2. Ordinal Numbers

In this chapter we introduce ordinal numbers and prove the Transfinite Recursion Theorem.

Linear and Partial Ordering

Definition 2.1. A binary relation $<$ on a set P is a *partial ordering* of P if:

(i) $p \not< p$ for any $p \in P$;
(ii) if $p < q$ and $q < r$, then $p < r$.

$(P, <)$ is called a *partially ordered set*. A partial ordering $<$ of P is a *linear ordering* if moreover

(iii) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

If $<$ is a partial (linear) ordering, then the relation $\le$ (where $p \le q$ if either $p < q$ or $p = q$) is also called a partial (linear) ordering (and $<$ is sometimes called a *strict* ordering).

Definition 2.2. If $(P, <)$ is a partially ordered set, X is a nonempty subset of P, and $a \in P$, then:

a is a *maximal* element of X if $a \in X$ and $(\forall x \in X)\, a \not< x$;
a is a *minimal* element of X if $a \in X$ and $(\forall x \in X)\, x \not< a$;
a is the *greatest* element of X if $a \in X$ and $(\forall x \in X)\, x \le a$;
a is the *least* element of X if $a \in X$ and $(\forall x \in X)\, a \le x$;
a is an *upper bound* of X if $(\forall x \in X)\, x \le a$;
a is a *lower bound* of X if $(\forall x \in X)\, a \le x$;
a is the *supremum* of X if a is the least upper bound of X;
a is the *infimum* of X if a is the greatest lower bound of X.

The supremum (infimum) of X (if it exists) is denoted $\sup X$ ($\inf X$). Note that if X is linearly ordered by $<$, then a maximal element of X is its greatest element (similarly for a minimal element).

If $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is *order-preserving* if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order-preserving function is also called *increasing*.

A one-to-one function of P onto Q is an *isomorphism* of P and Q if both f and f^{-1} are order-preserving; $(P, <)$ is then *isomorphic* to $(Q, <)$. An isomorphism of P onto itself is an *automorphism* of $(P, <)$.

Well-Ordering

Definition 2.3. A linear ordering $<$ of a set P is a *well-ordering* if every nonempty subset of P has a least element.

The concept of well-ordering is of fundamental importance. It is shown below that well-ordered sets can be compared by their lengths; ordinal numbers will be introduced as order-types of well-ordered sets.

Lemma 2.4. *If $(W, <)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

Proof. Assume that the set $X = \{x \in W : f(x) < x\}$ is nonempty and let z be the least element of X. If $w = f(z)$, then $f(w) < w$, a contradiction. $\square$

Corollary 2.5. *The only automorphism of a well-ordered set is the identity.*

Proof. By Lemma 2.4, $f(x) \geq x$ for all x, and $f^{-1}(x) \geq x$ for all x. $\square$

Corollary 2.6. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique.* $\square$

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an *initial segment* of W (given by u).

Lemma 2.7. *No well-ordered set is isomorphic to an initial segment of itself.*

Proof. If $\operatorname{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to Lemma 2.4. $\square$

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;*
(iii) *W_2 is isomorphic to an initial segment of W_1.*

Proof. For $u \in W_i$, $(i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 : W_1(x) \text{ is isomorphic to } W_2(y)\}.$$

Using Lemma 2.7, it is easy to see that f is a one-to-one function. If h is an isomorphism between $W_1(x)$ and $W_2(y)$, and $x' < x$, then $W_1(x')$ and $W_2(h(x'))$ are isomorphic. It follows that f is order-preserving.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case (i) holds.

If $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. Thus if $\mathrm{ran}(f) \neq W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$, we have $\mathrm{ran}(f) = W_2(y_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we would have $(x_0, y_0) \in f$, where $x_0 =$ the least element of $W_1 - \mathrm{dom}(f)$. Thus case (ii) holds.

Similarly, if $\mathrm{dom}(f) \neq W_1$, then case (iii) holds.

In view of Lemma 2.7, the three cases are mutually exclusive. □

If W_1 and W_2 are isomorphic, we say that they have the same *order-type.* Informally, an ordinal number is the order-type of a well-ordered set.

We shall now give a formal definition of ordinal numbers.

Ordinal Numbers

The idea is to define ordinal numbers so that

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \in \beta, \qquad \text{and} \quad \alpha = \{\beta : \beta < \alpha\}.$$

Definition 2.9. A set T is *transitive* if every element of T is a subset of T. (Equivalently, $\bigcup T \subset T$, or $T \subset P(T)$.)

Definition 2.10. A set is an *ordinal number* (an *ordinal*) if it is transitive and well-ordered by $\in$.

