5. The Axiom of Choice and Cardinal Arithmetic

The Axiom of Choice

***Axiom of Choice* (AC).** *Every family of nonempty sets has a choice function.*

If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that

$$f(X) \in X \tag{5.1}$$

for every $X \in S$.

The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1).

The Axiom of Choice differs from other axioms of ZF by postulating the existence of a set (i.e., a choice function) without defining it (unlike, for instance, the Axiom of Pairing or the Axiom of Power Set). Thus it is often interesting to know whether a mathematical statement can be proved without using the Axiom of Choice. It turns out that the Axiom of Choice is independent of the other axioms of set theory and that many mathematical theorems are unprovable in ZF without AC.

In some trivial cases, the existence of a choice function can be proved outright in ZF:

(i) when every $X \in S$ is a singleton $X = \{x\}$;
(ii) when S is finite; the existence of a choice function for S is proved by induction on the size of S;
(iii) when every $X \in S$ is a finite set of real numbers; let $f(X) =$ the least element of X.

On the other hand, one cannot prove existence of a choice function (in ZF) just from the assumption that the sets in S are finite; even when every $X \in S$ has just two elements (e.g., sets of reals), we cannot necessarily prove that S has a choice function.

Using the Axiom of Choice, one proves that every set can be well-ordered, and therefore every infinite set has cardinality equal to some $\aleph_\alpha$. In particular, any two sets have comparable cardinals, and the ordering

$$|X| \le |Y|$$

is a well-ordering of the class of all cardinals.

Theorem 5.1 (Zermelo's Well-Ordering Theorem). *Every set can be well-ordered.*

Proof. Let A be a set. To well-order A, it suffices to construct a transfinite one-to-one sequence $\langle a_\alpha : \alpha < \theta \rangle$ that enumerates A. That we can do by induction, using a choice function f for the family S of all nonempty subsets of A. We let for every α

$$a_\alpha = f(A - \{a_\xi : \xi < \alpha\})$$

if $A - \{a_\xi : \xi < \alpha\}$ is nonempty. Let θ be the least ordinal such that $A = \{a_\xi : \xi < \theta\}$. Clearly, $\langle a_\alpha : \alpha < \theta \rangle$ enumerates A. $\square$

In fact, Zermelo's Theorem 5.1 is equivalent to the Axiom of Choice: If every set can be well-ordered, then every family S of nonempty sets has a choice function. To see this, well-order $\bigcup S$ and let $f(X)$ be the least element of X for every $X \in S$.

Of particular importance is the fact that the set of all real numbers can be well-ordered. It follows that $2^{\aleph_0}$ is an aleph and so $2^{\aleph_0} \ge \aleph_1$.

The existence of a well-ordering of $\boldsymbol{R}$ yields some interesting counterexamples. Well known is Vitali's construction of a nonmeasurable set (Exercise 10.1); another example is an uncountable set of reals without a perfect subset (Exercise 5.1).

If every set can be well-ordered, then every infinite set has a countable subset: Well-order the set and take the first ω elements. Thus every infinite set is Dedekind-infinite, and so finiteness and Dedekind finiteness coincide.

Dealing with cardinalities of sets is much easier when we have the Axiom of Choice. In the first place, any two sets have comparable cardinals. Another consequence is:

(5.2) *if f maps A onto B then $|B| \le |A|$.*

To show (5.2), we have to find a one-to-one function from B to A. This is done by choosing one element from $f_{-1}(\{b\})$ for each $b \in B$.

Another consequence of the Axiom of Choice is:

(5.3) *The union of a countable family of countable sets is countable.*

(By the way, this often used fact cannot be proved in ZF alone.) To prove (5.3) let A_n be a countable set for each $n \in \boldsymbol{N}$. For each n, let us *choose* an enumeration $\langle a_{n,k} : k \in \boldsymbol{N}\rangle$ of A_n. That gives us a projection of $\boldsymbol{N} \times \boldsymbol{N}$ onto $\bigcup_{n=0}^{\infty} A_n$:

$$(n, k) \mapsto a_{n,k}.$$

Thus $\bigcup_{n=0}^{\infty} A_n$ is countable.

In a similar fashion, one can prove a more general statement.

Lemma 5.2. $|\bigcup S| \leq |S| \cdot \sup\{|X| : X \in S\}$.

