up to some other one, e.g. the number 6, is always represented by this latter one itself. Thus the multitude of *all* numbers must be as large as the *last* of them and thus itself be a number and therefore not infinite.'

The deceptiveness of this argument disappears as soon as it is remembered that in the multitude of all numbers, in their natural series, there is *no last one*. Therefore the concept of a last (highest) number is an empty one because it is a self-contradictory concept. For according to the *rule of formation* given in the definition of such a series (§8) every one of its terms has a *succeeding one*. This paradox may therefore be considered as solved by this single remark.

§ 16

If the multitude of *numbers* (namely the so-called *whole* numbers) is infinite, then it is all the more certain that the multitude of quantities (according to the definition appearing in §6 and *Wissenschaftslehre*, §87) is *infinite*. For as a consequence of that definition not only are all numbers also quantities, but there are even more quantities than numbers, because also the fractions $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \ldots,$ and the so-called *irrational* expressions $\sqrt{2}, \sqrt[3]{2}, \ldots, \pi, e, \ldots$ designate quantities. In consequence of this definition it is no contradiction to speak of quantities which are *infinitely large*, and of others which are *infinitely small*, as long as one understands by an *infinitely large* quantity only one which, once given a basic unit, appears as a whole for which every finite multitude of these units is only a part; and one understands by an *infinitely small* quantity one for which the unit itself appears as a whole of which every finite plurality of this quantity constitutes only a part. The multitude of all numbers appears immediately as an indisputable example of an infinitely large *quantity*. I say as a quantity, but not, of course, as an example of an infinitely large *number*, for this infinitely large plurality is of course not to be called a number, as we have just remarked in the previous paragraph. On the other hand, if we now make the quantity which appeared *infinitely large* with respect to another one taken as the unit, now itself as the unit, and we measure the one previously considered as the unit with it, then this latter one will now be represented as *infinitely small*.

§ 17

A most important class of infinitely large quantities which nevertheless do not belong in the domain of the actual, although they can be *determinations* of the actual, are *time* and *space*. Neither time nor space is something actual, for they are neither *substances* nor *properties* of substances, but they occur merely as determinations of all incomplete (bounded, finite or—what amounts to the same thing—dependent, created) *substances*. This is because each of the latter must always be in a certain time and also in a certain space such that every simple substance must stay at every *time instant*, i.e. in every simple part of time, and in some simple part of space, i.e. in some point of [space]. Now in time, as well as in space, the multitude of simple parts or points, of which the former and latter consist is *infinite*. Not only is the multitude of simple parts from which the whole of time and the whole of space is composed, i.e. the multitude of moments and points[g] that there are in general, infinitely large, but also the multitude of moments which lie between every two moments α and β, however close to one another, and in the same way the multitude of points which lie between every two points *a* and *b* however close to one another, is infinite. I need hardly go into a justification of these propositions since there is scarcely any mathematician, who, supposing he does not deny every infinity in general, would not concede them to us. But the opponents of *all* infinity, in order not to have to admit the infinity presented here so clearly escape behind the pretext, 'that we can of course always *think of adding* to the points in time and space, more than we have already thought of, but that the multitude of those which there are in *reality*, always remains only a finite multitude.' But I reply to this that neither time nor space, therefore also neither the simple parts of time nor those of space, are something real, so that it is absurd to speak of a finite multitude of them which exists in *reality*. And it is even more absurd to imagine that these parts only obtain their reality through our *thinking*. For it would follow from this that the properties of time, as well as those of space, depend on our thinking [them] or accepting [them] as true, and that therefore the ratio of the diameter to the circumference of a circle was rational as long as we mistakenly regarded it as if it were rational, and that space would have all those properties which we would get to know subsequently, would also then be accepted! But if the opponents rectify the above expression [by saying] that only thinking which is in accordance with truth may determine the true properties of time and space, then they say something completely tautological, namely that what is true, is true. From this then there is certainly not the slightest thing that can be concluded against the infinity of time and space as we have claimed.

[g] *Zeitpuncte* and *Raumpuncte* have been translated as 'moments' and 'points', respectively.

It is, in any case, inept to say that time and space contain only as many points as we imagine.

§ 18

Although every quantity, and generally every object, which counts for us as infinite in some respect, must be able to be considered in this respect as a whole consisting of an infinite multitude of parts, it does not hold conversely that every quantity which we consider as the sum of an infinite multitude of other quantities, which are all finite, must itself be an infinite quantity. For example, it is generally recognized, that irrational quantities, like $\sqrt{2}$, are finite quantities with respect to their underlying unit, although they can be viewed as composed from an infinite multitude of fractions of the form

$$\frac{14}{10} + \frac{1}{100} + \frac{4}{1\,000} + \frac{2}{10\,000} + \cdots,$$

of which the numerator and denominator are whole numbers. Equally the sum of the infinite series of summands of the form: $a + ae + ae^2 + \cdots$ *in inf*. is equal to the finite quantity $\frac{a}{1-e}$ as long as $e < 1$.* Therefore there is certainly nothing

* Since the usual proof for the summation of this series does not seem to be completely strict, it may be permitted to sketch out the following on this occasion. If we take $a = 1$ and e positive (because the application to other cases follows directly), and if we put as a symbolic equation

$$S = 1 + e + e^2 + \cdots \textit{in inf}. \tag{1}$$

then at least it is certain that S designates a positive quantity, no matter whether it is finite or infinitely large. But also for every arbitrary whole numbered value of n,

$$S = 1 + e + e^2 + \cdots + e^{n-1} + e^n + e^{n+1} + \cdots \textit{in inf}.$$

or also

$$S = \frac{1 - e^n}{1 - e} + e^n + e^{n+1} + \cdots \textit{in inf}. \tag{2}$$

for which we can also write

$$S = \frac{1 - e^n}{1 - e} + \overset{1}{P} \tag{3}$$

if we designate the value of the infinite series $e^n + e^{n+1} + \cdots$ *in inf*. by $\overset{1}{P}$, for which we certainly know at least this, that $\overset{1}{P}$ designates a quantity, measurable or unmeasurable, but at any rate positive, which is dependent on e and n. But we can represent the same infinite series in the following way:

$$e^n + e^{n+1} + \cdots \textit{in inf}. = e^n[1 + e + \cdots \textit{in inf}.].$$

Now here the sum consisting of infinitely many terms in the brackets on the right-hand side of the equation, namely

$$[1 + e + e^2 + \cdots \textit{in inf}.]$$

has completely the appearance of the series put forward in the symbolic equation $(1) = S$, but nevertheless it is not to be regarded as identical with it, since the *multitude* of summands here and in (1), although in both cases infinite, is not the same, rather here it is indisputably *n terms less* than in (1).

contradictory in the assertion that a sum of infinitely many finite quantities may itself be only a finite quantity, because otherwise it could not be proved to be true. But the paradox that might be perceived in this, is only produced because it is forgotten how the terms being added here become ever smaller and smaller. For that a sum of summands [*Addenden*], each successive one of which takes, for example, half the value of its predecessor, can never become more than double the first term, cannot really upset anybody because for each of the terms of this series, however much later, the series is always short of that double value by exactly as much as this last term.

§ 19

Even with the examples of the infinite considered so far it could not escape our notice that not all infinite multitudes are to be regarded as *equal to one another in respect of their plurality*, but that some of them are *greater* (or *smaller*) than others, i.e. another multitude is contained as a part in one multitude (or on the contrary one multitude occurs in another as a mere part). This also is a claim which sounds to many *paradoxical*. And of course everyone who defines infinity as something such that it is capable of no further increase, must find it not only paradoxical but directly *contradictory*, that one infinity may be greater than another one. However, we have already found above, that this view rests on a concept of infinity which does not coincide at all with the normal use of the word. After our definition, which corresponds not only to usage but also to the purpose of science, no one

Therefore with complete confidence we can only put the equation $[1 + e + e^2 + \cdots \textit{in inf.}] = S - \overset{2}{P}$ in which we may assume that $\overset{2}{P}$ designates a quantity which is dependent on n and always positive. Accordingly we obtain

$$S = \frac{1 - e^n}{1 - e} + e^n \left[S - \overset{2}{P}\right] \tag{4}$$

or

$$S[1 - e^n] = \frac{1 - e^n}{1 - e} - e^n \overset{2}{P},$$

or finally

$$S = \frac{1}{1 - e} - \frac{e^n}{1 - e^n}.\overset{2}{P}. \tag{5}$$

Combining the two equations (3) and (5) gives

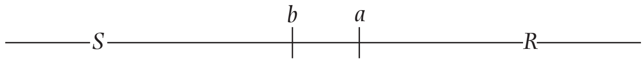
$$\frac{-e^n}{1 - e} + \overset{1}{P} = \frac{-e^n}{1 - e^n}.\overset{2}{P}$$

or

$$\overset{1}{P} + \frac{e^n}{1 - e^n}.\overset{2}{P} = +\frac{e^n}{1 - e}$$

from which we see that if we take n arbitrarily great, and thereby the value of $\frac{e^n}{1-e}$ is brought down below every arbitrary quantity $\frac{1}{N}$ however small, then also each of the quantities $\overset{1}{P}$ and $\frac{e^n}{1-e^n}.\overset{2}{P}$ must itself fall below every arbitrary value. But if this is so then each of the equations (3) and (5) shows that, because S has only an unchanging value for the same value of e and therefore cannot depend on n, $S = \frac{1}{1-e}$.

can find anything controversial or even noteworthy in the idea that one infinite multitude should be greater than another one. For example, to whom must it not be clear, that the length of the straight line



continuing without limit in the direction *aR* is an infinite length? But that the straight line *bR* going in the same direction from the point *b* may be called greater than *aR*, by the piece *ba*? And that the straight line continuing without limit on both sides *aR* and *aS* may be called greater by a quantity which is itself infinite? And so on.

§ 20

Let us now turn to the consideration of a highly remarkable peculiarity which can occur, indeed actually always occurs, in the relationship of two multitudes *if they are both infinite*, but which previously has been overlooked to the detriment of knowledge of some important truths in metaphysics, as well as physics and mathematics. Even now, as I am stating it, it will be found paradoxical to such a degree that it might be very necessary to dwell on its consideration somewhat longer. I claim that two multitudes, that are both infinite, can stand in such a relationship to each other that, *on the one hand*, it is possible to combine each thing belonging to one multitude, with a thing of the other multitude, into a pair, with the result that no single thing in both multitudes remains without connection to a pair, and no single thing appears in two or more pairs, and also, *on the other hand* it is possible that one of these multitudes contains the other in itself as a mere *part*, so that the pluralities which they represent if we consider the members of them all as equal, i.e. as units, have the *most varied relationships* to one another.

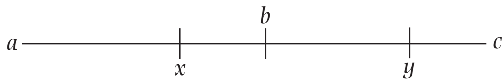
I shall offer the proof of this claim through two examples, in which what has been said indisputably occurs.

1. If we take two arbitrary (abstract) quantities, e.g. 5 and 12, then it is clear that the multitude of quantities which there are between zero and 5 (or which are smaller than 5) is infinite, likewise also the multitude of quantities which are smaller than 12 is infinite. And equally certainly the latter multitude is greater since the former is indisputably only a part of it. If we put any other quantity in the place of the quantities 5 and 12, we cannot avoid the judgement that those two multitudes do not always have the same relationship to one another but rather the most varied kinds of relationships occur. However, no less true than all these things is the following: if x denotes any quantity lying between zero and 5, and we determine the relationship between x and y by the equation

$$5y = 12x,$$

then also the value of y is a quantity lying between zero and 12, and conversely whenever y lies between zero and 12, then x lies between zero and 5. It also follows from that equation that to every value of x there belongs only one value of y, and conversely. From these two things it is clear that to every quantity $=x$, in the multitude of quantities lying between zero and 5 there is one quantity, $=y$, in the multitude of quantities lying between zero and 12, which can be combined with the former into a pair with the result that no single one of the things of which these two multitudes consist, remains without combination into a pair, and also no single one occurs in two or more combinations.

2. The second example will be taken from a spatial object. Whoever already knows that the properties of space are based on those of time, and the properties of time are based on those of abstract numbers and quantities did not of course need to learn from an example that there are such infinite multitudes as we have found generally among quantities also in time and space. Yet it is on account of the correct application of our proposition which we have to make subsequently that it is necessary to consider individually at least one case where there exist such multitudes. Therefore let a, b, c be three arbitrary points in a straight line, and let the ratio of the distances $ab : ac$ also be completely arbitrary, suppose ac denotes the greater of the two. Then although the multitude of points which lie in ab and ac are both infinite,



nevertheless the multitude of points which lie in ac exceeds that of the points in ab, because in ac as well as all the points of ab there also lie all those of bc, which do not occur in ab. We cannot even help admitting that if the ratio of the distances $ab : ac$ is altered arbitrarily then the ratio of these two multitudes will become very different. Nevertheless the same holds for these two multitudes which was proved before for the two multitudes of quantities which lie between 0 and 5 and between 0 and 12 in respect of the pairs which can be formed from each of the things from one multitude and each of the things from the other multitude. For let x be some point in ab, then if we take the point y in the direction ax, so that the ratio

$$ab : ac = ax : ay$$

holds, then y will also be a point in ac. And conversely if y is a point in ac, and if we determine ax from ay by the same equation, then x will be a point of ab. And every other x will determine another y, and conversely every other y will determine another x. But these two truths again show that for every point of ab a point of ac can be chosen, and for every point of ac, a point of ab can be chosen, with the result that of the pairs which we form from every two such points, it can be asserted that there is no single point in the multitude of points of ab, or in the multitude of points of ac, which does not appear in one of these pairs, and also none which appears in two or more pairs.

§ 21

Therefore merely for the reason that two multitudes A and B stand in such a relation to one another that to every part[h] a occurring in one of them A, we can seek out according to a certain rule, a part b occurring in B, with the result that all the pairs $(a + b)$ which we form in this way contain everything which occurs in A or B and contains each thing only once—merely from this circumstance we can—as we see—in no way conclude *that these two multitudes are equal to one another if they are infinite* with respect to the plurality of their parts (i.e. if we disregard all differences between them). But rather they are able, in spite of that relationship between them that is the same for both of them, to have a relationship of inequality in their plurality, so that one of them can be presented as a whole, of which the other is a part. An equality of these multiplicities may only be concluded if some other reason is added, such as that both multitudes have exactly the same determining grounds[i] [*Bestimmungsgründe*], e.g. they have exactly the same way of being formed [*Entstehungsweise*].

§ 22

The paradox which, as I do not deny at all, is attached to these assertions, arises solely from the circumstance that that mutual relationship which we find with the two multitudes being compared with one another, consisting in [the fact] that we can put together the parts of them in pairs in the way already mentioned several times, is indeed sufficient to define them as completely equal in respect of the *plurality* of their parts in every case where these multitudes are *finite*. Namely, if two finite multitudes are of such a nature that to every thing a of one of them we can find a b of the other one and join them into a pair so that in neither of the two multitudes does there remain a thing for which there is nothing corresponding in the other one, and there is also nothing which appears in two or more pairs, are always equal to one another in their plurality. It therefore appears that this should also be the case if these multitudes, instead of being finite, are infinite.

I say, it appears, because more exact consideration shows that in no way does it need to be so, since the reason why it happens for all finite multitudes lies precisely in their finiteness, and is therefore lacking with the infinite multitudes. Namely if both multitudes A and B are finite, or (for this is also sufficient) we know only of one of them A, that is it finite, and we disregard all differences between the things of which they consist, in order to consider now both multitudes only in respect of their plurality, then, while we designate some arbitrary thing in the multitude A by 1, some other arbitrary thing by 2 etc., in such a way that for every successive thing we always give for its designation the number of the things which we have considered so far (including this one itself), we must sometime arrive at

[h] Sometimes, as in §5, 'part' [*Teil*] is used in the sense of 'subset', here and in §23 it is used in the sense of 'element'.

[i] Steele, *PI* uses 'mode of specification' for this interesting concept.

a thing in A after the designation of which nothing more remains which is still undesignated. This is a direct consequence of the concept of a finite or countable plurality. Now let this last [thing] just spoken of in A get the number n for its designation, then the number of things in $A = n$. Now because to every thing in A there should be one found in B, that can be combined with it in a pair, then if we designate each of the things from B with exactly the symbol which that thing from A has with which it is paired, it must happen that there are also n things in B which we have used up in this way, since every one of [that multitude] gets a symbol which shows how many we have used so far. Therefore it is clear that of the things in B there are certainly not fewer than n, for this number corresponds to an actual [thing] (the one which we used last). But there are also no more of them, for if there was a single one beyond those used up so far, then there would be for this one nothing in A with which it could be combined in a pair, which contradicts the assumption. Accordingly the number of things in B is neither smaller nor greater than n, therefore $= n$. Therefore both multitudes have one and the same plurality, or as one can also say, *equal* plurality. Obviously this conclusion becomes void as soon as the multitude of things in A is an *infinite* multitude, for now not only do we never reach, by *counting*, the last thing in A, but rather, by virtue of the definition of an infinite multitude, in itself there is no such *last* thing in A, i.e. however many have already been designated, there are always others to designate. Therefore, in spite of the fact that likewise there never lack things in the multitude B which can be joined with those in A into new pairs, any reason to conclude that the two multitudes have one and the same plurality does not apply.

§ 23

What has now been said does show that the *reason*, which produces the necessary equality of finite multitudes as soon as the much discussed relationship holds between them, *becomes void with infinite multitudes*. But it does not show us, how and why it is that with the latter an inequality may often arise. This will only become clear from consideration of the examples mentioned. These show us, in fact, that the parts a and b taken from the two multitudes being compared, and which we combine into a pair $(a + b)$, *do not appear in their multitudes in completely the same way*. For if the parts a' and b' form a second pair and we compare the relationships in which a and a' appear in the multitude A and in which b and b' appear in the multitude B, then it is immediately evident that they are different. Let us take (in the first example) two quantities[j] quite arbitrarily from the multitude of quantities which lie between 0 and 5, say 3 and 4, then the [parts] in B belonging

[j] The word 'quantity' is used in this section to reflect the use of *Größe*, although 'number' might appear more natural.

to them (forming a pair with them) are obviously

$$\frac{12}{5}.3 \quad \text{and} \quad \frac{12}{5}.4, \qquad \text{i.e.} \qquad 7\frac{1}{5} \quad \text{and} \quad 9\frac{3}{5}.$$

Now if we understand (as we should) by the *relationship* between two things the collection into a whole of *all* known properties, then we may take into account for the relationship in which the parts 3 and 4 stand to one another in the one multitude, and $7\frac{1}{5}$ and $9\frac{3}{5}$ in the other multitude, not merely that ratio which is usually called *geometric*, but rather looking at everything which belongs here, therefore also in particular at the *arithmetic* difference between the quantities 3 and 4, which is quite different from between the quantities $7\frac{1}{5}$ and $9\frac{3}{5}$, since the former $= 1$ and the latter $= 2\frac{2}{5}$. Therefore although every quantity in A or B can be joined with one and only one unique [quantity] in B or A into a pair nevertheless the multitude of quantities in B is different (greater) than in A because the *distance* which every two such quantities in B have from one another is different (greater) than the *distance* which separates the two corresponding quantities in A from one another. Hence it follows naturally that every two of these quantities in B have *between them* a different (greater) multitude of such quantities than is the case in A, and therefore it is no surprise that the *whole* multitude of quantities in B is different (greater) than in A. It is completely similar in the two examples, therefore we wish to say no more about these than that the points in ab that are joined in thought with the points of ac in pairs, are all standing *nearer* to one another than the corresponding ones in ac, since the distance of every two there to the distance of every two here is always in the ratio of $ab : ac$.