We shall denote ordinals by lowercase Greek letters α, β, γ, ... The class of all ordinals is denoted by *Ord*.

We define

$$\alpha < \beta \quad \text{if and only if} \quad \alpha \in \beta.$$

Lemma 2.11.

(i) *$0 = \emptyset$ is an ordinal.*
(ii) *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*
(iii) *If $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*
(iv) *If α, β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$.*

Proof. (i), (ii) by definition.

(iii) If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta : \xi < \gamma\} = \gamma$, and so $\alpha \in \beta$.

(iv) Clearly, $\alpha \cap \beta$ is an ordinal, $\alpha \cap \beta = \gamma$. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$, and $\gamma \in \beta$, by (iii). Then $\gamma \in \gamma$, which contradicts the definition of an ordinal (namely that $\in$ is a *strict* ordering of α). □

Using Lemma 2.11 one gets the following facts about ordinal numbers (the proofs are routine):

(2.1) $<$ is a linear ordering of the class *Ord*.

(2.2) For each α, $\alpha = \{\beta : \beta < \alpha\}$.

(2.3) If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$.

(2.4) If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$.

(2.5) For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$.

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$ (the *successor* of α). In view of (2.4), the class *Ord* is a proper class; otherwise, consider $\sup \mathit{Ord} + 1$.

We can now prove that the above definition of ordinals provides us with order-types of well-ordered sets.

Theorem 2.12. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. The uniqueness follows from Lemma 2.7. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the Replacement Axioms, $F(W)$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exist). If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ. $\square$

If $\alpha = \beta + 1$, then α is a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$; α is called a *limit ordinal*. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

The existence of limit ordinals other than 0 follows from the Axiom of Infinity; see Exercise 2.3.

Definition 2.13 (Natural Numbers). We denote the least nonzero limit ordinal ω (or $\boldsymbol{N}$). The ordinals less than ω (elements of $\boldsymbol{N}$) are called *finite ordinals*, or *natural numbers*. Specifically,

$$0 = \emptyset, \qquad 1 = 0 + 1, \qquad 2 = 1 + 1, \qquad 3 = 2 + 1, \qquad \text{etc.}$$

A set X is *finite* if there is a one-to-one mapping of X onto some $n \in \boldsymbol{N}$. X is *infinite* if it is not finite.

We use letters n, m, l, k, j, i (most of the time) to denote natural numbers.

Induction and Recursion

Theorem 2.14 (Transfinite Induction). *Let C be a class of ordinals and assume that*:

(i) $0 \in C$;
(ii) *if* $\alpha \in C$, *then* $\alpha + 1 \in C$;
(iii) *if* α *is a nonzero limit ordinal and* $\beta \in C$ *for all* $\beta < \alpha$, *then* $\alpha \in C$.

Then C is the class of all ordinals.

Proof. Otherwise, let α be the least ordinal $\alpha \notin C$ and apply (i), (ii), or (iii). □

A function whose domain is the set $\boldsymbol{N}$ is called an (*infinite*) *sequence* (A *sequence in* X is a function $f : \boldsymbol{N} \to X$.) The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

or variants thereof. A *finite sequence* is a function s such $\operatorname{dom}(s) = \{i : i < n\}$ for some $n \in \boldsymbol{N}$; then s is a *sequence of length* n.

A *transfinite sequence* is a function whose domain is an ordinal:

$$\langle a_\xi : \xi < \alpha \rangle.$$

It is also called an α-*sequence* or a *sequence of length* α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an *enumeration* of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup \{(\alpha, x)\}.$$

Sometimes we shall call a "sequence"

$$\langle a_\alpha : \alpha \in Ord \rangle$$

a function (a proper class) on *Ord*.

"Definition by transfinite recursion" usually takes the following form: Given a function G (on the class of transfinite sequences), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

such that

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$.

We shall give a general version of this theorem, so that we can also construct sequences $\langle a_\alpha : \alpha \in Ord \rangle$.

Theorem 2.15 (Transfinite Recursion). *Let G be a function (on V), then (2.6) below defines a unique function F on Ord such that*

$$F(\alpha) = G(F{\restriction}\alpha)$$

for each α.

In other words, if we let $a_\alpha = F(\alpha)$, then for each α,

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle).$$

(Note that we tacitly use Replacement: $F{\restriction}\alpha$ is a set for each α.)