Proof. Let $\kappa = |S|$ and $\lambda = \sup\{|X| : X \in S\}$. We have $S = \{X_\alpha : \alpha < \kappa\}$ and for each $\alpha < \kappa$, we choose an enumeration $X_\alpha = \{a_{\alpha,\beta} : \beta < \lambda_\alpha\}$, where $\lambda_\alpha \leq \lambda$. Again we have a projection

$$(\alpha, \beta) \mapsto a_{\alpha,\beta}$$

of $\kappa \times \lambda$ onto $\bigcup S$, and so $|\bigcup S| \leq \kappa \cdot \lambda$. □

In particular, the union of $\aleph_\alpha$ sets, each of cardinality $\aleph_\alpha$, has cardinality $\aleph_\alpha$.

Corollary 5.3. *Every $\aleph_{\alpha+1}$ is a regular cardinal.*

Proof. This is because otherwise $\omega_{\alpha+1}$ would be the union of at most $\aleph_\alpha$ sets of cardinality at most $\aleph_\alpha$. □

Using the Axiom of Choice in Mathematics

In algebra and point set topology, one often uses the following version of the Axiom of Choice. We recall that if $(P, <)$ is a partially ordered set, then $a \in P$ is called *maximal* in P if there is no $x \in P$ such that $a < x$. If X is a nonempty subset of P, then $c \in P$ is an *upper bound* of X if $x \leq c$ for every $x \in X$.

We say that a nonempty $C \subset P$ is a *chain* in P if C is linearly ordered by $<$.

Theorem 5.4 (Zorn's Lemma). *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Proof. We construct (using a choice function for nonempty subsets of P), a chain in P that leads to a maximal element of P. We let, by induction,

$$a_\alpha = \text{an element of } P \text{ such that } a_\alpha > a_\xi \text{ for every } \xi < \alpha \text{ if there is one.}$$

Clearly, if $\alpha > 0$ is a limit ordinal, then $C_\alpha = \{a_\xi : \xi < \alpha\}$ is a chain in P and a_α exists by the assumption. Eventually, there is θ such that there is no $a_{\theta+1} \in P$, $a_{\theta+1} > a_\theta$. Thus a_θ is a maximal element of P. □

Like Zermelo's Theorem 5.1, Zorn's Lemma 5.4 is equivalent to the Axiom of Choice (in ZF); see Exercise 5.5.

There are numerous examples of proofs using Zorn's Lemma. To mention only of few:

Every vector space has a basis.
Every field has a unique algebraic closure.
The Hahn-Banach Extension Theorem.
Tikhonov's Product Theorem for compact spaces.

The Countable Axiom of Choice

Many important consequences of the Axiom of Choice, particularly many concerning the real numbers, can be proved from a weaker version of the Axiom of Choice.

The Countable Axiom of Choice. *Every countable family of nonempty sets has a choice function.*

For instance, the countable AC implies that the union of countably many countable sets is countable. In particular, the real line is not a countable union of countable sets. Similarly, it follows that $\aleph_1$ is a regular cardinal. On the other hand, the countable AC does not imply that the set of all reals can be well-ordered.

Several basic theorems about Borel sets and Lebesgue measure use the countable AC; for instance, one needs it to show that the union of countably many F_σ sets is F_σ. In modern descriptive set theory one often works without the Axiom of Choice and uses the countable AC instead. In some instances, descriptive set theorists use a somewhat stronger principle (that follows from AC):

***The Principle of Dependent Choices* (DC).** *If E is a binary relation on a nonempty set A, and if for every $a \in A$ there exists $b \in A$ such that $b\ E\ a$, then there is a sequence $a_0, a_1, \ldots, a_n, \ldots$ in A such that*

$$a_{n+1}\ E\ a_n \quad \text{for all } n \in \boldsymbol{N}. \tag{5.4}$$

The Principle of Dependent Choices is stronger than the Countable Axiom of Choice; see Exercise 5.7.

As an application of DC we have the following characterization of well-founded relations and well-orderings:

Lemma 5.5.

(i) *A linear ordering $<$ of a set P is a well-ordering of P if and only if there is no infinite descending sequence*

$$a_0 > a_1 > \ldots > a_n > \ldots$$

in A.

(ii) *A relation E on P is well-founded if and only if there is no infinite sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ in P such that*

$$a_{n+1}\ E\ a_n \quad \textit{for all } n \in \boldsymbol{N}. \tag{5.5}$$

Proof. Note that (i) is a special case of (ii) since a well-ordering is a well-founded linear ordering.

If $a_0, a_1, \ldots, a_n, \ldots$ is a sequence that satisfies (5.5), then the set $\{a_n : n \in \boldsymbol{N}\}$ has no E-minimal element and hence E is not well-founded.