§ 24

If we may now regard the proposition of §20 as sufficiently proved and clarified by the foregoing, then it follows as the next consequence of it *that we may not immediately put equal to one another, two sums of quantities which are equal to one another pair-wise* (i.e. every one from one with every one from the other), *if their multitude is infinite*, unless we have convinced ourselves that the infinite plurality of these quantities in both sums is the same. That the summands determine their sums, and that therefore equal summands also give equal sums, is indeed completely indisputable, and holds not only if the multitude of these summands is finite but also if it is infinite. But because there are different infinite multitudes, in the latter case it must also be proved that the infinite multitude of these summands in the one sum is exactly the same as in the other. But by our proposition it is in no way sufficient, to be able to conclude this, if in some way one can discover for every term occurring on one sum, another equal to it in the other sum. Instead this can only be concluded with certainty if *both multitudes have the same basis for their determination*. If this is overlooked we shall see subsequently, from some examples, what absurdities may be involved in calculation with infinity.

§ 25

I now come to the claim that there is an infinity not merely among the things which have no reality, but *also in the area of reality itself*. Whoever has arrived through a series of arguments from purely conceptual truths, or in some other way, to the highly important conviction *that there is a God*, a being which has the ground of his being in nothing else, and just for this reason is an *altogether perfect* being, i.e. all perfections and powers which can be present together, and each of them in the highest degree in which they can be together, are combined in him, who therefore takes on the existence of a being which has infinitude in more than one respect, in his *knowing*, his *willing*, his *external effect* (his power). He *knows infinitely many things* (namely the universe of truths), he *wills infinitely many things* (namely the sum of all possible good things), and *everything, which he wants, he puts into reality* through his power to produce external effect. From this last property of God arises the further consequence that there are beings outside of him, namely *created* beings which we call, in contrast to him, *finite beings*, of which nevertheless some infinite things can be proved. For already the *multitude* of these beings must be an infinite one, likewise the multitude of the *circumstances* which each single one of these beings experiences during however short a time, must be infinitely great (because each such time contains infinitely many moments) etc. Therefore we also meet with infinity everywhere in the area of reality.

§ 26

Nevertheless several of those scholars who realize they cannot deny infinity with those things which have no reality (like the mere propositions and truths in themselves) refuse to admit this. For to admit an infinity also in the area of reality, would, they think, be forbidden by the ancient principle *that all reality must have a general definiteness* [*durchgängige Bestimmtheit*]. However I believe I have already shown in the *Wissenschaftslehre* (Bd. I, §45) that this principle also holds of the unreal things in exactly the sense in which it holds of all *real* things. Namely, it holds generally simply in the sense that for every two *contradictory* properties, one must belong to each object (each arbitrary thing), and the other must be denied of it. Therefore if it were established that we violate this principle by the acceptance of an infinity of things which have reality, then we might also not speak of any infinity of the unreal objects of our thought, therefore we might not even admit an infinite multitude of truths in themselves or of mere numbers. But we do not violate the principle referred to at all when we declare something as infinite. We are only saying that in a certain respect there are in this object a plurality of parts which is greater than every arbitrary number, therefore indeed a multiplicity *which cannot be determined by a mere number*. But from this it does not follow at all that this plurality is *something which cannot be determined in any way*; it certainly does not follow that there is even a single pair of contradictory properties *b* and not-*b* of which both of them would have to be denied. What has no colour, e.g. a proposition, that may of course not be determined by the statement of its colour, whatever has no sound, cannot be determined by the statement of its sound etc. But on that account such things are certainly not incapable of being determined and do not make an exception to the principle that of the two predicates *b* or not-*b* (blue or not-blue, harmonious or unharmonious etc.), if we interpret them thus, as we must, so that they remain contradictory, one of them belongs to each thing. In just the same way as not being blue, or not being fragrant is a determination of Pythagoras' theorem (of course, only a very wide one), also the mere statement that the multitude of points between *m* and *n* is infinite is a determination of this multitude. And it may often not need many statements in order to determine such an infinite multitude of things *completely*, i.e. so that *all* its properties follow merely from the few that have been stated. Thus we have the infinite multitude of points just mentioned between *m* and *n* determined in the most complete way as soon as we only determine the two points *m* and *n* themselves (say by an intuition referring to them). For then it is decided precisely for every other point, merely by those few words, whether it belongs to this multitude or not.

§ 27

If in the foregoing I have been allowed to defend many assumptions of infinity against incorrect denials of them, I must now acknowledge with equal candour that many scholars especially among *mathematicians*, have gone too far in the opposite direction. They have adopted sometimes an *infinitely large*, and sometimes an *infinitely small*, where according to my own conviction there is none.

1. I have no objection to the assumption of an *infinitely large time interval*, if one understands by it a time interval which has either no start or no end or even neither the one nor the other (the whole of time or the collection of all moments of time in general is such). But I find it necessary to think of the *ratio* which one time interval, or distance between two moments, has to every other time interval or distance between two moments, as a merely finite ratio completely determined by mere concepts, therefore never to assume a time interval bounded by beginning and end as infinitely greater or smaller than another such time interval. But it is well known that many mathematicians do exactly this since they speak not only of infinitely large amounts of time, which nevertheless are to be bounded on both sides, but even more often of *infinitely small parts of time*, in comparison with which every *finite* time interval, e.g. a second, would have to be regarded as infinitely large.
2. A similar thing holds of the *distances between every two points in space*, which in my view can always stand in a merely finite relationship (completely determinable by pure concepts) to one another while nothing is more usual with our mathematicians than to speak of *infinitely large* and *infinitely small distances*.
3. Finally also with the *forces* in the universe which are assumed in metaphysics as well as physics, none of which we must suppose to be infinitely greater or smaller than another one but all are in a relationship to every other that is completely

determinable by mere concepts however often one allows oneself to do the opposite. The reasons for which I claim all these things I will not of course be able to make completely clear to anyone here who does not even know the concepts which I connect with the words *intuition* and *concept*, *derivability* of a proposition from others, *objective consequence* of a truth from other truths, and many others, finally also the *definitions* of time and space. Nevertheless whoever has at least read the two works, *Versuch einer objektiven Begründung der Lehre von der Zusammensetzung der Kräfte*,* and *Versuch einer objektiven Begründung der Lehre von der drei Dimensionen des Raumes*,** should find the following proof not entirely unintelligible.

From the definitions of time and space it follows directly that all *dependent* (i.e. created) substances always have a mutual effect on one another. Also, it may be allowed that of every two moments α and β, however near or far they may be from one another, the state of the world in the earlier one α may be considered as a *cause*, and the state of the world in the later one β as an *effect* (at least indirectly), as long as the direct actions of God which occurred in the intervening time $\alpha\beta$ are counted into the cause. Hence it follows further that from the statement of the two moments α and β, from the statement of all the *forces* which the created substances have in the moment α, from the statement of the *places* where each of them is, and finally a statement of the divine influences which one or other of those substances experienced within $\alpha\beta$—then as well as the *forces* which these substances experience at the moment β, also the *places* which belong to them are derivable in the same way as an *effect* must be derivable (equally whether directly or indirectly) from its complete cause. Now this further requires that all properties of the effect can be derived from the properties of its cause, by means of a principle [*Obersatz*], composed from nothing but pure concepts, of the form: Every cause with the property $u, u', u'', \ldots$ has an effect with the property $w, w', w'', \ldots$. An easy consequence from this which we require now for our purpose, is: Every circumstance of the cause which *does not hold equally* for the effect, i.e. which is of such a nature that the effect does not remain the same however the circumstance changes, must *be completely determined* through mere concepts for which at most some intuitions which are also required for the determination of the effect are taken as their basis.

Now after these preliminaries, our assertions made above are easily established. For if there were:

1. even only two moments α and β whose distance from one another was infinitely many times greater or smaller than the distance of two others γ and δ, then the absurdity would follow from this that the state of the world which is to occur at the moment β can absolutely not be determined from that state which occurs at the moment α together with the divine actions occurring in the time interval and also the size of the time interval $\alpha\beta$. Also for the determination of the state in which the created being exists, indeed only the *magnitudes of its forces* in a single

* Prague, 1842, published by Kronberger & Řiwnač.

** Prague, 1843, published by Kronberger & Řiwnač.

moment α, the basis of a proper time unit is necessary. For because these forces are merely *forces of change* then their magnitude cannot be judged other than with respect to a given time interval within which they bring about a given effect. Therefore if we take the time interval $\gamma\delta$ as this time unit (which we must be allowed to do), then even in the most favourable case, if with this time unit, all forces of the created substances, as they are at the moment α can be determined precisely, and if everything else which belongs to the complete cause of the state of the world ocurring at the moment β can be determined precisely, nevertheless the distance at which this moment itself stands from α cannot be determined by that time unit in that it proves to be infinitely large or infinitely small. Therefore *conversely* if it is to be allowed that every arbitrary state of the world (under the conditions already mentioned several times) should be considered as cause of every arbitrary later [state], then there may not be two moments α and β whose distance from one another compared with the distance in which another pair γ and δ stand proves to be infinitely great or small.

2. If there were even only two points in space a and b, whose distance from one another in comparison with the distance of another pair c and d was infinitely large or small, then for the determination of the state of the world belonging to some moment α would belong, among other things, the determination of the magnitude of the force (perhaps of attraction or of repulsion) which the substance A, occurring at that moment in the place a, exerts on B occurring in the place b. But if we adopt (as is always permitted) the distance cd as the unit of length, then this would, even in the most favourable case, when we were successful with all other forces, prove for this one to be a force which is impossible. For if the force of attraction or repulsion which substance A exerts on a substance completely similar to B at the distance $(= cd)$ taken for the unit of length, were to have a completely determinate magnitude, then directly from the fact that this magnitude is determinate, the magnitude of the attraction or repulsion with which A acts on B is indeterminate if the ratio of the distances $ab : cd$ on which it depends were to be infinite and therefore indeterminate.

3. Finally if there were even a single force k which appeared to be infinitely large or small in comparison with another one l, then if we denote the moment when this ratio holds by α, for this moment even in the most favourable case where all other forces had been shown to be finite for the units of time and space chosen for their measurement, and where also l was finite, the quantity k would, just for this reason, turn out to be an infinitely large or small quantity, i.e. as indeterminate. But thereby the whole state of the world at the moment α would appear indeterminate, therefore the derivation of some later state of the world as an effect produced by it would be impossible.

§ 28

Now I believe in the foregoing I have established the basic rules according to which all strange-sounding theories which we have to set out in the following, can be judged. It must be decided whether they should be renounced as errors or must be retained as propositions, which in spite of their appearance of contradiction, are nevertheless truths. The order in which we set out these paradoxes may determine the scientific area to which they belong, and their true importance, greater or lesser.

The first and most comprehensive science in whose domain we meet with paradoxes of the infinite is—as some examples have already shown—*the general theory of quantity* where such paradoxes are not missing even in *number theory*. Therefore it is with these that we shall begin.

Even the *concept* of a *calculation of the infinite* has, I admit, the appearance of being self-contradictory. To want to *calculate* something means to attempt a *determination of something* through numbers. But how can one determine the infinite through numbers—that infinite which according our own definition must always be something which we can consider as a multitude consisting of infinitely many parts, i.e. as a multitude which is greater than every number, which therefore cannot possibly be determined by the statement of a mere number? But this doubtfulness disappears if we take into account that a calculation of the infinite done correctly does not aim at a calculation of that which is determinable through no number, namely not a calculation of the infinite plurality in itself, but only a determination of the *relationship* of one infinity to another. This is a matter which is feasible, in certain cases at any rate, as we shall show by several examples.

§ 29

Whoever admits that there are infinite pluralities and therefore also infinite quantities generally, cannot also deny that there are infinite quantities which differ from one another according to their quantity (magnitude) in various ways. For example if we denote the series of natural numbers by

$$1, 2, 3, 4, \ldots, n, n+1, \ldots \textit{in inf}.$$

then the expression

$$1 + 2 + 3 + 4 + \cdots + n + (n+1) + \cdots \textit{in inf}.$$

will be the *sum* of these natural numbers, and the following expression

$$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n+1)^0 + \cdots \textit{in inf}.$$

in which the single summands, $1^0, 2^0, 3^0, \ldots$ all represent mere units, represents just the *number* [*Menge*] of all natural numbers. If we designate this by $\overset{0}{N}$ and therefore form the merely symbolic equation

$$1^0 + 2^0 + 3^0 + \cdots + n^0 + (n+1)^0 + \cdots \textit{in inf}. = \overset{0}{N} \qquad (1)$$

and in the same way we designate the number of natural numbers from $(n+1)$ by $\overset{n}{N}$, and therefore form the equation

$$(n+1)^0 + (n+2)^0 + (n+3)^0 + \cdots \textit{in inf.} = \overset{n}{N}. \tag{2}$$

Then we obtain by subtraction the certain and quite unobjectionable equation

$$1^0 + 2^0 + 3^0 + \cdots + n^0 = n = \overset{0}{N} - \overset{n}{N} \tag{3}$$

from which we therefore see how two infinite quantities $\overset{0}{N}$ and $\overset{n}{N}$ sometimes have a completely definite finite difference.

On the other hand if we designate the quantity which represents the *sum* of all natural numbers by $\overset{0}{S}$, or assert the merely symbolic equation

$$1 + 2 + 3 + \cdots + n + (n+1) + \cdots \textit{in inf.} = \overset{0}{S} \tag{4}$$

then we will certainly realize that $\overset{0}{S}$ must be far greater than $\overset{0}{N}$. But it is not so easy to determine precisely the difference between these two infinite quantities or even their (geometrical) *ratio* to one another. For if, as some people have done, we wanted to form the equation

$$\overset{0}{S} = \frac{\overset{0}{N} \cdot (\overset{0}{N} + 1)}{2}$$

then we could hardly justify it on any other ground than that for every finite multitude of terms the equation

$$1 + 2 + 3 + \cdots + n = \frac{n.(n+1)}{2}$$

holds, from which it appears to follow that for the complete infinite multitude of numbers n just becomes $\overset{0}{N}$. However it is in fact not so, because with an infinite series it is absurd to speak of a last term which has the value $\overset{0}{N}$.

The purely symbolic equation (4) underlying all this will surely allow the derivation, through successive multiplication of both sides by $\overset{0}{N}$, of the following equations:

$$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \cdots \textit{in inf.} = (\overset{0}{N})^2$$

$$1^0.(\overset{0}{N})^2 + 2^0.(\overset{0}{N})^2 + 3^0.(\overset{0}{N})^2 + \cdots \textit{in inf.} = (\overset{0}{N})^3 \quad \text{etc.}$$

from which we are convinced that there also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times. But it also certainly follows from this there are infinite quantities which have every arbitrary rational, as well as irrational, ratio $\alpha : \beta$ to one another, because, as long as $\overset{0}{N}$ denotes some infinite quantity which always remains the same, $\alpha.\overset{\circ}{N}$ and $\beta.\overset{\circ}{N}$ are likewise a pair of infinite quantities which are in the ratio $\alpha : \beta$.

It is no less clear that it will be found that the whole *multitude* (plurality) of quantities which lie between two given quantities, e.g. 7 and 8, although it is equal to an *infinite* [multitude] and therefore cannot be determined by any number however great, depends solely on the magnitude of the distance of those two boundary quantities from one another, i.e. on the quantity $8 - 7$, and therefore must be an equal [multitude] whenever this distance is equal. Assuming this, if we designate the multitude of all quantities lying between a and b by

$$\text{Mult.}\ (b - a)$$

there will be innumerable equations of the following form:

$$\text{Mult.}\ (8 - 7) = \text{Mult.}\ (13 - 12)$$

and also of the form

$$\text{Mult.}\ (b - a) : \text{Mult.}\ (d - c) = b - a : d - c$$

against the correctness of which no valid objection can be made.

§ 30

Now as these few examples are sufficient to show that a *calculation with the infinitely large* may exist, so one with the *infinitely small* may also exist. For if $\overset{\circ}{N}$ is infinitely large, then indeed

$$\frac{1}{\overset{\circ}{N}}$$

necessarily represents a quantity which is infinitely small, and at least in the *general* theory of quantity, we shall have no reason to describe such an idea as altogether empty. In order to give a single example, if the question is raised of what is the probability if someone who shoots a bullet at random, shoots it in such a way that its centre goes precisely through the centre of that apple hanging on this tree, then everyone must admit that the multitude of all possible cases of an equal or still smaller probability is infinite whence it follows that the degree of that probability has a magnitude $=$ or $< \frac{1}{\infty}$. But with this it is proved that we have infinitely many of the infinitely small quantities, of which they have every arbitrary ratio one to another. In particular, it can even be infinitely greater. Therefore there also exist infinitely many orders among the infinitely large, as also among the infinitely small quantities, and by following certain rules it will indeed be possible to find very often correct equations between quantities of this kind.

For example, if it is first decided that the value of variable quantity y depends on another x in such a way that the equation,

$$y = x^4 + ax^3 + bx^2 + cx + d$$

always holds between them, and it is compatible with the nature of that special kind of quantity which x and y designate here, that they can also become infinitely small and therefore can take an infinitely small increment, then if we can increase x by an infinitely small part designated by dx, and the change which y undergoes is designated by dy, then also the following equation must necessarily hold,

$$y + \mathrm{d}y = (x + \mathrm{d}x)^4 + a(x + \mathrm{d}x)^3 + b(x + \mathrm{d}x)^2 + c(x + \mathrm{d}x) + d,$$

from which also follows without contradiction,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4x^3 + 3ax^2 + 2bx + c) + (6x^2 + 3ax + b)\mathrm{d}x + (4x + a)\mathrm{d}x^2 + \mathrm{d}x^3$$

which represents the ratio of the two infinitely small quantities as a quantity dependent not only on a, b, c and x but also on the value of the variable dx itself.

§ 31

However most mathematicians who ventured to calculate with the infinite have gone much further than is allowed by the principles established here. Not only did they permit the assumption, without thinking, sometimes of an infinitely large and sometimes of an infinitely small among quantities which in their nature are incapable of such (of which examples are to be mentioned subsequently) but they even presumed sometimes to make quantities which arise from the summation of infinite series equal to one another, sometimes to set one as greater or smaller than the other, merely because in both of them corresponding terms which stand in such relationships of equality or inequality can be found although their multitudes were obviously unequal. They ventured to state that not only does every infinitely small quantity, or also one of a *higher* order next to one of a *lower* order, *vanish like a mere zero* in the summation with a finite [quantity], but also every infinitely *great* quantity of lower order in the summation next to one of a *higher* order *vanishes like a mere zero*. In order to justify to some extent their method of calculation based on this proposition, they think of the claim that it is permissible to consider a mere zero as divisor and that the quotient

$$\frac{1}{0}$$

basically denotes nothing but an *infinitely large quantity*, but the quotient $\frac{0}{0}$ denotes a completely *indeterminate quantity*. We must show how false and misleading these concepts are because even these days they are still more or less fashionable.