Corollary 2.16. *Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ such that* $\operatorname{ran}(G) \subset X$ *there exists a unique θ-sequence $\langle a_\alpha : \alpha < \theta \rangle$ in X such that $a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$ for every $\alpha < \theta$.* □

Proof. Let

$$F(\alpha) = x \leftrightarrow \text{there is a sequence } \langle a_\xi : \xi < \alpha \rangle \text{ such that:} \quad (2.6)$$
$$\text{(i)}\ \ (\forall \xi < \alpha)\, a_\xi = G(\langle a_\eta : \eta < \xi \rangle);$$
$$\text{(ii)}\ \ x = G(\langle a_\xi : \xi < \alpha \rangle).$$

For every α, if there is an α-sequence that satisfies (i), then such a sequence is unique: If $\langle a_\xi : \xi < \alpha \rangle$ and $\langle b_\xi : \xi < \alpha \rangle$ are two α-sequences satisfying (i), one shows $a_\xi = b_\xi$ by induction on ξ. Thus $F(\alpha)$ is determined uniquely by (ii), and therefore F is a function. It follows, again by induction, that for each α there is an α-sequence that satisfies (i) (at limit steps, we use Replacement to get the α-sequence as the union of all the ξ-sequences, $\xi < \alpha$). Thus F is defined for all $\alpha \in Ord$. It obviously satisfies

$$F(\alpha) = G(F{\restriction}\alpha).$$

If F' is any function on Ord that satisfies

$$F'(\alpha) = G(F'{\restriction}\alpha)$$

then it follows by induction that $F'(\alpha) = F(\alpha)$ for all α. □

Definition 2.17. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a *nondecreasing* sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \le \gamma_\eta$). We define the *limit* of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}.$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is *normal* if it is increasing and *continuous*, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$.

Ordinal Arithmetic

We shall now define addition, multiplication and exponentiation of ordinal numbers, using Transfinite Recursion.

Definition 2.18 (Addition). For all ordinal numbers α

(i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β,
(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$.

Definition 2.19 (Multiplication). For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β,
(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.

Definition 2.20 (Exponentiation). For all ordinal numbers α

(i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β,
(iii) $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$.

As defined, the operations $\alpha + \beta$, $\alpha \cdot \beta$ and α^β are normal functions in the second variable β. Their properties can be proved by transfinite induction. For instance, $+$ and $\cdot$ are associative:

Lemma 2.21. *For all ordinals α, β and γ,*

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
(ii) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

Proof. By induction on γ. □

Neither $+$ nor $\cdot$ are commutative:

$$1 + \omega = \omega \neq \omega + 1, \qquad 2 \cdot \omega = \omega \neq \omega \cdot 2 = \omega + \omega.$$

Ordinal sums and products can be also defined geometrically, as can sums and products of arbitrary linear orders:

Definition 2.22. Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The *sum* of these linear orders is the set $A \cup B$ with the ordering defined as follows: $x < y$ if and only if

(i) $x, y \in A$ and $x <_A y$, or
(ii) $x, y \in B$ and $x <_B y$, or
(iii) $x \in A$ and $y \in B$.

Definition 2.23. Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The *product* of these linear orders is the set $A \times B$ with the ordering defined by

$$(a_1, b_1) < (a_2, b_2) \text{ if and only if either } b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2).$$

Lemma 2.24. *For all ordinals α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ are, respectively, isomorphic to the sum and to the product of α and β.*

Proof. By induction on β. □

Ordinal sums and products have some properties of ordinary addition and multiplication of integers. For instance:

Lemma 2.25.

(i) *If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.*
(ii) *If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta = \beta$.*
(iii) *If $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.*
(iv) *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.*
(v) *If $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$.*

Proof. (i), (iii) and (v) are proved by induction on γ.

(ii) Let δ be the order-type of the set $\{\xi : \alpha \le \xi < \beta\}$; δ is unique by (i).

(iv) Let β be the greatest ordinal such that $\alpha \cdot \beta \le \gamma$. □

For more, see Exercises 2.10 and 2.11.

Theorem 2.26 (Cantor's Normal Form Theorem). *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $n \ge 1$, $\alpha \ge \beta_1 > \ldots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

Proof. By induction on α. For $\alpha = 1$ we have $1 = \omega^0 \cdot 1$; for arbitrary $\alpha > 0$ let β be the greatest ordinal such that $\omega^\beta \le \alpha$. By Lemma 2.25(iv) there exists a unique δ and a unique $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$; this δ must necessarily be finite. The uniqueness of the normal form is proved by induction. □

In the normal form it is possible to have $\alpha = \omega^\alpha$; see Exercise 2.12. The least ordinal with this property is called ε_0.

Well-Founded Relations

Now we shall define an important generalization of well-ordered sets.

A binary relation E on a set P is *well-founded* if every nonempty $X \subset P$ has an E-*minimal* element, that is $a \in X$ such that there is no $x \in X$ with $x \mathrel{E} a$.