Conversely, if E is not well-founded, then there is a nonempty set $A \subset P$ with no E-minimal element. Using the Principle of Dependent Choices we construct a sequence $a_0, a_1, \ldots, a_n, \ldots$ that satisfies (5.5). □

Cardinal Arithmetic

In the presence of the Axiom of Choice, every set can be well-ordered and so every infinite set has the cardinality of some $\aleph_\alpha$. Thus addition and multiplication of infinite cardinal numbers is simple: If κ and λ are infinite cardinals then

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

The exponentiation of cardinals is more interesting. The rest of Chapter 5 is devoted to the operations 2^κ and κ^λ, for infinite cardinals κ and λ.

Lemma 5.6. *If $2 \le \kappa \le \lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$.*

Proof.

$$2^\lambda \le \kappa^\lambda \le (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda. \tag{5.6}$$ □

If κ and λ are infinite cardinals and $\lambda < \kappa$ then the evaluation of κ^λ is more complicated. First, if $2^\lambda \ge \kappa$ then we have $\kappa^\lambda = 2^\lambda$ (because $\kappa^\lambda \le (2^\lambda)^\lambda = 2^\lambda$), but if $2^\lambda < \kappa$ then (because $\kappa^\lambda \le \kappa^\kappa = 2^\kappa$) we can only conclude

$$\kappa \le \kappa^\lambda \le 2^\kappa. \tag{5.7}$$

Not much more can be claimed at this point, except that by Theorem 3.11 in Chapter 3 ($\kappa^{\operatorname{cf}\kappa} > \kappa$) we have

$$\kappa < \kappa^\lambda \quad \text{if } \lambda \ge \operatorname{cf}\kappa. \tag{5.8}$$

If λ is a cardinal and $|A| \ge \lambda$, let

$$[A]^\lambda = \{X \subset A : |X| = \lambda\}. \tag{5.9}$$

Lemma 5.7. *If $|A| = \kappa \ge \lambda$, then the set $[A]^\lambda$ has cardinality κ^λ.*

Proof. On the one hand, every $f : \lambda \to A$ is a subset of $\lambda \times A$, and $|f| = \lambda$. Thus $\kappa^\lambda \le |[\lambda \times A]|^\lambda = |[A]^\lambda|$. On the other hand, we construct a one-to-one function $F : [A]^\lambda \to A^\lambda$ as follows: If $X \subset A$ and $|X| = \lambda$, let $F(X)$ be some function f on λ whose range is X. Clearly, F is one-to-one. □

If λ is a limit cardinal, let

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu \text{ is a cardinal and } \mu < \lambda\}. \tag{5.10}$$

For the sake of completeness, we also define $\kappa^{<\lambda^+} = \kappa^\lambda$ for infinite successor cardinals λ^+.

If κ is an infinite cardinal and $|A| \ge \kappa$, let

$$[A]^{<\kappa} = P_\kappa(A) = \{X \subset A : |X| < \kappa\}. \tag{5.11}$$

It follows from Lemma 5.7 and Lemma 5.8 below that the cardinality of $P_\kappa(A)$ is $|A|^{<\kappa}$.

Infinite Sums and Products

Let $\{\kappa_i : i \in I\}$ be an indexed set of cardinal numbers. We define

$$\sum_{i\in I} \kappa_i = \Big|\bigcup_{i\in I} X_i\Big|, \tag{5.12}$$

where $\{X_i : i \in I\}$ is a disjoint family of sets such that $|X_i| = \kappa_i$ for each $i \in I$.

This definition does not depend on the choice of $\{X_i\}_i$; this follows from the Axiom of Choice (see Exercise 5.9).

Note that if κ and λ are cardinals and $\kappa_i = \kappa$ for each $i < \lambda$, then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \kappa.$$

In general, we have the following

Lemma 5.8. *If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup\nolimits_{i<\lambda} \kappa_i. \tag{5.13}$$

Proof. Let $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. On the one hand, since $\kappa_i \le \kappa$ for all i, we have $\sum_{i<\lambda} \kappa \le \lambda \cdot \kappa$. On the other hand, since $\kappa_i \ge 1$ for all i, we have $\lambda = \sum_{i<\lambda} 1 \le \sigma$, and since $\sigma \ge \kappa_i$ for all i, we have $\sigma \ge \sup_{i<\lambda} \kappa_i = \kappa$. Therefore $\sigma \ge \lambda \cdot \kappa$. □

In particular, if $\lambda \le \sup_{i<\lambda} \kappa_i$, we have

$$\sum_{i<\lambda} \kappa_i = \sup_{i<\lambda} \kappa_i.$$

Thus we can characterize singular cardinals as follows: An infinite cardinal κ is singular just in case

$$\kappa = \sum_{i<\lambda} \kappa_i$$

where $\lambda < \kappa$ and for each i, $\kappa_i < \kappa$.