§ 32

It was only in 1830, in *Gergonne*'s *Annales de Mathématique* (Vol. 20, No. 12), someone with the signature *M. R. S.* attempted to prove that the well-known infinite series

$$a - a + a - a + a - a + \cdots \textit{in inf}.$$

has the value $\frac{a}{2}$, since he believed, having put this value $= x$, he may conclude that

$$x = a - a + a - a + \cdots \textit{in inf.} = a - (a - a + a - a + \cdots \textit{in inf.})$$

and the series enclosed in the brackets is identical with the one to be calculated therefore may again be put $= x$ which gives,

$$x = a - x$$

and therefore

$$x = \frac{a}{2}.$$

The false inference here is not deeply hidden. The series in the brackets obviously does not have the same multitude of terms as the one put $= x$ at first, rather it is lacking the first a. Therefore its value, supposing it could actually be stated, would have to be denoted by $x - a$. But this would have given the identical equation

$$x = a + x - a.$$

'But,' it might be said, 'there is something paradoxical here in that this series which is certainly not infinitely large, should have no exactly determinable, measurable value, the more so since it may arise through an indefinitely continued division by $2 = 1 + 1$ into a: an origin which speaks entirely for the correctness of the assumption that its true value is $\frac{a}{2}$.'

I may draw attention to the fact, which is not in itself incomprehensible, that there may be *quantity expressions* which designate *no actual quantity*, as we generally accept, and must accept, zero itself as one such expression.

In particular a *series*, if we want to consider it only as a quantity, namely only as the *sum* of its terms must, by virtue of the *concept* of a sum (which belongs to multitudes, i.e. to those totalities for which no attention is paid to the *order* of their parts) have such a nature that it undergoes no change in value when we make a change in the order of its terms. With quantities especially it must be that:

$$(A + B) + C = A + (B + C) = (A + C) + B.$$

This property now offers us a clear proof that the expression [*Zeichnung*] under discussion:

$$a - a + a - a + a - a + \cdots \textit{in inf.}$$

is not an expression of an actual quantity. For we should surely change nothing in the quantity represented here, supposing one was represented, if we altered that expression thus:

$$(a - a) + (a - a) + (a - a) + \cdots \textit{in inf.} \tag{1}$$

because here nothing else has happened than that every two adjacent terms in a partial sum have been combined. This certainly must be possible, because the given series should actually have no *last* term. But then we obtain

$$0 + 0 + 0 + \cdots \textit{in inf}.$$

which is obviously only $= 0$.

Nevertheless, just as little can anything be altered in the quantity which that expression represents, supposing it does actually represent one, if we re-arrange it thus:

$$a + (-a + a) + (-a + a) + (-a + a) + \cdots \textit{in inf}. \tag{2}$$

where, with the omission of the first [term], we combine every two successive terms in a partial sum, or also:

$$-a + (a - a) + (a - a) + (a - a) + \cdots \textit{in inf}. \tag{3}$$

which is obtained from (1) if the terms in each pair are transposed and in the expression obtained the same change is made as that by which (2) arises from (1). Therefore if the given quantity expression were *not empty* then the expressions (1), (2) and (3) would all have to denote the same quantity, because it is clear that the idea of a sum of one and the same multitude of quantities cannot represent several quantities *different from one another*, as is the case for example with the ideas $\sqrt{+1}$, $\arcsin = \frac{1}{2}$ etc. However, the quantity idea [*Größenvorstellung*] under consideration here:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots \textit{in inf}.$$

if it is not to be altogether empty, with the same justification with which we wanted to put it equal to zero (which is usually called a quantity albeit in a figurative sense), it must also be put $= +a$, and also $= -a$. This is altogether absurd and therefore justifies the conclusion that we have here an absolutely empty idea.

It is true that the series under discussion is produced by an indefinitely continued division of $2 = 1 + 1$ into a, but all series which are produced in such a way can, of course, just because that division always leaves a remainder (here alternately $-a$ and $+a$), only give the true value of the quotient (here $\frac{a}{2}$), if the remainders arising from further division become smaller than every quantity however small. This occurs in the case of the series, $a + ae + ae^2 + \cdots$ *in inf*., considered in §18, which is produced by the division of $1 - e$ into a provided $e < 1$. But if, as in the previous case, $e = 1$, or even if $e > 1$, where therefore the remainder rises ever higher the further the division is continued, nothing could be more understandable than that the value of the series cannot become equal to the quotient $\frac{a}{1-e}$. Or how should, for example, the series with alternating signs:

$$1 - 10 + 100 - 1\,000 + 10\,000 - 100\,000 + \cdots \textit{in inf}.$$

which arises through the indefinitely continued division of $1 + 10$ into 1, be able to become $= \frac{1}{11}$? Whoever really wanted to put the series

$$1 + 10 + 100 + 1\,000 + \cdots \textit{in inf}.$$

composed of purely positive terms, equal to the negative value $-\frac{1}{9}$, merely because $\frac{1}{1-10}$ expands into this series? Nevertheless the person *M. R. S.* mentioned before still defends such summations and wants to prove, for example, the correctness of the equation

$$1 - 2 + 4 - 8 + 16 - 32 + 64 - 128 + \cdots \textit{in inf.} = \frac{1}{3}$$

only for the reason that

$$\begin{aligned} x &= 1 - 2 + 4 - 8 + 16 - 32 + 64 - \cdots \\ &= 1 - 2(1 - 2 + 4 - 8 + 16 - 32 + \cdots) \\ &= 1 - 2x. \end{aligned}$$

Here it is again overlooked that the series contained in the brackets is not the same one as taken originally because it no longer has the same multitude of terms. It is also clear that this number expression is empty in a similar way as with the one considered earlier, because it leads to contradictory results. For on the one hand, it would have to be that:

$$\begin{aligned} &1 - 2 + 4 - 8 + 16 - 32 + 64 - \cdots \\ &\quad = 1 + (-2 + 4) + (-8 + 16) + (-32 + 64) + \cdots \\ &\quad = 1 + 2 + 8 + 32 + 64 + \cdots \end{aligned}$$

on the other hand, equally certainly:

$$\begin{aligned} &= (1 - 2) + (4 - 8) + (16 - 32) + (64 - 128) + \cdots \\ &= -1 - 4 - 16 - 64 - \cdots \end{aligned}$$

so that therefore, by a doubly justified procedure, the same expression results in one time an infinitely large positive value and another time an infinitely large negative value.

§ 33

Therefore if we wish to avoid getting onto the wrong track in our calculations with the infinite then we may never allow ourselves to declare two infinitely large quantities, which originated from the summation of the terms of two infinite series, as equal, or one to be greater or smaller than the other, because every term in the one is either equal to one in the other series, or greater or smaller than it. We may, just as little, declare such a sum as the greater just because it includes all the terms of the other and in addition many, even infinitely many, terms (which are all positive), which are absent in the other. For even in spite of that it can be smaller, even infinitely smaller, than the latter. An example is supplied by the very well-known sum of the *squares* of all natural numbers compared with the sum of the *first powers* of these numbers. Certainly no one can deny that every term of the series of all *squares*

$$\left.\begin{array}{l} 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + \cdots \textit{in inf.} = \\ 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + \cdots \textit{in inf.} \quad = \end{array}\right\} \overset{2}{S}$$

because it is also a natural number, also appears in the series of first powers of the natural numbers

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+\cdots \textit{in inf.} = \overset{1}{S}$$

and likewise in the latter series $\overset{1}{S}$, together with all the terms of $\overset{2}{S}$ there appear many (even infinitely many) terms which are missing from $\overset{2}{S}$ because they are not square numbers. Nevertheless $\overset{2}{S}$, the sum of all square numbers, is not smaller but is indisputably greater than $\overset{1}{S}$, the sum of the first powers of all numbers. For first of all, in spite of all appearance to the contrary, the *multitude of terms* [*Gliedermenge*] in both series (not considered as sums, and therefore not divisible into arbitrary multitudes of parts) is certainly the same. By the fact that we raise every single term of the series $\overset{1}{S}$ to the square into the series $\overset{2}{S}$, we alter merely the nature (the magnitude) of these terms not their plurality. But if the multitude of terms in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the *first* term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, namely:

$$0, 2, 6, 12, 20, 30, 42, 56, \ldots, n(n-1), \ldots \textit{in inf.},$$

in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. If we therefore subtract from this remainder the series $\overset{1}{S}$ for the second time, then we obtain as the *second* remainder a series of the same number of terms

$$-1, 0, 3, 8, 15, 24, 35, 48, \ldots, n(n-2), \ldots \textit{in inf.}$$

in which, with the exception of the first *three* terms, all the following terms are greater than the corresponding ones in $\overset{1}{S}$, so that also this third remainder is without contradiction greater than $\overset{1}{S}$. Now since these arguments can be continued without end it is clear that the sum $\overset{2}{S}$ is infinitely greater than the sum $\overset{1}{S}$, while in general we have

$$\overset{2}{S} - m\overset{1}{S} = (1 - m) + (2^2 - 2m) + (3^2 - 3m) + (4^2 - 4m) \\ + \cdots + (m^2 - m^2) + \cdots + n(n - m) + \cdots \textit{in inf.}^{\text{k}}$$

In this series only a finite multitude of terms, namely the first $m - 1$ are negative and the mth is 0, but all succeeding ones are positive and increase indefinitely.

§ 34

Before we can put the incorrectness of the other assertions mentioned in §31 in a proper light we must determine [*bestimmen*] the concept of zero rather more precisely than is usually done.*

All mathematicians indisputably wish to know that only such a concept is connected with the symbol 0 that the two equations

$$A - A = 0 \qquad \text{I}$$

$$A \pm 0 = A \qquad \text{II}$$

may always be written, whatever kind of quantity expression A is, regardless whether it corresponds to an actual quantity or is quite empty. Now here everyone will admit that this can only be permitted if we consider the symbol 0 itself not as the idea of an actual quantity, but rather as the mere absence of a quantity and the notation $A \pm 0$ as a demand for the possible quantity, which A denotes, if in truth *we wish neither to add nor subtract something*. But it would be wrong to believe that the mere explanation that zero is an empty quantity idea is sufficient for the complete determination of the concept which mathematicians associate with this symbol. For obviously there are other notations for quantities in mathematics, as in particular the sign $\sqrt{-1}$ which has become so very important in analysis, which are likewise empty, which nevertheless we may not view and deal with as equivalent to 0. But if we determine the meaning of the symbol 0 more precisely by the definition: it is to be understood in such a way that the two equations I and II hold generally, then we establish a concept which on the one hand is quite wide enough for what is required by previous usage and the interests of science and yet on the other hand is also sufficiently narrow to prevent any misuse of it.

But, on further consideration, it is not merely the concept of zero which is determined in a special way by stipulating the general validity of the two equations I and II, but also the concepts of *addition* and *subtraction*, which appear here

* I very gladly grant to *Herr M. Ohm* the merit, in his very valuable *Versuche eines vollkommen konsequenten Systems der Mathematik* (2. Aufl., Berlin, 1828), to have been the first to have drawn the attention of the mathematical public to the difficulties in the concept of zero.

k The German first edition has only $\overset{2}{S} - \overset{1}{S}$ on the left-hand side of this equation.

with the symbols $+$ and $-$, undergo a particular extension which is very much to the advantage of science.

Furthermore, the same advantage of science requires that the concept of *multiplication* may be understood so broadly that whatever A is (whether finite, or an infinitely large or infinitely small quantity, or even a merely empty quantity idea like $\sqrt{-1}$ or 0) the equation:

$$0 \times A = A \times 0 = 0 \qquad \text{III}$$

can be formed.

Finally, also in the interests of science, we must require that the concept of *division* be conceived as generally as possible so as not to contradict one of the three equations already established, therefore also in the equation:

$$B \times \left(\frac{A}{B}\right) = \left(\frac{A}{B}\right) \times B = A \qquad \text{IV}$$

to give the symbol B such a wide range as those three equations allow in the generality already belonging to them. Now all these permit that B may designate any arbitrary finite, as well as infinitely large or infinitely small actual quantity, as well as the imaginary $\sqrt{-1}$, but absolutely not that B may become put $= 0$, i.e. that at any time we do not use zero, or some expression equivalent to zero, as a *divisor*. For since by III it must be that $0(A) = 0$, whatever A is, then if we put $B = 0$ in IV it would also have to be that $B\left(\frac{A}{B}\right) = 0$ which would agree with the equation $B\left(\frac{A}{B}\right) = A$ required in IV only in the single case when $A = 0$. Therefore in order not to fall into contradiction we must establish the rule *that zero or an expression equivalent to zero may never be used as a divisor in an equation which is to be anything other than a mere identity*, as perhaps

$$\frac{A}{0} = \frac{A}{0}.$$

That the observation of this rule is absolutely necessary is proved, apart from what has just been said, from the highly absurd consequences which arise from completely correct premisses as soon as we allow ourselves divisions by zero.

Let a be any kind of real quantity, then if division by an expression equivalent to zero, e.g. $1 - 1$, is to be permitted, then by the well-known and certainly quite correct method of division, the following equation arises:

$$\frac{a}{1-1} = a + a + \cdots + a + \frac{a}{1-1}$$

where arbitrarily many of the summands of the form a can appear. Now if we subtract from both sides the same quantity expression $\frac{a}{1-1}$ the highly absurd equation arises:

$$a + a + \cdots + a = 0.$$

If a and b are a pair of different quantities then the two identical equations hold:

$$a - b = a - b$$
$$b - a = b - a$$

Therefore also by addition $\quad a - a = b - b$

or $\quad a(1 - 1) = b(1 - 1).$

Now if it is permitted to divide the two sides of an equation by a factor equivalent to zero, then we obtain the absurd result $a = b$, whatever a and b may be. Nevertheless it is generally known that an incorrect result may be reached much too easily with larger calculations if a common factor is cancelled from both sides of an equation without first being convinced that it is not zero.

§ 35

It will now be easy to show how wrong is the assertion put forward by so many that not only does an infinitely small quantity of higher order *vanish like a mere zero* in combination by addition or subtraction with another of lower order or with a finite quantity, but also every finite quantity, and even every infinitely large quantity of each arbitrarily high order in combination by addition or subtraction with another infinitely large quantity of higher order *vanishes like a mere zero*. Now if this is to be understood—and in the usual expositions which read rather carelessly, like the expressions just used, one is not warned against such misinterpretation—if this, I say, is to be explained in such a way that from the combination [*Komplexe*] of the two quantities $M \pm m$, of which the first is infinitely greater than the second, the latter may be dropped altogether, even if in the course of the calculation the quantity M itself may disappear (maybe by subtraction of a quantity equal to it), then I hardly need to prove the error of this rule.

Nevertheless it will be said that this is not what is meant. If the quantities M and $M \pm m$ are said to be equal, then it is not meant that they yield an equal result if in further calculation they enter into new combinations by additions or subtractions, but rather their equality only consists in this, that in the process of *measuring*, namely by a quantity N which has equal status with them and stands in a finite (therefore completely determinable) ratio to one of them, e.g. to M, they give equal results. This would in fact be the least that one is justified to require in a definition of a pair of quantities being *equally great*. But do M and $M \pm m$ achieve even this much? If one of them, e.g. M, stands in an irrational ratio to the measure N, then it can certainly happen that, with the most usual method of measuring, which seeks, for every arbitrary number q however large, another number p with the property that

$$\frac{M}{N} > \frac{p}{q} < \frac{p+1}{q}$$ [1]

[1] It is printed thus in the first edition. It clearly means what we would now write as $\frac{p}{q} < \frac{M}{N} < \frac{p+1}{q}$. Similarly for the equation two lines later.

and it can happen that $\frac{M \pm m}{N}$ always remains within the same limits, i.e. that also

$$\frac{M \pm m}{N} > \frac{p}{q} < \frac{p+1}{q}.$$

But if the ratio $\frac{M}{N}$ is rational then there is a q for which

$$\frac{M}{N} = \frac{p}{q}$$

and on the other hand $\frac{M \pm m}{N}$ is either $>$ or $< \frac{p}{q}$, where therefore there is a difference made known between these quantities even in comparison with mere *numbers* (finite quantities). Therefore how can we call them equal to one another?

§ 36

In order to avoid such contradictions several mathematicians have taken refuge, following *Euler*'s procedure, in the explanation that infinitely small quantities were in fact *mere zeros* but that the infinitely large quantities were the quotients which arise from a finite quantity from division by a mere zero. With this statement the vanishing or dispensing of an infinitely small quantity in combination by addition with a finite quantity was more than justified. But it was all the more difficult to make comprehensible the existence of infinitely large quantities, likewise the possibility of the emergence of a finite quantity from the division of two infinitely small or large quantities, and the existence of infinitely small and infinitely large quantities of higher order. For on this view the infinitely large quantities arose from a division by zero or a quantity expression equivalent to zero (which is actually an empty idea), therefore in a way forbidden by the laws of calculation. But to all those finite or even infinite quantities which could arise by division of an infinite quantity into another infinite quantity there cling the many blemishes of illegitimate birth.

What seems to support best the correctness of this calculation with zeros is surely the way in which the value of a quantity y, that is dependent on the variable x and is to be determined by the equation

$$y = \frac{Fx}{\Phi x}$$

may be calculated in the special cases when a certain value of $x = a$ makes either the denominator alone of this fraction equal to zero, or the denominator and numerator together equal to zero. In the first case, if $\Phi a = 0$ but Fa remains a finite quantity it is concluded that y has become *infinitely large*. On the other hand, in the second case, when $Fa = 0$ as well as $\Phi a = 0$, then it is concluded that the two expressions Φx and Fx contain a factor of the form $(x - a)$ once or several times and therefore must be of the form

$$\Phi x = (x-a)^m.\phi x; \quad Fx = (x-a)^n.fx$$

where ϕx or fx can possibly also represent constants. Now if $m > n$ then it is concluded that after removing the common factors in the denominator and the numerator (which does not change the value of the fraction $\frac{Fx}{\Phi x}$), the former still becomes zero for $x = a$, and therefore the assertion still holds that the value $x = a$ gives an infinitely large y. But if $m = n$ then, since it must be that $\frac{Fx}{\Phi x} = \frac{fx}{\phi x}$, the finite quantity which $\frac{fa}{\phi a}$ expresses, is viewed as the correct value of y. And finally if $m < n$ then it is concluded that because now

$$\frac{Fx}{\Phi x} = \frac{(x-a)^{n-m}.fx}{\phi x}$$

becomes zero for $x = a$, that the value $x = a$ makes the quantity y zero.