Clearly, a well-ordering of P is a well-founded relation.

Given a well-founded relation E on a set P, we can define the *height* of E, and assign to each $x \in P$ an ordinal number, the *rank* of x in E.

Theorem 2.27. *If E is a well-founded relation on P, then there exists a unique function ρ from P into the ordinals such that for all $x \in P$,*

$$\rho(x) = \sup\{\rho(y) + 1 : y \mathrel{E} x\}. \tag{2.7}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number. This ordinal is called the *height* of E.

Proof. We shall define a function ρ satisfying (2.7) and then prove its uniqueness. By induction, let

$$P_0 = \emptyset, \qquad P_{\alpha+1} = \{x \in P : \forall y\,(y \mathrel{E} x \to y \in P_\alpha)\},$$
$$P_\alpha = \bigcup_{\xi<\alpha} P_\xi \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Let θ be the least ordinal such that $P_{\theta+1} = P_\theta$ (such θ exists by Replacement). First, it should be easy to see that $P_\alpha \subset P_{\alpha+1}$ for each α (by induction). Thus $P_0 \subset P_1 \subset \ldots \subset P_\theta$. We claim that $P_\theta = P$. Otherwise, let a be an E-minimal element of $P - P_\theta$. It follows that each $x \mathrel{E} a$ is in P_θ, and so $a \in P_{\theta+1}$, a contradiction. Now we define $\rho(x)$ as the least α such that $x \in P_{\alpha+1}$. It is obvious that if $x \mathrel{E} y$, then $\rho(x) < \rho(y)$, and (2.7) is easily verified. The ordinal θ is the height of E.

The uniqueness of ρ is established as follows: Let ρ' be another function satisfying (2.7) and consider an E-minimal element of the set $\{x \in P : \rho(x) \neq \rho'(x)\}$. □

Exercises

2.1. The relation "$(P,<)$ is isomorphic to $(Q,<)$" is an equivalence relation (on the class of all partially ordered sets).

2.2. α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, for every β.

2.3. If a set X is inductive, then $X \cap \mathit{Ord}$ is inductive. The set $\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}$ is the least limit ordinal $\neq 0$.

2.4. (Without the Axiom of Infinity). Let $\omega =$ least limit $\alpha \neq 0$ if it exists, $\omega = Ord$ otherwise. Prove that the following statements are equivalent:

(i) There exists an inductive set.
(ii) There exists an infinite set.
(iii) ω is a set.

[For (ii) $\rightarrow$ (iii), apply Replacement to the set of all finite subsets of X.]

2.5. If W is a well-ordered set, then there exists no sequence $\langle a_n : n \in \boldsymbol{N} \rangle$ in W such that $a_0 > a_1 > a_2 > \ldots$.

2.6. There are arbitrarily large limit ordinals; i.e., $\forall \alpha \, \exists \beta > \alpha \, (\beta$ is a limit).
[Consider $\lim_{n \to \omega} \alpha_n$, where $\alpha_{n+1} = \alpha_n + 1$.]

2.7. Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord \rangle$ has arbitrarily large *fixed points*, i.e., α such that $\gamma_\alpha = \alpha$.
[Let $\alpha_{n+1} = \gamma_{\alpha_n}$, and $\alpha = \lim_{n \to \omega} \alpha_n$.]

2.8. For all α, β and γ,

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,
(ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$,
(iii) $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

2.9. (i) Show that $(\omega + 1) \cdot 2 \neq \omega \cdot 2 + 1 \cdot 2$.
(ii) Show that $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.

2.10. If $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$,

2.11. Find α, β, γ such that

(i) $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$,
(ii) $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$,
(iii) $\alpha < \beta$ and $\alpha^\gamma = \beta^\gamma$.

2.12. Let $\varepsilon_0 = \lim_{n \to \omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.

A limit ordinal $\gamma > 0$ is called *indecomposable* if there exist no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$.

2.13. A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α.

2.14. If E is a well-founded relation on P, then there is no sequence $\langle a_n : n \in \boldsymbol{N} \rangle$ in P such that $a_1 \, E \, a_0$, $a_2 \, E \, a_1$, $a_3 \, E \, a_2$, $\ldots$.

2.15 (Well-Founded Recursion). Let E be a well-founded relation on a set P, and let G be a function. Then there exists a function F such that for all $x \in P$, $F(x) = G(x, F \restriction \{y \in P : y \, E \, x\})$.

Historical Notes

The theory of well-ordered sets was developed by Cantor, who also introduced transfinite induction. The idea of identifying an ordinal number with the set of smaller ordinals is due to Zermelo and von Neumann.