An infinite product of cardinals is defined using infinite products of sets. If $\{X_i : i \in I\}$ is a family of sets, then the *product* is defined as follows:

$$\prod_{i\in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}. \tag{5.14}$$

Note that if some X_i is empty, then the product is empty. If all the X_i are nonempty, then AC implies that the product is nonempty.

If $\{\kappa_i : i \in I\}$ is a family of cardinal numbers, we define

$$\prod_{i\in I} \kappa_i = \Bigl|\prod_{i\in I} X_i\Bigr|, \tag{5.15}$$

where $\{X_i : i \in I\}$ is a family of sets such that $|X_i| = \kappa_i$ for each $i \in I$. (We abuse the notation by using $\prod$ both for the product of sets and for the product of cardinals.)

Again, it follows from AC that the definition does not depend on the choice of the sets X_i (Exercise 5.10).

If $\kappa_i = \kappa$ for each $i \in I$, and $|I| = \lambda$, then $\prod_{i\in I} \kappa_i = \kappa^\lambda$. Also, infinite sums and products satisfy some of the rules satisfied by finite sums and products. For instance, $\prod_i \kappa_i^\lambda = (\prod_i \kappa_i)^\lambda$, or $\prod_i \kappa^{\lambda_i} = \kappa^{\sum_i \lambda_i}$. Or if I is a disjoint union $I = \bigcup_{j\in J} A_j$, then

$$\prod_{i\in I} \kappa_i = \prod_{j\in J}\Bigl(\prod_{i\in A_j} \kappa_i\Bigr). \tag{5.16}$$

If $\kappa_i \ge 2$ for each $i \in I$, then

$$\sum_{i\in I} \kappa_i \le \prod_{i\in I} \kappa_i. \tag{5.17}$$

(The assumption $\kappa_i \ge 2$ is necessary: $1+1 > 1\cdot 1$.) If I is finite, then (5.17) is certainly true; thus assume that I is infinite. Since $\prod_{i\in I} \kappa_i \ge \prod_{i\in I} 2 = 2^{|I|} > |I|$, it suffices to show that $\sum_i \kappa_i \le |I| \cdot \prod_i \kappa_i$. If $\{X_i : i \in I\}$ is a disjoint family, we assign to each $x \in \bigcup_i X_i$ a pair (i, f) such that $x \in X_i$, $f \in \prod_i X_i$ and $f(i) = x$. Thus we have (5.17).

Infinite product of cardinals can be evaluated using the following lemma:

Lemma 5.9. *If λ is an infinite cardinal and $\langle \kappa_i : i < \lambda \rangle$ is a nondecreasing sequence of nonzero cardinals, then*

$$\prod_{i<\lambda} \kappa_i = (\sup_i \kappa_i)^\lambda.$$

Proof. Let $\kappa = \sup_i \kappa_i$. Since $\kappa_i \le \kappa$ for each $i < \lambda$, we have

$$\prod_{i<\lambda} \kappa_i \le \prod_{i<\lambda} \kappa = \kappa^\lambda.$$

To prove that $\kappa^\lambda \le \prod_{i<\lambda} \kappa_i$, we consider a partition of λ into λ disjoint sets A_j, each of cardinality λ:

$$\lambda = \bigcup_{j<\lambda} A_j. \tag{5.18}$$

(To get a partition (5.18), we can, e.g., use the canonical pairing function $\Gamma : \lambda \times \lambda \to \lambda$ and let $A_j = \Gamma(\lambda \times \{j\})$.) Since a product of nonzero cardinals is greater than or equal to each factor, we have $\prod_{i\in A_j} \kappa_i \ge \sup_{i\in A_j} \kappa_i = \kappa$, for each $j < \lambda$. Thus, by (5.16),

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda} \Big(\prod_{i\in A_j} \kappa_i \Big) \ge \prod_{j<\lambda} \kappa = \kappa^\lambda. \qquad \square$$

The strict inequalities in cardinal arithmetic that we proved in Chapter 3 can be obtained as special cases of the following general theorem.