My opinion of this procedure is as follows. If the value of y belonging to $x = a$ in the specified cases is declared to be infinitely large, then that can obviously only happen to be true if the quantity y is of a kind which *can* become infinitely large. In the first place, it remains true that this result does not arise from the given expression, which here calls for a division by zero. Merely from the circumstance that is stated, the value of y is always the one which the given expression $\frac{Fx}{\Phi x}$ specifies, we can only argue for the nature of the quantity y for all those values of x which represent a real quantity, but not for those with which this expression is *empty*, as is the case if its denominator becomes zero, or even only its numerator is zero, and certainly if both are zero at once. It could well be said that the quantity y, in the case mentioned first where only $\Phi x = 0$, may become *greater* than every given quantity, and in the second case where only $Fx = 0$, it may become *smaller* than every given quantity. Finally in the third case where $\frac{Fx}{\Phi x}$ contains an equal number of factors of the form $(x - a)$ in the denominator and numerator, [the quantity y] can approach as close as desired to the value $\frac{fa}{\phi a}$ while x approaches as close to the value a as desired. However nothing follows from all this about the nature of this value when the expression $\frac{Fx}{\Phi x}$ is *empty*, i.e. represents no value at all, because it either takes the value 0 itself, or the form $\frac{c}{0}$, or indeed the form $\frac{0}{0}$. For the proposition about the equality of the value of two fractions of which one differs from the other only by the removal of a common factor in the denominator and numerator holds indeed in all cases except in the case where this factor is a *zero*. Because otherwise with the same justification with which we claim to maintain that $\frac{2.0}{3.0} = \frac{2}{3}$, it might also be maintained that any arbitrary quantity, e.g. $1000 = \frac{2}{3}$. For it is certain that $3000.0 = 0$ as well as $2.0 = 0$. Therefore if $\frac{2.0}{3.0}$ may be put $= \frac{2}{3}$, then also

$$\frac{2 \times (3000.0)}{3 \times (2.0)} = \frac{(2.3000).0}{(3.2).0} = \frac{2.3000}{3.2} = 1000.$$

The fallacy which is obvious here, attracted less attention above because the division with a factor $(x - a)$ equivalent to zero is done in a form which disguises this zero value. And because the removal of this is allowed in every other case, it is assumed all the more confidently that it may also be allowed in this case, because the resulting value for y is just as one is entitled to expect, that is, if it is a *finite* value, exactly as the law of continuity requires it, zero, if the neighbouring values decrease towards zero, and infinitely large if the neighbouring values increase indefinitely. But it is being forgotten here that the law of continuity may not be followed by all variable quantities. So a quantity which becomes as small as desired while x is brought as near as desired to the value a does not on this account have to become zero for $x = a$, and just as little, if it grows indefinitely as x approaches the value a, may it actually become infinite for $x = a$. Particularly in geometry there are numerous quantities which do not follow any law of continuity, for example the magnitudes of lines and angles which serve for the determination of the circumscribing lines and surfaces of polygons and polyhedra etc.

§ 37

Although we can reproach, not unjustly as I believe, previous presentations of the *theory of the infinite* with many important defects, it is nevertheless well known that *mostly quite correct results* are obtained if the rules which are generally established for calculation with the infinite are followed with suitable care. Such results could never have arisen if there were not a way of understanding and using these methods of calculation which is actually perfectly correct. I am happy to think that it might have been fundamentally this that the clever discoverers of that method had in mind although they were not immediately in a position to explain their ideas on it perfectly clearly, a matter which is generally only achieved in difficult cases after repeated attempts.

Let me outline briefly here how I believe this method of calculating has to be understood so that it may be completely justified. It will be sufficient to speak of the procedure which is followed with the so-called *differential* and *integral calculus*, for the method of calculating with the *infinitely large* arises easily through mere contrast, especially after all that *Cauchy* has achieved on this already.

Therefore I definitely do not need here the narrow assumption, which normally would be regarded as necessary, that the quantities used in calculation can become *infinitely small*, a restriction whereby all limited [*begrenzte*] temporal and spatial quantities, also all forces of finite matter, therefore basically all quantities whose determination we are mostly concerned with are excluded in advance from the scope of this method of calculation. I require nothing other than that these quantities in case they are *variable*, and yet not freely variable but *dependent* on one or more other quantities, have their *derivatives* (*une fonction derivée* according to the definition of Lagrange) if not for all values of their *determining parts* at least for all those for which the calculation is to be validly applied. In other words, if x is a freely variable quantity and $y = fx$ designates a quantity dependent on it, then if our calculation is to give a correct result for all values of x lying between $x = a$ and $x = b$, y must depend on x in such a way that for all values of x lying between a and b the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - fx}{\Delta x}$$

which arises when we divide the increase in y by the increase in x belonging to it, approaches a quantity $f'x$, which is either constant or depends on x alone, as close as desired provided Δx is taken small enough and then it remains as close, or approaches even closer, if Δx is made still smaller.*

If an equation between x and y is given then it is usually a very easy and well-known business to find this derivative of y. For example, if it were that

$$y^3 = ax^2 + a^3 \tag{1}$$

then one would have here for every Δx, which is not zero,

$$(y + \Delta y)^3 = a(x + \Delta x)^2 + a^3 \tag{2}$$

which gives by well-known rules

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{2ax + a\Delta x}{3y^2 + 3y\Delta y + \Delta y^2}\\ &= \frac{2ax}{3y^2} + \frac{3ay^2\Delta x - 6axy\Delta y - 2ax\Delta y^2}{9y^4 + 9y^3\Delta y + 3y^2\Delta y^2}.\end{aligned} \tag{3}$$

And the required *derived function* of y or (in *Lagrange*'s notation) y' would be

$$\frac{2ax}{3y^2},$$

a function which arises from the expression of

$$\frac{\Delta y}{\Delta x}$$

if, after its proper development, namely one in which we separate the terms in the numerator and the denominator which are multiplied by Δx or by Δy from the others, therefore in the expression

$$\frac{2ax + a\Delta x}{3y^2 + 3y\Delta y + \Delta y^2},$$

we put both Δx as well as $\Delta y = 0$.

* It can be shown that all *dependent variable quantities*, provided they are generally *determinable*, must be bound by this law in the sense that exceptions to it, if in an infinite multitude, may always only occur for *isolated values* of its *free variables*.[m]

[m] This claim that determinable functions are differentiable with possibly the exception of infinitely many isolated values may appear to be in conflict with the function Bolzano defined in F § 111. The function is intuitively determinable but it is proved in F § 135 that it is not differentiable on a dense set of values. It depends, as van Rootselaar points out in his detailed note (*PU*(5), p. 142), on exactly what Bolzano meant by 'determinable' and by 'isolated'. By drawing on material from the diaries he shows it to be 'plausible' that Bolzano meant by a determinable function one that is

I do not need to speak of the many uses there are for the finding of this *derivative*, of the ways in which for every finite increase in x the corresponding finite increase in y can be calculated by means of such derivatives, and how, if conversely only the derivative $f'x$ is given, also the original function fx can be determined up to a constant.

But because, as would be noticed just now, we obtain the derived function of a dependent quantity y with respect to its variable x, supposing it was first developed so that neither Δx nor Δy appeared anywhere as divisors, as soon as put in the expression

$$\frac{\Delta y}{\Delta x}$$

the Δx as well as the $\Delta y = 0$, then it might not be inappropriate to represent the derivative by a notation as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

providing we explain here *on the one hand* that all the Δx, Δy, or perhaps the $\mathrm{d}x$, $\mathrm{d}y$ written in their place, which appear in the development of $\frac{\Delta y}{\Delta x}$ are to be treated and viewed as *mere zeros*. But *on the other hand* the notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not to be viewed as a *quotient* of $\mathrm{d}y$ by $\mathrm{d}x$, but only as a *symbol* of the derivative of y by x.

It is clear that in no way could the objection be made to such a procedure that it assumes ratios between quantities which do not even exist (zero to zero), for that notation is known to be regarded as nothing but a *mere sign*.

Furthermore it will be just as perfectly correct if the *second* derived function of y by x, i.e. that quantity dependent merely on x (or perhaps also completely constant) which the quotient

$$\frac{\Delta^2 y}{\Delta x^2}$$

approaches as closely as desired as long as Δx may also be taken as small as desired, is denoted by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

and this is interpreted so that the quantities Δx, $\Delta^2 y$ appearing in the development of $\frac{\Delta^2 y}{\Delta x^2}$ are treated and considered as mere zeros, and the notation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ must not be regarded as a division of zero into zero but only the *symbol* of the function into which the development of $\frac{\Delta^2 y}{\Delta x^2}$ proceeds following the required change just described.

piecewise monotonic. The Bolzano function does not have this property, thus resolving the conflict. However, there remains much of interest, requiring further investigation, about Bolzano's notion of a determinable function. See also on this $F+$ § 38 on p. 586. For the history of this problem see Berg (1962) p. 26.

Once these meanings of the symbols $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}, \ldots$ are assumed we can prove strictly that every variable quantity dependent on another free variable x in a determinable way

$$y = fx,$$

is governed, with at most the exception of certain isolated values of x and Δx, by the equation

$$f(x + \Delta x) = fx + \Delta x.\frac{dfx}{dx} + \frac{\Delta x^2}{1.2}.\frac{d^2fx}{dx^2} + \frac{\Delta x^3}{1.2.3}.\frac{d^3fx}{dx^3} + \cdots + \frac{\Delta x^n}{1.2\ldots n}.\frac{d^nf(x + \mu\Delta x)}{dx^n}$$

in which $\mu < 1$.*

No one is unaware of how many important truths of the general theory of quantity [*Größenlehre*] (especially in the so-called higher analysis) can be established through this single equation. But also in applied mathematics, in the theory of space (geometry) and the theory of forces (statics, mechanics etc.) this equation paves the way for the solution of the most difficult problems e.g. the rectification of lines, the complanation of surfaces, the cubature of solids without needing some contradictory assumption of the infinitely small, as well as another alleged axiom such as the well-known Archimedean axiom and several others.

But if it is permitted to put forward equations in the previously defined sense of such a kind as, for example, the formula for the rectification of curves with a rectangular co-ordinate system

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]}$$

then it will also be possible, without danger of error, to write down equations of the following kind

$$d(a + bx + cx^2 + dx^3 + \cdots) = b\,dx + 2cx\,dx + 3dx^2dx + \cdots;$$

$$ds^2 = dx^2 + dy^2 + dz^2;$$

or if r denotes the radius of the circle of curvature of a line of simple curvature,

$$r = -\frac{ds^3}{d^2y.dx}$$

and many others, in which we consider the signs dx, dy, dz, ds, d^2y etc. not as signs of actual quantities but rather we consider them as equivalent to zero, and we see in the whole equation nothing other *than a complex sign which is so*

* The proof of this theorem for every kind of dependency of y on x, no matter if known to us and representable by the symbols used so far, or not, has already been written down by the author for a long time, and will perhaps soon be published.

constructed that if we make only genuine changes in it which algebra allows with all signs for actual quantities (therefore here also a division with dx etc.)—*an incorrect result is never produced if we eventually manage to see the signs* dx, dy *etc. disappear on both sides of the equation.*

It is easy to comprehend that this is so and must be so. For if, for example, the equation

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]}$$

is perfectly correct, how could it be that the equation

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

is not also perfectly correct since the former can also be derived from the latter directly by the kind of procedure just mentioned?

Finally, it is easy to think that also no error could be produced if, in some equation which contains the signs dx, dy, . . . , then for the abbreviation of all those summands for which we know for certain, in advance, that they will be omitted at the conclusion of the calculation as equivalent to zero, we omit them directly at the outset. For example, if we come across in some calculation the equation (arising from (1) and (2)),

$$3y^2.\Delta y + 3y\Delta y^2 + \Delta y^3 = 2ax\Delta x + a\Delta x^2$$

which, with the transition of the symbols equivalent to zero, takes the form

$$3y^2.\mathrm{d}y + 3y.\mathrm{d}y^2 + \mathrm{d}y^3 = 2ax\,\mathrm{d}x + a.\mathrm{d}x^2.$$

We can immediately see that the summands which contain the higher powers dy^2, dy^3, dx^2 will, at any rate eventually, be omitted and therefore we can immediately put

$$3y^2\mathrm{d}y = 2ax\,\mathrm{d}x$$

from which then the required derivative of y with respect to x arises directly

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2ax}{3y^2}.$$

This whole procedure, to say it finally in one word, rests on quite similar principles to those on which calculation with the so-called *imaginary quantities* (which are mere notations just like our dx, dy, . . .) rests, or also the abbreviated methods of division discovered in recent times and other similar calculation abbreviations. It is sufficient here, just as it is there, to prove the justification of the procedure, that we give to the signs introduced

$$\left(\mathrm{d}x, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots, \sqrt{-1}, \left(\sqrt{-1}\right)^3, \frac{\sqrt{-1}}{-\sqrt{-1}}\text{etc.}\right)$$

only such meanings, and we allow ourselves to make only such changes to them that *it is always the case in the end that if finite signs appear instead of the empty* *signs so that they signify actual quantities, then both sides of the equation are actually equivalent to one another.*

§ 38

If we turn to the applied part of mathematics we encounter the first paradoxes in the field of the *theory of time* in the concept of *time itself*, especially in so far as it is to be a *continuous extension*. But it rests on the *apparent contradictions* famous from ancient times which are believed to be found in the concept of a *continuous extension* of a continuum, in the same way in the temporal, as in the spatial, even in the material. Therefore we shall consider them all together.

It is very well known that everything extended, according to its concept, must be composed of parts; it is further recognized that the being [*Dasein*] of what is extended can be explained without circularity from the composition of parts which are themselves extended. Nonetheless some also claim to find a contradiction in the assumption that extension arises from parts which have no extension but are absolutely simple (points in time, or space, atoms, i.e. simple substances in the universe in the realm of reality).

If it would be asked what is found objectionable in this last explanation then they would sometimes say that a property which is lacking in all the parts cannot belong to the whole, and sometimes that every two points, in time as well as in space, and likewise also every two substances, always have a distance from one another, therefore can never form a *continuum*.

But it really does not need much reflection to see the absurdity in these objections. An attribute which is lacking in all the parts should also not belong to the whole? Precisely the converse! Every whole has, and must have, some properties which the parts lack. An automaton has the attribute of imitating almost perfectly certain movements of a living human being, but the individual parts, the springs, small wheels etc. lack this property. That every two instants in time are separated by an infinite multitude of instants in between, that likewise between every two points in space there is an infinite multitude of points lying in between them, that even in the realm of reality between every two substances there is an infinite multitude of others—is of course conceded, but what follows from this which contains a contradiction? Only this much follows, that with two points alone, or even with three, four or any merely *finite* multitude of them, no extension is produced. We admit all this, indeed we admit that even an infinite multitude of points is not always sufficient to produce a continuum, e.g. a line however short, if these points do not also have the proper *arrangement*. If we attempt to gain a clear awareness [*Bewußtsein*] of the concept which we designate by the expression a *continuous extension* or a *continuum* then we cannot help defining a continuum to exist where, and only where, a collection of simple objects (of points in time or space or even of substances) occurs which are so arranged that every single one of them has at least one neighbour in this collection at every distance however small. If this is not the case, for example, if among a given collection of points in space even only a single one occurs which is not so thickly surrounded by neighbours that for every distance, provided it is taken small enough, a neighbour can be indicated, then we say that this point stands *single* (isolated) and that that collection accordingly does not present a perfect continuum. On the other hand, if there is not a single point that is isolated in this sense in the collection of points under consideration, therefore each of them has at least one neighbour for every distance however small, then there remains nothing which could justify us denying to this collection the name of continuum. For what more would we require?

'This, *that every point has one which it touches directly!*' comes the reply. Here however, something is required which is an obvious impossibility, which ends in a contradiction in itself. For when do you want to say that a pair of points touch one another? Perhaps if the boundary of one of them (say the right-hand side of it) coincides with the boundary of the other one (say the left-hand side of it)? But surely points are *simple* parts of space, they therefore have no boundaries, no right and left sides. If one had only a part in common with the other then it would be absolutely the same as it, and if it is to have something different from it, then both must lie completely outside one another and there must therefore be space for another point lying between them. Indeed because the same holds of these intermediate points in comparison with those two, [there is space] for an infinite multitude of points.

'*But that is all incomprehensible!*' they say. Certainly, it cannot be grasped in one's fingers, or perceived with one's eyes, but it will surely be known by the understanding, and known as something which can be necessarily so and not otherwise, so that a contradiction is then only assumed if it is presented as other [than it is], if it is presented incorrectly.

However, it is continued: 'How incomprehensible it is, to imagine in the smallest line an accumulation of infinitely many points, even an infinite multitude of such accumulations of points, as must be done in the usual theory! For even the smallest line can be divided into an infinite multitude of other lines, since it may first be divided into two halves, then these again may be divided into two halves and so on without end!' I find nothing wrong and nothing strange in this whole chain of ideas up to the single expression of *a smallest line* which many people could only miss from lack of attention because there is no such thing and cannot be such, and of this very thing being considered it is immediately explained that it can be divided into smaller ones. Every infinite multitude, not only the points in a line, can be divided into parts which themselves contain infinite multitudes, indeed into infinitely many such parts. For if ∞ denotes an infinite multitude, then also $\frac{\infty}{2}, \frac{\infty}{4}, \frac{\infty}{8}, \ldots$ are *infinite multitudes*. Thus it is with the concept of the infinite.

In case the previous discussions, after further consideration, were to turn out satisfactorily, the following might eventually be said. 'But how should we interpret the claim of those mathematicians who explain that extension cannot be produced by any accumulation of points however large, and that by division into a multitude of parts, however large, extension can never be resolved into simple points?' Strictly speaking, it should of course be said *on the one hand*, that a finite multitude can never provide an extension and an infinite multitude only does so if the condition, already mentioned several times, that every point has certain neighbours at every sufficiently small distance. *On the other hand*, it should be admitted that not *every division* of a given spatial thing into parts, namely no division into such parts whose number is only finite, not even every one such as goes on indefinitely (e.g. by continued bisection), as we saw before, reaches the simple parts. Nevertheless, one must insist on the fact that ultimately every continuum can arise from nothing else but from points and only points. And provided they are correctly understood, both things are perfectly compatible.

§ 39

It can be anticipated that the properties of that particular continuous extension, *time*, may give rise to particular problems. Especially for those philosophers who, like the sceptics, purposely aim at confusing and finding apparent contradictions everywhere instead of elucidating human concepts, the theory of time must offer welcome material. But we shall only mention here the most important things, not everything which would arise here concerns the concept of infinity.

The question may be raised whether time is something *actual*, and if so, whether it is substance or attribute, and in the first case whether it is created or uncreated? 'If the former,' one might think, 'it must have had a beginning, also at some future time it must have an end, therefore it must change, accordingly there needs to be another time in which it changes. It would be even more absurd to define it as *God himself*, or as an *attribute* belonging to him. Certainly *time* may be compared here *to eternity*—what is it? How is it possible that an infinite multitude not only of moments but *of complete time intervals* may be contained in a single small period however short, e.g. in a single glance of the eyes [*Blick mit dem Auge*], of which every simple instant is called a *moment* [*Augenblick*]? But in fact (it is said, in the end) *time does not exist at all!* For the time that is passed, just because is has passed, is obviously there no more, and the future, because it is future, is not here now. Finally, what is present is nothing other than a *mere moment* in the strictest sense of the word which has no *duration*, therefore no claims to the *name* of time.'

As a consequence of my ideas time is certainly *nothing actual* in the proper sense of the word where we attribute being actual only to *substances* and their *forces*. I therefore regard it neither as God himself, nor as a created substance, nor even as an *attribute* either of God, or of some other created substance, or of a collection of several substances. It is also certainly nothing *variable*, but rather it is that *in which* all variation takes place. If the opposite is said, as in the proverb, '*times change*', then it was already noticed long ago that by time here is understood only the things and circumstances which occur in it. Now to state this more precisely time itself, is that *determination* occurring in every (variable or what is equivalent) dependent substance whose idea we must add to the idea of this substance in order to be able, for every two *contradictory attributes* b and not-b, to attribute one of them to it truly and to disallow the other. More precisely the *determination* mentioned here is *a single simple part* of time, an instant or moment, in which we have to imagine the substance x, to which we want to attribute with certainty one of two contradictory attributes b and not-b, in such a way that our decision must therefore actually state that x at time instant t has either attribute b or not-b. If it is admitted that this is a correct definition of the concept of a moment then I can also state clearly what time itself, and indeed *the whole time*, or *eternity* is, namely that whole [*Ganze*] to which all moments belong as parts. And every *finite* time, i.e. every *time interval* or *time period* contained within two given moments I define as the collection of all the moments which lie between those two boundary moments. As a consequence of these definitions there is therefore no difference between time and eternity if the former is understood (as often happens) as not restricted to a finite time but rather to the whole of time (endless in both directions). But there is indeed a great difference in the way that God, and the variable or created beings, occur in this time. Namely, the latter are in time *in that they themselves vary in it*, but God is for all time completely constant. This has given rise to him alone being called *eternal*, but the other beings, his creatures, being called *temporal beings*. To represent the fact that every small period, however short, like a glance, already contains an infinite multitude of complete time intervals in a form accessible to the senses may be a difficult problem for *our imagination*. It is sufficient if our *mind* can grasp it and recognize it as something which cannot be otherwise. The objective reason for this can be seen from the concept of time which we have indicated here, but the analysis of it here would be too lengthy. It would be absurd if we had claimed that the same multitude of moments may be put into a short time as into a longer time, or that the infinitely many time intervals into which the former can be divided were of an equal length as for some longer time.

Finally, the fallacy which wishes to destroy completely the reality of the concept of time is so obvious that it scarcely needs one word for its refutation. We admit indeed that time generally is nothing existing and that neither past nor future time exists, even the present has no existence, but how should it follow from this that time is *nothing*? For are not propositions and truths in themselves also something although it may not occur to anyone to claim that they were something existing—providing they were not being confused with their conception in the consciousness of a thinking being, therefore with actual *ideas or judgements*?

§ 40

It is well known with regard to the paradoxes in the *theory of space* that is not even known how to define space. It is frequently held to be something existing, it is sometimes confused with the substances which occur in it, and sometimes it is even identified with God or at least with an attribute of the Deity. Even the great *Newton* thought of defining space as 'the sensorium of the Deity' [*das Sensorium der Gottheit*]. Not only do the substances occurring in space often *move*, but space itself can change, i.e. the locations of their positions. It used to be believed (since *Descartes*) that not all substances, but only the so-called material substances, may occur in space, until finally *Kant* got the unfortunate idea, still repeated by many today, of considering space as well as time not to be something objective, but to be a mere (subjective) *form of our intuition*. Since that time the question has been raised whether other beings may not have another space, e.g. with two or four dimensions. Finally *Herbart* has moreover presented us with a double space, both *rigid and continuous*, and a similarly double [concept of] time. I have already explained myself about all these things in other places.

For me space, in a similar way as for time, is *not an attribute* of substances but only a *determination* of them. Indeed, I call those determinations of created substances which state the reason why, with their attributes, they produce exactly these changes in one another at a certain time, the *places* at which they occur. The collection of all places I call *space*, the whole of space. This definition puts me in the position of deriving the theory of space from that of time *objectively*, and therefore showing, for example, both that, and why, space has three dimensions, and several other things.

Therefore the paradoxes which have been found already in the *concept* of space, in that *objectivity* which befits it despite the fact that it is nothing actual, in the infinite multitude of its parts and in the continuous whole which they form among one another despite the fact that no two of these simple parts (points) touch one another directly—these apparent contradictions I believe I should not discuss further but may consider them as dealt with.

The first thing which still requires further illumination might well be the concept of the *magnitude* [*Größe*] of a spatial extension. There is no dispute that *magnitude* applies to all extension. And it is also agreed that the magnitudes appearing (in the one temporal dimension[n] or the *three* spatial dimensions) can be determined only by their ratio to the one which has been adopted arbitrarily as the *unit of measurement*. Also [it is agreed] that this extension adopted for the unit must be of just the same kind as that which is being measured by it, therefore a *line* for *lines*, a surface for surfaces, a solid for solids.* But if we now ask in what this really consists, what we call the magnitude of a spatial extension, then one would surely like [to say], especially since such an extension consists of nothing

* Perhaps it is not unwelcome to some people to read here the definition of these three kinds of spatial extension. If the definition given in §38 of *extension in general* is admitted as correct (and it has the merit that it can easily be extended to those quantities of the *general theory of quantity*, which are called *continuously variable*) then I say a spatial extension is extended *simply*, or is a line, if every point, for every sufficiently small distance, has one or several, but never so many neighbours that their collection *in itself alone* forms an extension; I say further that a spatial extension is *doubly* extended, or a *surface*, if every point, for every sufficiently small distance, has a whole line of points for its neighbours. Finally I say that a spatial extension is *triply* extended, or a *solid*, if every point for every sufficiently small distance, has a whole surface full of points for its neighbours.

n German *Ausdehnung* also means 'extension' which is how the word has usually been rendered.

but points arranged according to a certain rule, that with a magnitude we should never look at the order but only at the multitude of points. We should be very inclined to conclude that it is just this *multitude of points* that we think of by the magnitude of any spatial thing. *The name itself* also seems to confirm this when we call the magnitude of a surface or a solid, the *content* [*Inhalt*] of this spatial thing. Nevertheless a closer consideration shows that this is not so. Or how otherwise could we assume, as we do generally and unhesitatingly, that the magnitude of a spatial thing, e.g. a cube, does not change in the least whether we include in the calculation of the content, the boundary of it, here therefore the surface area of the cube (which itself has a magnitude), or not? And thus we proceed without any question if we find the magnitude of a cube of side 2 to be eight times as large as a cube whose side $= 1$, in spite of the fact that the former one has a surface area of size which is 12 square units less than the latter since by their composition into a single cube of 24 such squares, half of them—those in the interior of the larger cube—are lost. Hence it follows that by the magnitude of a spatial extension, whether it is a line, surface or solid, we really think of nothing but a quantity which is derived from an extension assumed as unit of the same kind as that to be measured by such a rule that if, proceeding by this rule we derive from the piece M the quantity m and from the piece N the quantity n, and following the same rule we obtain from the piece produced by the combination of the pieces M and N, the quantity $m + n$, equally whether we take the boundaries of M and N and the whole consisting of the two, $M + N$, into account or not. That the most general formulae which the science of space [*Raumwissenschaft*] has for the rectification, complanation and cubature, can in fact be derived from this concept without needing any other assumption, especially not the misnamed 'axioms' of Archimedes, has already been shown in the work mentioned in §37.

§ 41

Relying on the definitions given so far we may now, without fear of being charged with a contradiction, put forward propositions like the following, paradoxical as some of them may appear to the usual mode of thinking.

1. The collection of all points which lie between the two points a and b, represents an extension of simple kind, or a line. This is so just as much when we include the points a and b, when it is a *bounded* straight line, as when one or the other or both of the boundary points is not included, when it is therefore *unbounded*. But in each case the length of it is always as before. Every such unbounded straight line has, on the side where its boundary point is missing, for this reason, no *extreme* (furthest) point, but beyond each one there is a further one, although its distance always remains finite.
2. The perimeter of a triangle, abc can be composed, (1) from the straight line ab bounded on both sides, (2) from ac bounded only on one side, at c, and (3) from bc unbounded on both sides; but its length is equal to the sum of the three lengths of ab, bc, and ca.

3. Let us imagine that the straight line *az* is bisected by the point *b*, the piece *bz* is again bisected by the point *c*, that *cz* is again bisected by the point *d*, and this is continued without end. If we assume that these infinitely many points of bisection *b*, *c*, *d*, . . . and the point *z* are to be thought of as omitted from the collection of points lying between *a* and *z*, then the collection of all the remaining ones still deserves the name of a *line*, and its magnitude is to be the same as before. But if we include *z* in the collection then the whole is no longer a continuous extension, for the point *z* is isolated, because for it there is no distance, however small, of which it could be said that for this distance and for all smaller distances it has neighbours in this collection of points. Namely for all distances of the form $\frac{az}{2^n}$ a neighbour is missing for *z*.
4. If the distance of the points *a* and *b* equals the distance of the points α and β, then the multitude of points between *a* and *b* must be assumed equal to the multitude of points between α and β.
5. Extensions which have an equal multitude of points are also of equal magnitude, but not conversely, *two* extensions, which are of equal magnitude, need not have equally many points.
6. For a pair of spatial things which are perfectly similar to one another, the multitudes of their points must be related exactly as their magnitudes.
7. Therefore if the ratio of magnitudes between two perfectly similar spatial things is irrational, then the ratio between the multitudes of their points is irrational. Therefore there are *numbers* (indeed only infinite ones) *whose ratio, in every way that is chosen, is irrational.*

§ 42

Among these propositions, whose number (as may be seen) could easily be increased, to my knowledge only the sixth has been given any attention in the writings of mathematicians up till now. But only in the sense that the [following] proposition has been put forward contradicting it: *similar lines, however different in their magnitude, have an equal number of points. Dr J. K. Fischer* asserts such a thing (*Grundriß der gesamten höheren Mathematik*, Leipzig, 1809, Bd. II, §51, Anm.), particularly of similar and concentric circular arcs, for the additional reason that through every point of one a radius may be drawn which meets one point of the other. But it is well known that *Aristotle* has already considered this paradox. *Fischer*'s method of argument obviously reveals the opinion that a pair of multitudes, if they are also infinite, must be equal to one another as soon as every part of one can be joined together into a pair with one part of the other. After discovering this error no further refutation of that theory is needed. Moreover it cannot even be understood in this regard why, if it were correct, we would have to limit this assertion of the equal numbers of points only to circular arcs which are concentric and similar, since the same reason could also be given for all *straight* lines and for the most different kinds, no less than for similar curves.

§ 43

There is hardly any truth in the theory of space that teachers of this subject have sinned against more often than this one: *that every distance lying between two points in space, as also every straight line bounded on both sides, is only finite*, i.e. it stands to every other in a relationship which is precisely determinable by mere concepts. For there will scarcely be a geometer who has not at times spoken of *infinitely large* distances and a straight line, which is to have its boundary points on both sides [but which] under certain circumstances would have become *infinitely large*. As an example it is sufficient here to refer to that well-known pair of lines which are called (understood in a geometrical sense of the words) the tangent and secant of an angle or arc. According to the explicit definition these should be a pair of straight lines which are bounded on both sides, and yet how little there is which teaches us to doubt that for a right angle the tangent, as well as the secant, would become *infinitely large*. Nevertheless one is punished immediately for this mistaken theory by the embarrassment into which one lands as soon as one is to state whether these two infinitely large quantities are to be viewed as *positive* or as *negative*? For obviously the same reason holds, which could be quoted for the one, as for the other; because as is well known a straight line drawn through the centre of a circle parallel to a tangent line of it, has a completely equal relationship to both sides of this touching, therefore it intersects it on one side as little as it does on the other side. Also in the quantity expression for these two lines which $= \frac{1}{0}$, since zero can be viewed as neither positive nor negative, lies not the least reason to define this supposedly infinite quantity as positive or as negative. Therefore it is not merely paradoxical, but quite false, to assume the existence of an infinitely large tangent of a right angle, as also of all angles of the form

$$\pm n\pi + \frac{\pi}{2}.$$

That there is, strictly speaking, also for the angle $= 0$ or for the angle $= \pm n.\pi$ neither sine nor tangent, is only mentioned occasionally. The difference in these two assumptions is merely that with the latter no false result arises if one considers the products as not existing at all in cases where these number expressions appear as factors, but where they occur as divisors, one concludes that the calculation requires something unlawful.

§ 44

It was an equally unjustified procedure, which nevertheless has fortunately found few imitators, when *Joh. Schultz* wanted to calculate *the magnitude of a whole infinite space*, from the circumstance that from every given point *a*, on all sides outwards, i.e. in every direction which there is, straight lines can be imagined drawn indefinitely, and from the further circumstance that every conceivable point *m* of the universe must lie on one and only one of these lines. The conclusion drawn that the whole of infinite space may be viewed as a *sphere* which is described from the arbitrarily chosen point *a* with a radius of magnitude $= \infty$, whence it immediately arises that the whole infinite space has precisely the magnitude $\frac{4}{3}\pi\infty^3$.

Without doubt it would be one of the most important theorems of the science of space if this could be justified as true. And hardly anything well-founded can be said against the two premisses (which nevertheless I did not present here precisely according to the exposition of *Schultz* which I do not have to hand). For if someone wished to say that the second premiss must be wrong because from it a very unequal distribution of points in space would follow, namely a much denser accumulation around the arbitrarily chosen centre point *a*, he should recognize that he has not overcome the prejudice we attacked in §21 [and following sections]. *Schultz* is mistaken, and quite obviously mistaken, just in this, that he assumes that the straight lines which have to be drawn out in all directions from the point *a* into the unbounded, if every point of space is to lie on some one of them, are nevertheless *radii*, therefore lines bounded on both sides. For only on this assumption does the spherical form of space follow and the calculation of its magnitude as $= \frac{4}{3}\pi\infty^3$. But from this error the absurdity also follows that—because to every sphere there is a circumscribing cylinder, or even a circumscribing cube, indeed many other spatial things, e.g. there must be infinitely many other spheres of equal diameter enclosing it—the alleged whole space is not the whole, but a mere part, which again has infinitely many other spaces outside it.

The single remark that a line drawn out to infinity only on one side is, for this reason, not a line bounded on this side and that therefore we can speak of a boundary point of it just as little as of the point of a sphere, or the curve of a straight line or a single point, or the point of intersection of two parallel lines—this single remark, I say, is sufficient to show the invalidity of most of the paradoxes which *Boscovich* brought forward in his *Diss. de transformatione locorum geometricorum* (appended to his *Elem. univ. Matheseos*, T. III, Romae, 1754).

§ 45

Infinitely small distances and lines in space have also been assumed as often as the infinitely large, particularly if there is an apparent need to treat lines or surfaces of which no part (which is itself extended) is straight or flat, nevertheless as though they are straight or flat, for example, in order to be able to determine more easily their length, the magnitude of their curvature, or even perhaps some properties of them important for mechanics. Indeed one is allowed to speak in such cases even of distances which are supposed to be measured by infinitely small quantities of the second, third and other higher orders.

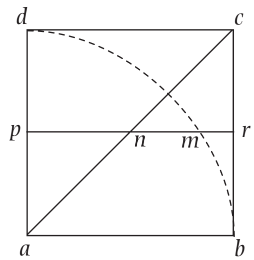
For the fact that one only rarely reaches a false result by this procedure, particularly in geometry, one must be grateful for the circumstance mentioned in §37 that the variable quantities which refer to spatial extensions which are *determinable*, must be of such a nature that they have, with at most the exception of single *isolated values*, a first, second and every successive *derived* function. For where such [things] exist, what is asserted of the so-called infinitely small lines, surfaces and solids, holds in common with all lines, surfaces and solids, which—although they always remain finite—can be taken as small as desired, i.e. can decrease (as they say) indefinitely. Therefore these variable quantities were really these for which, what had been stated falsely of the infinitely small distances, was valid.

But it is obvious that with such a presentation of the subject many paradoxes, and even completely wrong things, have had to be brought out and apparently be proved. How objectionable it sounded for example, when it was asserted of every curved line and surface that it is nothing other than an object composed from infinitely many straight lines and plane surfaces, which would have to be assumed infinitely small, especially if in addition infinitely small lines and surfaces would be acknowledged by them which are nevertheless curved. How strange it would be if it was asserted of lines which have no curvature at all at one of their points, but have, for example, a turning point, that its curvature in this point is infinitely small, and that its radius of curvature is therefore infinitely large; or of lines which, at one of their points taper into a cusp, that its curvature here was infinitely large, and that its radius of curvature was infinitely small, and similar things.

§ 46

As a really striking, and at the same time very simple, example of the subject and cause of the absurdities that the assumption of such infinitely small distances presents, let me quote here a proposition from the report of *Kästner* (*Anfangsgründe der höheren Analysis*, Bd. II, Vorr.) which had already been stated by *Galileo* in his *Discorsi e dimostrazioni matematiche* etc., surely only with a view to arousing some thinking, namely that the *circumference of a circle is as large as its centre*.

In order to gain an idea of the way that one might try and prove this, the reader may think of a square *abcd*, in which from *a* as the centre, with the radius $ab = a$, the quadrant *bd* is described, then



the straight line *pr* parallel to *ab* is drawn which cuts both sides of the square *ad* and *bc* in *p* and *r*, the diagonal *ac* in *n*, and the quadrant in *m*: in short, the well-known figure, by which one usually proves that a circle with the radius *pn* is equal to the annulus which is left behind after removing the circle on *pm* from that on pr, or that

$$\pi.pn^2 = \pi.pr^2 - \pi.pm^2.$$

If pr continually draws nearer to ab, obviously the circle on pn becomes continually smaller, and the annulus between the circles on pm and pr becomes ever thinner. Therefore geometers who did not find infinitely small distances objectionable, extended this relationship also to the case when pr drew infinitely close to ab, therefore for example, the distance ap becomes $= \mathrm{d}x$, when the equation

$$\pi.\mathrm{d}x^2 = \pi.a^2 - \pi(a^2 - \mathrm{d}x^2)$$

should hold, which is also in fact justified as a mere identity. But in this case their idea of the circle on pn has become an infinitely small thing of the second order; the annulus on the other hand which remains after the removal of the circle on pm from that on pr, now only has the width

$$mr = \frac{1}{2}.\frac{\mathrm{d}x^2}{a} + \frac{1}{2.4}.\frac{\mathrm{d}x^4}{a^3} + \cdots \text{[o]}$$

which itself was an infinitely small thing of the second order. Now if it were assumed especially that pr goes into ab completely, then the infinitely small circle on pn contracts into the single point a, and the infinitely thin annulus of width mr changes into the mere circumference of the circle with diameter ab. Therefore one seems to be justified in concluding that the mere centre a of every arbitrary circle on ab would be as large as its whole circumference.

The deceptiveness of this proof is chiefly produced by the introduction of the infinitely small. Through this the reader is led to a series of thoughts which lets him overlook much more easily how absurd are the assertions that, of the circle on pn, if instead of the point p the point a is finally considered, and no radius like pn exists any more, yet *the centre a still remains*, and that the annulus originating from the removal of the circle with the smaller radius pm from the circle with the larger radius pr, if both radii and therefore also the circles become equal to one another, becomes the circumference of the previously greater one. For indeed with infinitely small quantities one is used to the same quantities sometimes being considered as equal to one another, then one, as an infinitely small quantity of a higher order, being greater or smaller than the other, then also as being considered as being completely equal to zero. If we wish to proceed logically then we may conclude, from the correctly stated equation

$$\pi.pn^2 = \pi.pr^2 - \pi.pm^2$$

which compares the mere quantities (surface areas) of the circles concerned, nothing but that for the case when pr and pm become equal to one another, the circle on pn has no magnitude, therefore it does not exist at all.