Theorem 5.10 (König). *If $\kappa_i < \lambda_i$ for every $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. We shall show that $\sum_i \kappa_i \not\ge \prod_i \lambda_i$. Let T_i, $i \in I$, be such that $|T_i| = \lambda_i$ for each $i \in I$. It suffices to show that if Z_i, $i \in I$, are subsets of $T = \prod_{i\in I} T_i$, and $|Z_i| \le \kappa_i$ for each $i \in I$, then $\bigcup_{i\in I} Z_i \ne T$.

For every $i \in I$, let S_i be the projection of Z_i into the ith coordinate:

$$S_i = \{f(i) : f \in Z_i\}.$$

Since $|Z_i| < |T_i|$, we have $S_i \subset T_i$ and $S_i \ne T_i$. Now let $f \in T$ be a function such that $f(i) \notin S_i$ for every $i \in I$. Obviously, f does not belong to any Z_i, $i \in I$, and so $\bigcup_{i\in I} Z_i \ne T$. $\square$

Corollary 5.11. $\kappa < 2^\kappa$ *for every κ.*

Proof. $\underbrace{1 + 1 + \ldots}_{\kappa \text{ times}} < \underbrace{2 \cdot 2 \cdot \ldots}_{\kappa \text{ times}}$. $\square$

Corollary 5.12. $\operatorname{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Proof. It suffices to show that if $\kappa_i < 2^{\aleph_\alpha}$ for $i < \omega_\alpha$, then $\sum_{i<\omega_\alpha} \kappa_i < 2^{\aleph_\alpha}$. Let $\lambda_i = 2^{\aleph_\alpha}$.

$$\sum_{i<\omega_\alpha} \kappa_i < \prod_{i<\omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}. \qquad \square$$

Corollary 5.13. $\operatorname{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$.

Proof. We show that if $\kappa_i < \aleph_\alpha^{\aleph_\beta}$ for $i < \omega_\beta$, then $\sum_{i<\omega_\beta} \kappa_i < \aleph_\alpha^{\aleph_\beta}$. Let $\lambda_i = \aleph_\alpha^{\aleph_\beta}$.

$$\sum_{i<\omega_\beta} \kappa_i < \prod_{i<\omega_\beta} \lambda_i = (\aleph_\alpha^{\aleph_\beta})^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}. \qquad \square$$

Corollary 5.14. $\kappa^{\operatorname{cf}\kappa} > \kappa$ *for every infinite cardinal* κ.

Proof. Let $\kappa_i < \kappa$, $i < \operatorname{cf}\kappa$, be such that $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$. Then

$$\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i < \prod_{i<\operatorname{cf}\kappa} \kappa = \kappa^{\operatorname{cf}\kappa}. \qquad \square$$

The Continuum Function

Cantor's Theorem 3.1 states that $2^{\aleph_\alpha} > \aleph_\alpha$, and therefore $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$, for all α. The *Generalized Continuum Hypothesis* (GCH) is the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for all α. GCH is independent of the axioms of ZFC. Under the assumption of GCH, cardinal exponentiation is evaluated as follows:

Theorem 5.15. *If* GCH *holds and* κ *and* λ *are infinite cardinals then*:

(i) *If* $\kappa \leq \lambda$, *then* $\kappa^\lambda = \lambda^+$.
(ii) *If* $\operatorname{cf}\kappa \leq \lambda < \kappa$, *then* $\kappa^\lambda = \kappa^+$.
(iii) *If* $\lambda < \operatorname{cf}\kappa$, *then* $\kappa^\lambda = \kappa$.

Proof. (i) Lemma 5.6.

(ii) This follows from (5.7) and (5.8).

(iii) By Lemma 3.9(ii), the set κ^λ is the union of the sets α^λ, $\alpha < \kappa$, and $|\alpha^\lambda| \leq 2^{|\alpha|\cdot\lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$. $\square$

The *beth function* is defined by induction:

$$\beth_0 = \aleph_0, \qquad \beth_{\alpha+1} = 2^{\beth_\alpha},$$
$$\beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Thus GCH is equivalent to the statement $\beth_\alpha = \aleph_\alpha$ for all α.

We shall now investigate the general behavior of the *continuum function* 2^κ, without assuming GCH.

Theorem 5.16.

(i) *If* $\kappa < \lambda$ *then* $2^\kappa \leq 2^\lambda$.
(ii) $\operatorname{cf} 2^\kappa > \kappa$.
(iii) *If* κ *is a limit cardinal then* $2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa}$.