[o] The first edition has long dashes to the left of mr and in place of each of the two '+' signs.

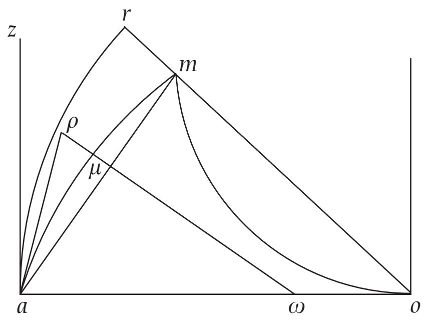
It is indeed true (and I have set out the premisses leading to this truth at §41) that there are circles with and without circumferences, and that this alters nothing in their magnitudes, which depend solely on the magnitude of their radii. And from this someone could perhaps want to derive a new apparent proof for *Galileo*'s proposition since he may start from the demand, surely allowable, that one should imagine the circle on *pm* as without circumference, but the circle on *pr* as together with its circumference. Then of course, after taking away the circle on *pm* from the one on *pr*, if we go from *pr* to *ab* in fact only the circumference of the circle on *ab* is left remaining. But also now no circle around *a* can be spoken of which has contracted into a single point, and still less would it be allowed to call on the above equation in order to deduce from it that the point *a* and that circumference were equally great since the equation stated only deals with the magnitudes of the three circles—they may be considered with or without circumferences.

§ 47

The example just discussed was, as already mentioned, not put forward by its discoverer in order to be wondered at as a truth. But it is taught as a serious truth about the common cycloid, that at the point where it meets its base line it has an infinitely large curvature or (what amounts to the same) an infinitely small radius of curvature, and it stands here in a vertical direction. This is also completely correct if it is understood that the radius of curvature decreases indefinitely while the cycloidal arc approaches the base line indefinitely. And also it should be understood that its direction at the point of starting is itself a vertical direction. What was said of the radius of curvature having become infinitely small or zero, consists (when expressed correctly) merely in this, that (because the curve, as is well known, continues over its baseline on both sides indefinitely, and therefore has *no boundary points*) two pieces of arc meet one another in this point, and indeed in such a way that, because they are both vertical to the baseline, they form a *cusp* with one another, and indeed one such that both have one and the same direction, or (as already said no less correctly), their directions here form an *angle* of zero.

However, one can be convinced by calculation that all this is the case and yet not understand how it comes about or even how it is possible. In order to make clear by what means the paradox is to be resolved, we must understand first of all why the direction in which the common cycloid rises above its baseline is a vertical one.

From the way the common cycloid can be constructed, namely that from every point *o* of the basis one describes an arc of a circle touching the latter with the radius of the generating circle, and cutting off from this a piece *om* equal in length to the distance of the point *o* from the starting point *a*, *m* is considered as a point of the cycloid. It follows immediately that the angle *mao* comes ever closer to a right angle, the closer the point *o* is moved towards *a*, since the angle *moa*, whose size is the half arc *om*, gets ever smaller and the relationship of the two sides *oa* and *om*



in the triangle *moa* approaches ever more to the relationship of equality. Therefore the angle on the third side *am* differs less and less from a right angle. The actual calculation shows this quite clearly. Hence, moreover, it follows that the cycloidal arc *am* lies wholly on the same side of the chord *am*, namely between it and the vertical *at* raised from *a*, therefore that the latter denotes the direction of the curve at the point *a*. If further we describe from *o* as centre an arc of a circle proceeding from *a*, on *oa*, then it is obvious that this cuts the chord *om* first in a point *r* of its extension, because it must be that $or = oa > om$. Now if μ is some point of the curve lying nearer to *a* then there is for it an ω lying still nearer to *a* in *ao* of such a kind that the same holds of the chord $\omega\mu$ which has just been asserted of *om*, namely that an arc of a circle described from ω as centre with the radius ωa meets the extension of $\omega\mu$ somewhere beyond μ in ρ. But because $\omega a < oa$ the arc $a\rho$ lies within the arc *ar*, and therefore between the cycloid arc $a\mu$ and the circular arc *ar*. Therefore we see that to every circular arc *ar* described with the radius *oa*, which touches the cycloid *am* in *a*, there is another $a\rho$ which comes even closer to it in this region. In other words, there is no circle so small that it could be viewed as a measure for the curvature occurring at *a*, in the case that there is a curvature here. Therefore in truth there is no curvature here, but the curve, which does not end at this point, has here, as we already know, a cusp.

§ 48

It has frequently been found paradoxical that some spatial extensions, *which extend through an infinite space* (i.e. have points whose distances from one another exceed any given distance) nevertheless possess only a *finite magnitude*, and again others which are *limited to a quite finite space*, (i.e. whose complete multitude of points are situated so that their distances from one another do not exceed a given distance), yet they possess an *infinite magnitude*. Finally many a spatial extension has a finite magnitude although it *makes infinitely many rotations around one point*.

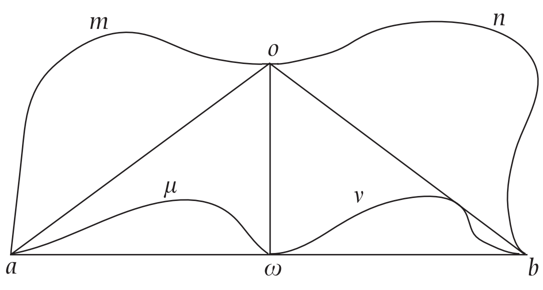
1. Before anything else we must distinguish here whether by the spatial extension of which we speak here is to be understood a whole consisting of several *separated*

parts (e.g. such is the hyperbola with four branches), or only an *absolutely connected whole*, i.e. only such an extension which has no individual part which itself represents another extension in which there was not at least *one point* which, in relation to the other parts, again forms an extension with them.

No one who thinks of the fact that an infinite series of quantities, if they decrease in geometric ratio, have a merely finite sum, will find it objectionable that an extension consisting of separate parts could spread itself out over an infinite space without thereby becoming infinitely large. In this sense therefore also a *line* can extend indefinitely and yet be only finite, like that which arises, if we draw a bounded straight line *ab* from a given point *a* in the given direction *aR*, then a straight line *cd* which remains always at an equal distance, and is only half so long as the previous one, and continue with the same rule indefinitely.

But if we speak only of such spatial extensions which provide a *connected whole*—and that should now always be the case in what follows—then clearly among *extensions of the lowest kind*, i.e. *lines*, none could be found which stretch indefinitely, without at the same time having an *infinite magnitude* (length). For this already necessarily follows from the well-known truth that the shortest absolutely connected line which is to join two given points to one another is just the straight line between them.*

* Because the proof of this truth is so short, I shall include it in this note. If the line *amonb* is not straight, then there must be some point *o* in it which lies outside the straight line *ab*, and if we drop from *o* the

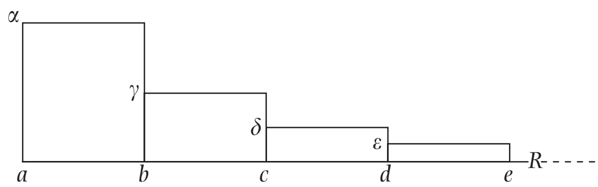


perpendicular $o\omega$ onto *ab*, then the distances are [related by],

$$a\omega < ao, \quad b\omega < bo.$$

But since all systems of two points are similar to one another, there is a line $a\mu\omega$ between the points *a* and ω similar to the piece *amo* of the given *amonb* which lies between the point *a* and *o*, and likewise a line $b\nu\omega$ between the points *b* and *o*, similar to the piece *bno* of the given *bnoma* lying between the points *b* and *o*. But this *similarity* also requires that the length of the straight line $a\omega$ is related to the length of $a\mu\omega$ as the length of the straight line *ao* is to the length of the piece *amo*, and the length of the straight line $b\omega$ is to the length $b\nu\omega$ as the length of the straight line *bo* is to the length of the piece *bno*. Now because $a\omega < ao$, then also it must be that $a\mu\omega < amo$ and because $b\omega < bo$, then it

It is different from the case of lines, with *surfaces*, which, with the same length can be made as small as desired merely by reducing their width, and with *solids*, which, for the same length and width, can become as small as desired merely by reducing their height. Hence it is understandable why surfaces with an infinite *length*, and *solids* with an infinite *length* and an infinite *width*, sometimes nevertheless only possess a finite magnitude. An example, which even the most inexperienced will find understandable, is provided if we require that he imagines drawn indefinitely on the indefinitely extended straight line aR the equal pieces $ab = 1 = bc = cd =$ etc.,



then on the first piece ab he will imagine the square $b\alpha$, on the second bc the rectangle $c\gamma$ that has only half the height of bc, and so for each subsequent one, a rectangle half as high as the directly preceding one, when he will certainly very soon recognize that the connected surface which is presented to him here extends indefinitely and yet is not bigger than 2. It will not be more difficult to him to imagine a cube whose side $= 1$, and to think of a second solid whose base area is a square of side 2, therefore four times as big as the base area of the previous cube, but the height comes to only $\frac{1}{8}$. Then to put after this one a third one whose base area is again four times as big as the directly preceding one but whose height comes to $\frac{1}{8}$ of the solid before, and to imagine that this is continued according to the same law indefinitely. He will understand that the length and width of the solids which are produced in succession increase indefinitely although their volume [*körperlicher Inhalt*] becomes ever smaller, indeed so that every successive one is half of the one directly preceding. Therefore the quantities make a whole in the shape of a pyramid that evidently never exceeds the volume $= 2$ in spite of its infinite basis.

2. As the case considered before, where an extension had something infinite in it (an infinite length, or even width), and yet was of only finite magnitude, occurred only with the two higher kinds of extension, *surfaces* and *solids*, but not with lines, then the opposite occurs in the case which we now come to discuss, where an extension which appears finite because it is restricted to a completely finite space, nevertheless in fact has an infinite magnitude. This case can only occur with the

must also be that $b\nu\omega < bno$. Consequently also the whole $a\mu\omega\nu b <$ the whole $amonb$. The curved line $amonb$ is therefore not the shortest between a and b, but $a\mu\omega\nu b$ is shorter.

two lower kinds of extension, *lines* and *surfaces*, but in no way can it happen with solids. A solid, in which there are no points whose distances from one another exceed any given quantity can certainly not be infinitely great. This follows directly from the well-known truth that among all solids with points whose distances from one another do not exceed a given distance ε, the greatest is a sphere of diameter ε. For this contains all those points, and its magnitude is only $\frac{\pi}{6}.\varepsilon^3$, every other solid not exceeding this space, must therefore necessarily be *smaller* than $\frac{\pi}{6}.\varepsilon^3$. On the other hand, there are infinitely many *lines* which can be drawn in the space of a single surface however small, e.g. a *square foot*, and to each of them we can attribute a magnitude which is at least finite, e.g. the length of a foot, also by the addition of one, or even infinitely many, connecting lines [we can] combine them all into a single absolutely connected line whose length then must certainly be infinite. And in completely the same way there are infinitely many *surfaces* which can be drawn in the space of a single *solid*, however small, e.g. a *cubic foot*, for each of which we can attribute a magnitude, e.g. a square foot, and by the addition of one or even infinitely many connecting surfaces we can combine all these surfaces into a single one whose magnitude will then undoubtedly be infinite. All this can surprise nobody who does not forget that it is not the same unit with which we measure lines, surfaces and solids, and that although the number of points in every line, however small, is infinite, in a surface this number must nevertheless be assumed infinitely many times greater than in a line, and finally in a solid, with equal certainty, it must be infinitely many times greater than in a surface.

3. The third paradox mentioned at the beginning of this paragraph is that there are also extensions which make an infinite number of revolutions [*Umläufen*] around a certain point, and nevertheless have a finite magnitude. If such an extension is to be *linear* then this can only happen, as we saw in no. 1, if the whole line is situated in a finite space. But on this condition there is absolutely nothing incomprehensible in the phenomenon that it has a finite length although it makes infinitely many revolutions around a given point, providing the further condition is satisfied that these revolutions beginning from a finite magnitude decrease in the appropriate way indefinitely. [This is] a requirement which is possible through the circumstance that it is a *mere point*, around which those revolutions are to occur. For if this is allowed, that the distances of the individual points of such a revolution have from this centre, and therefore also have from one another, can be decreased indefinitely, then the circle itself shows us that the length of this revolution can be reduced indefinitely. The *logarithmic spiral*, if merely that piece of it is to be observed, that, starting from a given point continually approaches the centre without actually reaching it, will have given our readers an example of a line like the one discussed here.

But if the spatial extension which makes infinitely many revolutions around a given point, is to be a *surface* or a *solid*, then the restriction is not even needed that none of the points of the spatial thing are further than a particular distance from its centre. For in order to make myself understandable in the shortest way, let the reader just imagine the spiral mentioned as a kind of abscissae-line from each point of which ordinates emanate at right angles to it and to its plane. The collection of all these ordinates then obviously forms a surface (of a cylindrical kind) which on the one side approaches the centre in infinitely many windings, without ever reaching it, but on the other side it is indefinitely far away. How large this surface is will depend on the rule according to which we allow the ordinates to increase or decrease. But the part going to the centre will always remain finite as long as we cannot increase the ordinates on this side indefinitely (i.e. over the finite branch of the abscissa) because every surface in which neither the abscissa nor the ordinate grows indefinitely, is finite. But also the part of the surface which stands above the other branch of the spiral extending indefinitely will remain finite as long as the ordinates decrease at a faster rate than the abscissae (i.e. the arclengths of the spiral) increase. Therefore if we choose for the abscissae-line the *natural* spiral, where the branch moving towards the centre with radius $= 1$ has length $\sqrt{2}$, and take for the boundary of the surface the arc of a hyperbola of higher kind for which the equation is $yx^2 = a^3$, then that part of this surface which belongs [to values of x] from $x = a$ to all higher values of x only has the magnitude a^2, while the other part, belonging to all smaller values of x, grows indefinitely. But if we take $a > \sqrt{2}$ and transfer the endpoint of the abscissa $x = a$ to the point of the spiral which has the radius 1 then its centre coincides with the endpoint of the abscissa $x = a - \sqrt{2}$, therefore it has a finite ordinate and the part of it which lies over this branch of the spiral is not greater than

$$a^3\left(\frac{1}{a} - \frac{1}{x}\right) = a^2 - \frac{a^3}{a - \sqrt{2}} = -\left(\frac{a^3}{a - \sqrt{2}} - a^2\right).$$

The whole surface covering the spiral on both sides (which we must obtain by adding both magnitudes in their positive value) is therefore

$$= a^2 + \left(\frac{a^3}{a - \sqrt{2}} - a^2\right) = \frac{a^3}{a - \sqrt{2}}.$$

Therefore, for instance, for $a = 2$ the whole surface comes to only $4(2 + \sqrt{2})$.

It is a very similar situation also with solid extensions. Only it is to be observed that here if one wants to allow the part of the solid moving towards the centre to increase its extension in breadth [*Breite*] and thickness it would encroach on the space of its own adjacent revolutions (to the right and the left). If one wanted to avoid this and have a solid all of whose parts are separate then among other ways to achieve the aim is that one adds, to a surface of such a kind as the one just considered which always increased its breadth on approach to the centre, a third dimension, a thickness, which diminishes towards the centre in such a ratio that it always amounts to less than half of the distance between two adjacent turnings of the spiral.

§ 49

Spatial extensions which have an infinite magnitude, precisely with regard to this magnitude itself, are of so many kinds and often have such paradoxical relationships that we must at least give some of them special consideration.

There is the fact that a spatial thing which contains an infinite multitude of points does not, on that account, have to be a continuous extension, as also the fact that with a continuous extension we do not even determine the number of points through its magnitude. Of two extensions which we regard as equally great, one can contain even infinitely many points more or less than the other. Indeed, a surface can contain infinitely many lines, and a solid infinitely many surfaces, more or less, than an extension of the same kind which is considered as equally great. We can consider all these things as sufficiently explained by what has been said already.

1. The first thing, to which we want to direct the readers' attention, is that the multitude of points which a single line *az*, however short, contains, is a multitude which must be considered as *infinitely greater* than that infinite multitude which we select from the former if, starting from one of its boundary points *a*, we extract a second *b* at a suitable distance, after this at a smaller distance a third *c*, and continue without end reducing those distances according to a rule such that the infinite multitude of them has a sum equal to, or smaller than, the distance *az*. For since also the infinitely many pieces *ab*, *bc*, *cd*, . . . , into which *az* is divided are all finite lines, then we can do with each of them what we have just required of *az*, that is we can prove that in each of them again there is such an infinite multitude of points as in *az*, and all these points are at the same time in *az*. Therefore the whole *az* must contain such an infinite multitude of points infinitely many times.

2. Every straight line, indeed every spatial extension in general, which is not only similar to another one but also (geometrically) *equal* (i.e. coincides with it in all *characteristics which are conceptually representable* [*begrifflich darstellbaren*] through comparison with a given distance) must also have an equal multitude of points belonging to it providing we also assume the two have the same kind of *boundary*, i.e. in two straight lines the endpoints may be included or not included. For the opposite could only occur if there were distances which, although equal, permitted an unequal multitude of points between the two points which are at these distances. But that contradicts the concept which we associate with the word *geometrically equal*, for we only call a distance *ac* unequal with another *ab*, indeed

greater than the latter, if, in the case that b and c both lie in the same direction, the point b is between a and c, and therefore all points lying between a and b also lie between a and c, but not conversely that all between a and c are also between a and b.

3. If we designate the multitude of points that lie between a and b, including a and b, by E, and take the straight line ab as the unit of all *lengths* then the multitude of points in the straight line ac, which has the length n (by which we now understand only a whole number) if its endpoints a and c are to be counted in, is $= nE - (n-1)$.

4. The multitude of points in a *square area* whose side is $= 1$, (the usual unit for areas), will be, if we include the perimeter, $= E^2$.

5. The multitude of points in every *rectangle*, of which one side has the length m and the other side has the length n, will, with the inclusion of the perimeter, be

$$= mnE^2 - [n(m-1) + m(n-1)]E + (m-1)(n-1).$$

6. The multitude of points in a *cube* whose side $= 1$ (the usual unit for a volume), will be, if we count in the points of the surface, $= E^3$.

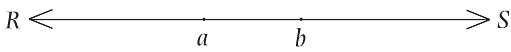
7. The multitude of points in a *parallelepiped* whose sides have the lengths m, n, r, with the inclusion of the surface, will be:

$$\begin{aligned} mnr.E^3 &- [nr(m-1) + mr(n-1) + mn(r-1)]E^2 \\ &+ [m(n-1)(r-1) + n(m-1)(r-1) + r(m-1)(n-1)]E \\ &- (m-1)(n-1)(r-1). \end{aligned}$$

8. We must ascribe to a straight line which extends indefinitely on both sides an infinite length and a multitude of points which is infinitely many times as great as the multitude of points $= E$ in the straight line taken as the unit. We must also grant to *all* such straight lines, equal lengths and *equal* multitudes of points, because the determining pieces, the two points by which any such pair of straight lines can be determined, through which they go, if we assume the distance between these points to be equally great, are not only similar to one another, but also (geometrically) equal.

9. The position of a point chosen arbitrarily in such a straight line is completely similar on both sides of the line, and clearly presents only such characteristics as are conceptually intelligible [*begrifflich erfassbaren*], as does the position of every other point of the kind. Nevertheless, it cannot be said that such a point divides the line into two *equally long* pieces, for if we said that of a point a, then we would have to assert it also of every other point b for the same reason, which is

self-contradictory since, if $aR = aS$ then also it could not be bR $(= ba + aR) = bS$ $(= aS - ab)$.



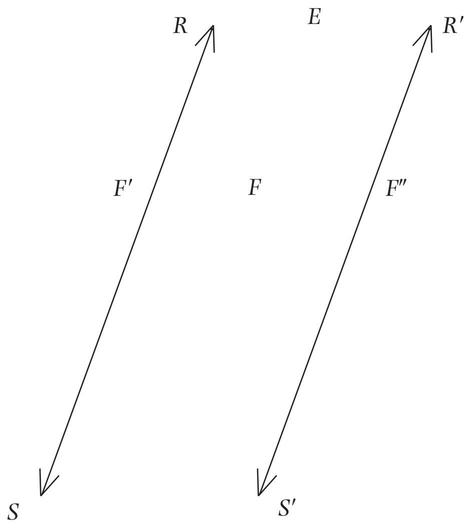
Therefore we must instead claim that a line, unbounded on both sides, *does not have a midpoint*, i.e. has no point which could be determined by a relationship to this line that is merely conceptually intelligible.

10. We must grant to the plane surfaces, which two parallel lines unbounded on each side *enclose* between them, (i.e. to the collection of all those points contained in the perpendiculars from every point of one of these parallels to the other), an *infinitely great* surface area, and a number of points which is infinitely greater than the number in the square assumed as unit area $= E^2$. We must also attribute to all such parallel strips, if they have the same *width* (length of the perpendicular), an equal magnitude and number of points. For they can be determined in such a manner that the determining pieces are not only similar to one another but also geometrically equal, e.g. if we determine them through the statement of an isosceles right-angled triangle of equal side for which we establish that one of these parallels goes through the base line and the other goes through the apex of the triangle.

11. The position of a perpendicular chosen arbitrarily from such parallel strips, is similar on both sides of the surface, and presents no other characteristics that are conceptually intelligible than those the position of every other such perpendicular presents. Nevertheless it cannot be said that such a perpendicular divides the surface into two *geometrically equal* pieces. For this assumption would involve us immediately in a completely similar contradiction as no.9, and this proves its falsity.

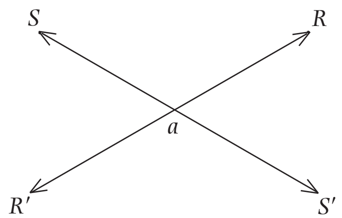
12. To a plane which extends in all directions indefinitely, we must allow an infinitely great surface area and a multitude of points which is infinitely greater than the multitude of points which are in one parallel strip. But as we allow all such parallel strips to be of equal width with one another, so we must allow all such boundless planes the same infinite multitude of points. For it also holds of them that they can be determined in a not merely similar, but also (geometrically) equal manner; as, e.g. if we determine each one through three points lying in it which form a similar and equal triangle.

13. The position of an unbounded line chosen arbitrarily in such an unbounded plane is completely similar on both sides of the plane. Moreover, it presents the same characteristics that are conceptually representable as the position of every other straight line of the kind.

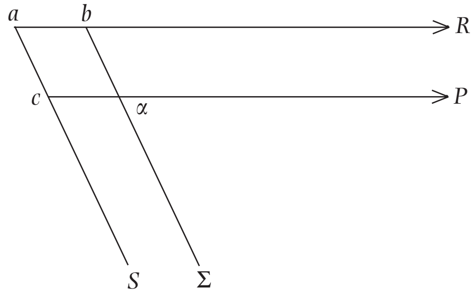


But this is not to say that such a straight line divides the plane into two *geometrically equally large* pieces. For if we were to assert that of one straight line RS, then we would have to admit it also of every other $R'S'$, which leads to an obvious contradiction as soon as we take these lines parallel to one another.

14. Two unbounded straight lines which lie in the same plane but do not run parallel to one another, therefore intersect somewhere and form four angles (equal in pairs), divide the whole surface of the unbounded plane into four pieces of which each two spanned by the *equal* (similar) angles $RaS = R'aS'$, $RaS' = R'aS$ are *similar* to one another.



Each of these four *angle spaces* [*Winkelräume*] contains an infinite multitude of *parallel strips* stretching indefinitely on one side, such as we considered in no. 11, of any arbitrary width. And after we remove, in thought, every finite multitude of them there still remains an angle space, spanned by an *equal angle* as at the beginning. However, after no. 9 and no. 11, we are as little justified in calling the arms of this angle, or even the parallel strips, which we can demonstrate as parts of its surface area, equal to one another, as (indeed for the same reasons as there) we are justified in calling these infinite angle spaces, even with equal (similar) angles, equal to one another, i.e. equally great. Thus it is obvious of the two angle



surfaces RaS and $P\alpha\Sigma$ that the first is greater than the second, although the angles themselves are equal to one another, if $b\,\Sigma \# aS$, $cP \# aR$.[p]

15. We must define the solid space, which two unbounded, parallel planes enclose (i.e. the collection of all those points, which all the perpendiculars falling from every point of one plane to the other, contain), this *unbounded solid layer* (as it could be called) as *infinitely large*, however the *width* of it (the length of one such perpendicular) may be. But with equal widths we may define these magnitudes, indeed also the multitude of points in two solid layers, as equal, according to the same arguments which we have already applied several times (no.s 8, 10, 12).

16. The position which a *parallel strip* perpendicular to its planes chosen arbitrarily in an unbounded *solid layer* [occupies] is completely similar on its two sides in that solid layer, and also the position which another parallel strip of this kind has in the same layer, or even in every other arbitrary unbounded solid layer, is similar. Nevertheless it cannot be said that those two parts into which the solid layer is divided by such a parallel strip would have to be of *equal magnitude*.

17. Two unbounded planes which *intersect* one another divide the whole infinite space into four infinitely large parts of which each opposite two are undoubtedly *similar*, but may not immediately be held to be of equal magnitude.

18. Just as little may the solid spaces, which enclose two *similar* or (as is commonly said) equal *solid corners* between their indefinitely extended side surfaces [*Seiteflächen*], be given as being equally great.

19. Also the two parts into which a *single infinite plane* divides the whole space, are, although similar, not to be considered as *geometrically equal*, i.e. as of equal magnitude, much less as consisting of equal multitudes of points.

[p] Evidently Bolzano is using the symbol # to mean 'is parallel to'.

§ 50

Now we still need to have a short discussion of those paradoxes which we meet with in the area of *metaphysics* and *physics*.

In these sciences I put forward the propositions: '*there are no two absolutely equal things, therefore also no two absolutely equal atoms*, or simple substances [*Substanzen*], in the universe; but such simple substances must necessarily be assumed as soon as *composite bodies* are accepted; finally one must also assume that all these simple substances are *variable* and change continually.' I assert all these because it seems to me they are truths which can be proved as strictly and clearly as some theorem of mathematics. Nevertheless I must be apprehensive that most physicists will only be shaking their heads while they listen to these propositions. They pride themselves on only putting forward truths which *experience* teaches them; but experience does not indicate any difference between the smallest particles of a solid, especially of the same kind, e.g. between the smallest particles of gold that we have obtained from this or from that mine. Furthermore, experience surely teaches us that every body is composite, but nobody has perceived atoms which would be absolutely simple and therefore also without any extension; finally, experience shows that the different elements, e.g. oxygen, hydrogen, etc., sometimes enter this combination with each other, sometimes the other, and as well as this, they display sometimes these, sometimes those, effects—but that they themselves would become altered in their inner nature, and that, for example, oxygen would gradually transform itself into another element, that would simply be invented.

1. In my view it is an error that *experience* teaches what is claimed here. Experience, merely, direct experience or perception, without being combined with certain purely conceptual truths, teaches us nothing but that we are having these, and those, intuitions or ideas. Where these ideas come to us from, whether through the effect of some object different from us, indeed whether they even need a cause at all, what properties these may have, about these things direct perception teaches us nothing at all, but we infer them only from certain conceptual truths which we have to supply through reason, and for the most part the inference is made according to a mere rule of probability, e.g. that this red which we are seeing just now, is produced by a diseased condition of our eyes, but that perfume is produced by a flower being nearby. On the other hand, in order to see that between every two things there must be some difference it does not need any inferences of the mere probability drawn from experience, but with a little thought we can recognize it with complete certainty. If A and B are two things then just for this reason the truth holds that the thing A is not the thing B, a truth which assumes that there are two ideas A and B, of which one represents only the thing A and not B, the other, only the thing B and not A. And in this fact, there is already a difference (and indeed an intrinsic one) between the things A and B. If we see in this way that every two things necessarily have certain differences, how can we believe ourselves justified in doubting such a difference merely because

sometimes we do not perceive it? Since a special sharpness of the senses, and many other circumstances, are needed for this perception.

2. It is correct that only experience teaches us that of the things which affect us, there are several, particularly all those which intuitions mediate to us, which are *composite*. Yet experience only teaches this under the assumption of certain purely conceptual truths: such as that different effects can only be produced by different causes etc. But no less certain are the conceptual truths, that every cause has to be something real, and everything real is either a substance or a collection of several substances, or an attribute of one or more substances, likewise, that attributes, that are something real, cannot exist without the existence of a substance in which they occur, and collections of substances [cannot exist] without simple [substances] which form the parts of these collections. But from this the existence of simple substances follows with strict necessity, and it will be ridiculous to wish not to accept the latter because they cannot be seen. And [it would be] all the more absurd when further reflection teaches that every body, which is to be perceivable by our senses, must be composite and composed from an infinite multitude of simple parts.

3. It is a similar fallacy from the non-perceivable to the non-existent, if one wishes not to admit that all finite substances are liable to an unceasing *change*. In our own *soul* we know well enough the changeability of its states, ideas, properties and powers; we are brought to infer a similar thing also about the souls of animals and about plants, by mere analogy. But that all those substances which show no noticeable change over a period of a century, do in fact change, we will assume justified only on the grounds of reason. Whoever disputes this, wants to deny it, at least with reference to the so-called *inanimate material* and in respect of its *simple parts* or atoms, may see himself driven to the claim that all changes which appear to us in this part of the creation, for example, if a piece of ice which a short time ago was solid, has now melted and in the next hour evaporates in vapour, that (I say) all these changes are nothing but mere changes in the spatial relationships of the smaller or larger particles of this solid, nothing thereby changes in the inner nature of those particles themselves. But how could one fail to notice that with this explanation one runs into a contradiction? For if nothing could change in the simple substances themselves (in their inner nature), then how could changes in their spatial relationships among one another be produced, and what consequences should these merely external changes have, what purpose should they serve, and how could it even be known? All these questions can only be answered rationally if we allow to the simple substances—those which are not altogether perfect, therefore can assume more forces [*Kräfte*][q] than they already have—just because of this, the capability of *change* through mutual effect on one another. And let us consider their *positions* as those determinations of

[q] The German *Kräfte* has occurred before but becomes more significant in the more metaphysical context of this, and succeeding sections. It can mean both 'forces' as here and 'powers' (as in the next section) though often neither is a very satisfactory translation.

them which contain the reason why, with exactly this size of forces in a given period, one produces exactly this change on the other, and not a larger or smaller change. Only under this assumption, which is so clear to common sense, does every contradiction in the theory of the universe disappear, and we only needed to rise above some almost obsolete *scholastic opinions*, in order to find everything in harmony.

§ 51

1. The first of these *scholastic opinions* which we must give up is the *dead* or merely *inert material* invented by the older physicists, whose simple parts, if it has such, should all be equal to one another and eternally unchangeable, and have no forces of their own at all, except the so-called *force of inertia*. Whatever is *actual*, must indeed also *act*, and therefore have *powers* to act. But a *limited* substance which for that reason is also variable, has no power which, by its nature, allows a change in its acting, therefore especially no power of creation, but it must have mere *powers of change*, which moreover can either be *inherent* like the power of *sensation*, or *transient*, like the power of motion.

At any rate, in order gradually to learn to estimate with sufficient precision the result of a certain combination of several bodies, it may still be permitted to imagine the case first of all far simpler and, instead of the infinite multitude of forces which are in fact acting together here, to assume only the presence of a few of them, indeed to imagine bodies and their properties which in reality are not even present, in order to determine what these would produce. Only we may not assume, without having first expressly taken the matter into account, that the result which would have to appear in this fictitious case, would coincide to a certain degree with what would occur in reality. The neglect of this precaution has been the cause of many famous paradoxes, as we shall see.

§ 52

2. It is another prejudice of the scholastics that *any assumption of a direct action of one substance on another in science is not permitted*. It is only true that we may never assume, without first having to prove it, that a certain action results directly. It is true that all scientific study would cease if with every phenomenon that we experienced we wanted to explain it by saying that it was produced directly. However, we obviously go too far and fall into a new, likewise very detrimental, error if we explain every action which one substance exerts on another as merely *indirect*, and so want to allow no direct action anywhere. For how could there be an indirect action if there is no direct action? Since this is clear enough we will not delay with this any longer, but it is enough to say how remarkable it is that such a great and cautious thinker as *Leibniz*, just from this motive—because no *means* was known to him whereby substances which are simple should be able to influence one another—came to that unfortunate *hypothesis of the pre-established harmony* which spoils his otherwise beautiful system of cosmology.

§ 53

3. Intimately related to this prejudice, and therefore already disproved, is that even older one, that there is no effect possible (no direct effect that is) of one substance on another situated *at a distance* from it. In the most blatant contradiction of this idea, I claim instead, that every effect of one substance (situated in space, therefore limited) on another is an *actio in distans*, for the perfectly simple reason that every two different substances in each moment also occupy two different simple places, therefore must have a *distance* between them. I have already discussed above the apparent contradiction which lies between this and one of our other claims, that space is to be continuously filled.

§ 54

4. But of course with this we also reject another, more recent, scholastic prejudice which wants to perceive a *penetration* of substances, especially in every *chemical* combination. I absolutely deny any possibility of such a penetration, because, as far as I see, it is already in the *concept of a simple place* (or point), that it is a place which can only accommodate a *single* (simple) substance. Where there are two atoms, there are also two places. From our *definition* of space, which has been repeated already several times, it also follows directly that only the magnitude of the distance between two atoms acting on one another determines the *magnitude* of the change which one produces on the other during a given time interval. If two or more substances could be in one and the same place for however short a time then the magnitude of their mutual influence would be absolutely indeterminable, and if it were even only a single moment, it would not determine its state in that moment.

§ 55

5. Since *Descartes* a new prejudice has arisen in the schools. Since he believed (from a very commendable motive) that he could not sufficiently state the difference between *thinking* and *non-thinking* substances (*mind* and *matter* as he called them) he came to that almost inconceivable claim, so striking to common sense, that a mental entity may be viewed not only not as something which is *extended*, i.e. as consisting of parts, but not even as something occurring in space, therefore not even a being occupying, by its presence, a single *point* in space. Now since at a later time, *Kant* went so far as to define space (no less than time) as a pair of pure forms of our sensibility [*Sinnlichkeit*] to which no object in itself corresponds; and since he directly contrasts *two worlds*, one *intelligible world* of the *mind*, and one *world of the senses* then it is not surprising if the prejudice of the non-spatial nature of mental entities is established so deeply, in Germany at least, that it still exists to the present day in our schools. With regard to the grounds on which I believe I have attacked this prejudice I must refer to other writings, chiefly to the *Wissenschaftslehre* and *Athanasia*. Everyone will have to admit this much, that the

view I put forward, as a consequence of which all created substances for the same reason must be located in space as well as in time, and any difference in their powers is merely a difference of degree, already recommends itself through its simplicity before any others known up until now.

§ 56

6. With this view is also removed the great paradox that has always been found previously in the *connection between the mental and the material substances*. How the material could influence the mind, and how the latter could influence the former, if both were of such different kinds, has been declared a mystery not open to human investigation. But from the above views it follows that this mutual influence must be, at least partly, a *direct* influence, to that extent therefore it could not be something mysterious and secret in itself. Yet with this we certainly do not wish to have said that there is not very much that is worthy of knowing and investigating in that part of this influence which is *mediated* in some way, especially *through organisms*.

§ 57

7. If, in times past, *substances without forces* were imagined, then conversely people in more recent times want to construct the universe from mere *forces without substances*. It was without doubt the fact, that every substance does not make its existence known other than through its effects, therefore through its *forces*, that caused the erroneous definition of the concept of substance that it was a collection of *mere* forces. And the sense-laden image to which the etymology of the words: *substance*, *substratum*, *subject*, *bearer*, and such like, points, appears to offer a clear proof that the generally prevailing theory, that for the existence of a substance it needs a proper *thing*, to which those forces belong as its *properties*, is a mere illusion of the sense faculty, for there is certainly no need here for a *bearer*, a *support* in the proper sense of the word. But must we then stay with this sense-based [*sinnlich*] interpretation? Every arbitrary something, even the mere *concept* of *nothing*, we must consider as an object to which belongs not merely one, but a whole collection of infinitely many properties. Do we therefore think of every arbitrary thing as a bearer in the proper sense? Certainly not! But if we think of a thing with the determination that it is an *actual thing* and such an actual thing that is no property of another *actual thing*, then we conceive it under the concept of a *substance* according to the correct definition of the word. And of such substances, apart from the one uncreated one, there is an infinite multitude of created ones. According to the prevailing usage, we call forces, all those properties of these substances which we must assume as most immediate (i.e. direct) ground of something else inside or outside the substance causing it. A force which is not found to be in any substance as an attribute of it, would, for this very reason, not be called a mere force, but a *substance* existing in itself. Because as a cause it

would have to be something actual, accordingly an actual thing which occurs in *no* other actual thing.

§ 58

That no *degree of existence is the highest*, and none is the *lowest* in God's creation, further that to every degree, however high, and at every time, however early, there have been creatures which by their rapid progress have already risen to this degree, but also that to every degree, however low, and at every time, however late, there will be creatures which, in spite of their continual progress occur now only at this degree—these paradoxes need no further justification after all that we said about similar relationships (§ 38 ff.) with time and space.

§ 59

However, more objectionable is the *paradox*: 'it could be, notwithstanding that the *entire infinite space of the whole universe* is filled at all times with substances in such a way that no single point or even a moment is without a substance inherent in it and that no single point accommodates two or more [substances]—that nevertheless there is an infinite multitude of different *degrees of density* with which different parts of space are filled at different times, in such a way that the same multitude of substances that at this moment, for example, occupies this cubic foot, at another times may be spread out through a space a million times bigger, and again at another time may become forced into a space a thousand times smaller, without any point standing empty with the expansion into a bigger space, nor with the compression into a smaller space any point needing to take two or more atoms.'

I know perfectly well that with this I claim something which seems to most physicists until now as an absurdity. For because they suppose that the fact of the *unequal density of solids* cannot be combined with the assumption of a continuously filled space, they assume a kind of *porosity* as a general property of all bodies, even those with which (as with gases and the aether) not the slightest observation speaks for it. And in these *pores*, of which the larger are generally to be filled with gases, therefore really only in the never-seen pores of the fluidities, physicists still assume up till now their so-called *vacuum dispertitum*, i.e. certain empty spaces in such multitudes and extension, that scarcely the billionth part of a space filled with mere aether contains true matter. Nevertheless I hope that if all that was said in §20 ff. is taken into consideration, it will become clear enough that it contains absolutely nothing impossible that the same (infinite) multitude of atoms may at one time be spread through a greater space and at another time drawn into a smaller space, without it causing in the first case even a single point in that space to be solitary, and without a single point in the second case having to take two atoms.

§ 60

And now one might scarcely take much offence at an assertion (which all the same has already been put forward in the older metaphysics in the theory *de nexu cosmico*) that every substance in the world stands in a state of continuous exchange with every other [substance], but so that the change which one produces in the other becomes all the smaller, the greater the distance lying between them; and that the *total result of the influence of all of them on every individual one* is a change which—apart from the case in which there is a direct act of God—proceeds according to the well-known *law of continuity*, because a deviation from this latter requires a force which in comparison with a continuous one would have to be *infinitely great*.

§ 61

The theory already put forward in the first edition of *Athanasia* (1829) of *dominant substances* can easily be derived from mere concepts. Equally easily people will discover paradoxes in it. Therefore it is necessary to say a few words about them here.

I start in the work mentioned above from the idea that, because it is well known that between every two substances in the universe at every time there must be some difference of finite magnitude, there are also substances at every time which are already so advanced in their powers that they exert a kind of superiority over all substances lying around them in an area, however small it may be. It would be an error, which immediately brought this assumption into the suspicion of an inner contradiction, if someone wanted to imagine that such a dominant substance would have to possess powers which exceeded those of the ones *dominated* by an infinite amount. But this is not at all the case. For if we suppose, in a space of finite magnitude, e.g. in that of a sphere, there occurs a substance (say at the centre of it) which in its forces surpasses each of the others in a finite ratio, as it would be for example, if each of the latter were to be only half as strong as it was. Now although there can be no doubt at all that the total effect of these infinitely many weaker substances there, where they happen *to combine* in their activity (as, for example, we shall soon hear, what tends to happen with the striving for the approach to a central body) surpasses infinitely many times the activity of the one stronger one, there can and must be other cases where those forces are not seeking the same goal, particularly this must be if we wish to keep in mind now merely that effect which every substance occurring in space exerts for itself alone on every other and experiences mutually from it—it can be stated as a rule, that this mutual effect, on the side of the stronger substance, is all the stronger in the same proportion to its strength. Therefore in this example, the substance which we assume as at least twice as strong as each of its neighbouring ones, will act on each of them at least twice as strongly as they act back on it. And this is all that we imagine when we say that it *dominates* the others.

§ 62

However, someone may perhaps say, if things behave in this way, then one must find not merely in some, but in every space, however small, indeed in every arbitrary collection of atoms, a dominant one. For there must be a strongest as well as a weakest atom in every collection of several atoms. Nevertheless I hope none of my readers needs the explanation that this holds at most of finite multitudes, but where an infinite multitude occurs, every member can have an even greater one above it (or an even smaller one below it), without it being that any one of them exceeds a given finite quantity (or sinks below it).

§ 63

These *dominant* substances which, according to their concept, appear in each finite space only in finite numbers, but each one surrounded with a shell [*Hülle*], sometimes greater, sometimes smaller, of merely inferior substances, are what, when combined into clusters of finite magnitude, form what we call the manifold *bodies* appearing in the world (gaseous, as well as liquid, solid, organic etc.). In contrast with them, I call all the other material of the world, which, without having distinguished [*ausgezeichnete*] atoms, fills all the spaces existing elsewhere, and therefore combines all bodies of the world, the *aether*. Here is not the place to discuss how many phenomena so far explained only imperfectly, or even not at all, from the previous assumptions (even if they were admitted as assumptions), may be explained with great facility. I must allow myself, in accord with the purpose of this writing, just a few indications by which apparent contradictions will be elucidated.

If all created substances are distinguished from one another only through the *degree* of their powers, therefore each of them must be allowed some, however little, degree of *sensitivity*, and all [of them] act on all [others], then nothing is more understandable than that for every two, however formed, and all the more certainly for every two distinguished substances, *not every distance* appears *equally acceptable to them* (equally good for them), because the magnitudes of the effects which they exert as well as those they experience, depend on the magnitude of the distance. If the distance at which they are situated is greater than that which is acceptable to one of them, then there will be a striving in it to shorten this distance, therefore a so-called *attraction*, but in the opposite case a *repulsion*. Neither the former nor the latter must we think of as always mutual, much less always as a result accompanying an actual change of position. But we may indeed assume with certainty that for every two substances in the universe there is a distance great enough such that for this one and for all greater, there is mutual attraction, and equally also a distance small enough that for it and for all smaller there is a mutual repulsion. But how much the magnitudes of the two distances spoken of here which are the boundaries of the attraction and the repulsion for the two given substances, are, may vary with the time, not only according to the nature of these substances themselves, but also according to the nature of the neighbouring substances lying in the whole region. It is indisputable that all the influence which two substances exert on one another in otherwise similar circumstances must decrease with the increase of their distance from one another, if only for the reason that the number of those which could take effect at the same distance and had a claim to the same effect, increases as the square of that distance. Further, as the superiority that every distinguished substance has over a merely inferior one, rises always only as a finite quantity, against which the number of the latter exceeds that of the former infinitely many times, so it is understandable that the magnitude of the attraction which all the substances in a given space exerts on one atom lying outside it, if the distance of it is sufficiently large, is close to the one which there would be if that space contained no distinguished substances but only an equally large number of common atoms. This combined with the earlier [assertions], leads to the important conclusion: *that a force of attraction exists between all bodies, providing their distance from one another is sufficiently great, which varies directly as the sum of their masses* (i.e. the multitude of their atoms), *and inversely as the square of their distance*. No physicist or astronomer in our day denies that this law may be observed in the universe. But it seems to have been rarely considered how difficult it is to square this with the usual view of the nature of the elementary parts of the different bodies. Namely if things actually behave as they have usually been represented up to now, that those 55 or more *simple elements* which our chemists have got to know from the earth, the masses of all the bodies occurring here would be formed in the way that each of them would be a mere collection of atoms of one, or another, or some of these elements put together. Thus, for example, gold would be a mere collection of nothing but gold atoms, sulphur would be a collection of nothing but sulphur atoms, etc. Then explain to me, whoever can, how it comes about that elements which are so different in their powers, especially in the degree of their mutual attraction, are nevertheless absolutely the same as one another in their *weight*, i.e. that their weights are proportional to their masses. For that this occurs proves directly the well-known experience that spheres of any arbitrary element, if they are of equal weights, on collision with one another behave exactly as bodies of equal mass would have to do, e.g. with equal speed (as long as the effect of the elasticity is mutual or taken into account) they bring one another to rest. But if we assume that all bodies actually consist of nothing other than an infinite multitude of aether [atoms] in which there occurs a quite negligible number of distinguished atoms whose forces surpass those of an aether atom only finitely many times, then it is understandable that the force of attraction which these bodies experience from the side of the whole globe can in no case be noticeably increased by the small number of those distinguished atoms, their weight must therefore only be proportional to their mass. Nevertheless there is now no lack of physicists who consider *caloric* [*Wärmestoffe*] (therefore fundamentally the same element which I identify with the aether) as a fluid, which occurs in all bodies and can never be completely expelled from them. Therefore if they had not unfortunately conceived the idea that this caloric was *imponderable* and if they had taken the view that the multitude of atoms which are present in every particular body compared with the caloric is insignificant (and how near to this they were when they sometimes postulated that the former separated by distances which, compared with their diameters, are infinitely large), then it would soon have become perfectly clear to them that it may be just its caloric which determines the weight of all bodies.

§ 64

It is easy to think that that domination which a distinguished substance exerts over its immediate neighbourhood, consists, if in nothing else, at least in a certain stronger attraction of its neighbouring atoms, as a consequence of which these are forced together and *denser* around it than they would otherwise be; and for this very reason they have a tendency to move themselves again, on a given opportunity, somewhat further from this point of attraction as well as from one another, therefore *to repel* one another. This is a matter to which much experience points, but for the explanation of it an *original* force of repulsion between the particles of the *aether* has quite unnecessarily been assumed.

§ 65

From this fact an easy proof follows of the proposition which I have put forward already in the *Athanasia*, *that no distinguished substance experiences in its shell such a change* that it may not keep a certain part (however small it may be) of its nearest surroundings. Certainly no one will be concerned that a distinguished substance *a* should be deprived of the aether atoms immediately surrounding it, if, among all the neighbours lying immediately round about it, of distinguished rank *b*, *c*, *d*, *e*, . . . , none changes its distance from *a*. But what may cause some concern is something like [what happens] if some of them, or even all of them, disappear. Even if this does happen only a part of the aether particles surrounding *a* can follow the escaping substances *b*, *c*, *d*, *e*, . . . , but a part, and indeed of those which are nearest to *a*, must always remain, although we do not only admit, but even assert it as necessary, that it will expand into a larger space. In fact according to the detailed circumstances, aether atoms from certain distant regions could flow into it, and press into those spaces which are filled with a comparatively more diffuse aether on account of the far too great distances into which the substances *a*, *b*, *c*, *d*, *e*, . . . have just dispersed. But there is no reason to suppose that this aether coming from a distance should push away and displace the [atoms] surrounding substance *a* at the moment. Instead of completely driving away the aether surrounding the substance *a* the aether flowing in must rather hinder its spreading further and force it together compactly until its density balances the attractive forces of all the surrounding atoms.

§ 66

Next some questions can be answered in a manner which could have been found paradoxical if it were not for the explanation provided by the foregoing material. Of such a kind is the question about the *boundaries of a body*: where exactly does one body end and another begin? I understand by the *boundary* of a body the collection of those *outermost* aether atoms that still belong to it, i.e. those that are attracted by the distinguished atoms of the body more strongly than they are by other dominant atoms in the neighbourhood. This happens in such a way that in so far as the body changes its position in relation to its neighbourhood (i.e. moves away from it), it will *pull them along* with it, perhaps not with the same speed, but so that no separation takes place and no extraneous atoms come in between. Assuming this concept of a boundary it is obvious that the boundary of a body is something very variable, indeed that it is changing almost continually. Any change whatsoever proceeds partly in the boundary itself, and partly in the neighbouring bodies because understandably all such changes can produce many a change in the magnitude, as well as in the direction, of the attraction which acts on the atoms of a body, not only the subsidiary [*dienenden*] ones but even its dominant ones. Thus for example, several particles of this quill which shortly before were being attracted by those of the rest of the quill more strongly than by the surrounding air, therefore belonged to it, but now are attracted by my fingers more strongly than by the material of the quill, and are themselves therefore torn away. More exact consideration shows that some bodies at certain places do not even have any boundary atoms, i.e. no atoms which are the *outermost* among those which still belong to it and would still be drawn along with it if its position changes. For in fact, whenever one of two neighbouring bodies has an outermost atom at some place, pulled along with it, for this very reason the other has no such outermost [atom] because all aether atoms occurring behind the former already belong to the latter.

§ 67

Here also the question may be answered of whether and when bodies are standing in direct *contact* with one another, or are separated by a space in between? If I may be allowed (as seems most expedient to me) the definition that a pair of bodies touch one another where the outermost atoms, which according to the definition of the previous paragraph, belong to one of them, form a continuous extension with certain atoms of the other one, then it will certainly not be denied that there are many bodies which touch each other mutually. This is not only so if one, or indeed both, are fluid, but also if they are solid, in so far as the air attaching to them, in the usual conditions on earth, is removed between them through strong pressure or in some other way. If a pair of bodies are not touching one another, then, because there is no completely empty space, the space in between must be filled with some other body or at least by mere aether. Therefore it can be asserted that really every body stands on all sides in contact with some other bodies, or for lack of them, with mere aether.

§ 68

In respect of the different kinds of motion occurring in the universe, it could be believed that in the circumstance (in our view) that no part of space is empty there is never any other motion possible than one in which the whole mass is moved simultaneously and forms a single extension where every part of the mass always only occupies a place which directly before another part of the mass had occupied. But whoever has kept in mind what was said in §59 about the different degrees of density with which space can be filled, will understand that many other motions can and must take place. Particularly one motion, the *vibrating motion*, must occur not only with all aether atoms, but also with nearly all distinguished atoms, almost unceasingly, for a reason which is so clear that I do not need to give it. After this, *rotating* motion must also be very common to these [distinguished atoms] especially with *solid* bodies. How this is to be imagined, how it may be explained, that, if the axis of rotation is a material line (which as a consequence of our views it must always be), the same atoms which now occur on this side of it, after a half turn get to the opposite side, without tearing off? [These problems] can only mislead those who forget that in a continuum, just as much as outside of it, every atom is at a certain distance from every other, and therefore can circle the latter without tearing off or having to turn around with it. The latter, the rotation around itself, with a simple spatial object, would be something self-contradictory.

§ 69

Without wanting to assert that even a single dominant or common atom in the universe at some time may describe a perfectly straight or perfectly circular path (which indeed, with the infinitely many perturbations which every atom has due to the effect of all the others, would have an infinitely great improbability), we may nevertheless not declare such motions as something which would be impossible in itself. But we may surely assert that the describing of, for example, a *broken line* could then only come about if the speed of the atom gradually reduced towards the end of the piece *ab*, so that at the point *b* it became zero, whereupon if the motion is not interrupted by a finite period of rest, in each of the moments following the arrival in *b* there must again be a speed (increasing from zero).

It is not so with certain other lines, as for example, with the logarithmic spiral. Even apart from all the perturbations from outside, it is contradictory that only that branch of this line which, starting from any point of it towards the centre, may be described by the motion of an atom in a finite time. And it is still more absurd to require that the moving atom finally arrive at the centre of the spiral. In order to prove this for the case where the atom proceeds in its path with uniform speed, imagine first of all that it moves alone. Then it is soon apparent that its progress in the spiral can be considered as if it were composed of two motions: one uniformly in the line directed towards the centre, and the other an angular turning around this centre whose speed, increasing uniformly, must become greater than every finite quantity if the atom is to get as near to the centre as desired. Therefore it is certain that there is no force in nature which could give it this speed, much less is there a force which could impart such a speed to a complete mass of atoms spread through three dimensions, as is required if that atom in it is to traverse in a finite time all the infinitely many turns of the spiral up to the centre. But even if it had this, could it really be said of it that it may reach the centre? I at least do not think so. For although it may be said that this centre forms a continuum with the points of the spiral (which belong to it quite undeniably) because a neighbour may be found among them for however small a distance, this linear extension lacks a second property which every one must have if it is to be able to be described through the motion of an atom: namely that at each of its points, it has one or more definite directions. This is well known not to be the case at the centre point.

Finally, this is also the place for the teasing question of whether, with our views of the infinity of the universe, there could also be a progression of the universe in some given direction, or even a rotation of it around a given world axis, or world centre? We reply, that neither the one, nor the other motion has been declared impossible because there cannot be found places for every atom to move to, but it surely must be declared impossible because of the lack of causes (forces) which should produce such a motion. For neither a *physical reason* or state of affairs [*Einrichtung*], which is absolutely necessary (i.e. which is a mere consequence of purely theoretical conceptual truths), nor a *moral reason* or state of affairs which is only *conditionally* necessary (i.e. which we only meet with in the world because God causes every beneficial event for the well-being of his creation)—can be conceived on the basis of which such a motion could occur in the world.

§ 70

Let us conclude these considerations with two paradoxes made especially famous by *Euler*. *Boscovich* has already drawn attention to the fact that to one and the same question, namely how an atom *a* moves if it is attracted by a force at *c* which is inversely proportional to the square of the distance, one obtains a different answer according to the case considered as one in which the elliptic motion changes gradually if its speed of projection decreases to zero, or if the matter is judged merely in itself quite apart from this fiction. If the atom *a* had obtained, through projection (or in some other way), a lateral speed at the beginning of its motion perpendicular to *ac*, then (apart from any resistance in the medium) it must describe an ellipse of which one focus is at *c*. If this lateral speed decreases indefinitely then also the smaller axis of this ellipse decreases indefinitely, on account of which *Euler* argued that in the case where the atom has no speed at all at the point *a*, an oscillation of it between the points *a* and *c* must occur, this is the only motion into which that elliptical motion may pass without breaking the law of continuity. On the other hand others, chiefly *Busse*, found it absurd,

that the atom, whose speed in the direction *ac* should increase indefinitely when approaching *c*, should here, be locked in its course and pushed back in the opposite direction without any reason being given (for the presence of an atom preventing the passage through this place, like an atom fixed and impenetrable, would certainly not be assumed). They therefore claimed that it must instead continue its motion in the direction *ac* beyond *c* but now with diminishing speed until it reaches the end of $cb = ca$, and then it returns in a similar way from *b* to *a* again, and so on without end. In my view nothing could be decided here through *Euler*'s appeal to the law of continuity. For the phenomenon which is disputed here violates the kind of continuity, which provably in fact dominates in the changes in the universe (in the growth or decline of the forces of individual substances), just as little if the oscillation of the atom occurred within the limits *a* and *b*, as if it occurred within *a* and *c*. But this rule is violated, in a way which is absolutely not justified, when forces are assumed here, namely a force of attraction, which grows indefinitely, and therefore it should not be a surprise if from contradictory premisses, contradictory conclusions can be derived. Hence one may nevertheless see that not only *Euler*'s but also *Busse*'s answer to the question is incorrect, because it assumes something which is in itself impossible, namely the infinitely great speed at the point *c*. If this mistake is corrected, so it is assumed that the speed with which the atom progresses changes according to such a rule that it always remains finite, and finally if it is considered that one can never speak of the motion of a single atom without a medium in which it moves, and assuming a greater or smaller multitude of atoms moving with it at the same time, then a quite different result comes to light, with whose more exact description we do not need to concern ourselves here.

The *second* paradox that we want to say a few words about, concerns *pendulum motion* and is as follows. The half period of a simple pendulum whose length $= r$, is well known to be $= \frac{\pi}{2}\sqrt{\frac{r}{g}}$, calculated through an infinitely small arc, while the time of descent over the chord of this arc, which is usually considered as of equal length with it, gives it as $= \sqrt{2}.\sqrt{\frac{r}{g}}$. That *Euler* saw a paradox in this surely rests solely on his incorrect idea of the *infinitely small* which he imagined as equivalent to zero. But in fact there are no infinitely small arcs any more than there are infinitely small chords, but what mathematicians assert of their so-called infinitely small arcs and chords are really only proved by them of arcs and chords which can become as small as desired. The above two equations understood correctly can have no other meaning than: the half period of a pendulum approaches the quantity $\frac{\pi}{2}\sqrt{\frac{r}{g}}$ as much as desired if the arc through which it can swing is taken as small as desired. But the time of descent on the chord of this arc, under the same circumstances, approaches as precisely as desired, the quantity $\sqrt{2}.\sqrt{\frac{r}{g}}$. Now that these two quantities are different, that therefore the arc and its chord differ in respect of the time of descent mentioned, however small they are taken, is something no more strange than many other differences between them whose vanishing no one expects as long as both just exist. For example, the arc always has a curvature, and indeed one whose magnitude we could measure with $\frac{1}{r}$, while the chord always remains straight, i.e. has no curvature.