Proof. (ii) By Corollary 5.12,

(iii) Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_i 2^{\kappa_i} \leq \prod_i 2^{<\kappa} = (2^{<\kappa})^{\operatorname{cf}\kappa} \leq (2^\kappa)^{\operatorname{cf}\kappa} \leq 2^\kappa. \qquad \square$$

For regular cardinals, the only conditions Theorem 5.16 places on the continuum function are $2^\kappa > \kappa$ and $2^\kappa \leq 2^\lambda$ if $\kappa < \lambda$. We shall see that these are the only restrictions on 2^κ for regular κ that are provable in ZFC.

Corollary 5.17. *If* κ *is a singular cardinal and if the continuum function is eventually constant below* κ, *with value* λ, *then* $2^\kappa = \lambda$.

Proof. If κ is a singular cardinal that satisfies the assumption of the theorem, then there is μ such that $\operatorname{cf}\kappa \leq \mu < \kappa$ and that $2^{<\kappa} = \lambda = 2^\mu$. Thus

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = (2^\mu)^{\operatorname{cf}\kappa} = 2^\mu. \qquad \square$$

The *gimel function* is the function

$$\gimel(\kappa) = \kappa^{\operatorname{cf}\kappa}. \tag{5.19}$$

If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then the cardinal $\lambda = 2^{<\kappa}$ is a limit of a nondecreasing sequence

$$\lambda = 2^{<\kappa} = \lim_{\alpha\to\kappa} 2^{|\alpha|}$$

of length κ. By Lemma 3.7(ii), we have

$$\operatorname{cf}\lambda = \operatorname{cf}\kappa.$$

Using Theorem 5.16(iii), we get

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda}. \tag{5.20}$$

If κ is a regular cardinal, then $\kappa = \operatorname{cf}\kappa$; and since $2^\kappa = \kappa^\kappa$, we have

$$2^\kappa = \kappa^{\operatorname{cf}\kappa}. \tag{5.21}$$

Thus (5.20) and (5.21) show that the continuum function can be defined in terms of the gimel function:

Corollary 5.18.

(i) *If* κ *is a successor cardinal, then* $2^\kappa = \gimel(\kappa)$.
(ii) *If* κ *is a limit cardinal and if the continuum function below* κ *is eventually constant, then* $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$.
(iii) *If* κ *is a limit cardinal and if the continuum function below* κ *is not eventually constant, then* $2^\kappa = \gimel(2^{<\kappa})$. $\square$

Cardinal Exponentiation

We shall now investigate the function κ^λ for infinite cardinal numbers κ and λ.

We start with the following observation: If κ is a regular cardinal and $\lambda < \kappa$, then every function $f : \lambda \to \kappa$ is bounded (i.e., $\sup\{f(\xi) : \xi < \lambda\} < \kappa$). Thus

$$\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda.$$

and so

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda.$$

In particular, if κ is a successor cardinal, we obtain the *Hausdorff formula*

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}. \tag{5.22}$$

(Note that (5.22) holds for all α and β.)

In general, we can compute κ^λ using the following lemma. If κ is a limit cardinal, we use the notation $\lim_{\alpha\to\kappa} \alpha^\lambda$ to abbreviate $\sup\{\mu^\lambda : \mu$ is a cardinal and $\mu < \kappa\}$.

Lemma 5.19. *If κ is a limit cardinal, and $\lambda \geq \operatorname{cf}\kappa$, then*

$$\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa}.$$

Proof. Let $\kappa = \sum_{i<\operatorname{cf}\kappa} \kappa_i$, where $\kappa_i < \kappa$ for each i. We have $\kappa^\lambda \leq (\prod_{i<\operatorname{cf}\kappa} \kappa_i)^\lambda = \prod_i \kappa_i^\lambda \leq \prod_i (\lim_{\alpha\to\kappa} \alpha^\lambda) = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} \leq (\kappa^\lambda)^{\operatorname{cf}\kappa} = \kappa^\lambda$. □

Theorem 5.20. *Let λ be an infinite cardinal. Then for all infinite cardinals κ, the value of κ^λ is computed as follows, by induction on κ:*

(i) *If $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *If $\kappa > \lambda$ and if $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then:*
  (a) *if* $\operatorname{cf}\kappa > \lambda$ *then* $\kappa^\lambda = \kappa$,
  (b) *if* $\operatorname{cf}\kappa \leq \lambda$ *then* $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$.

Proof. (i) Lemma 5.6

(ii) $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$.

(iii) If κ is a successor cardinal, we use the Hausdorff formula. If κ is a limit cardinal, we have $\lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf}\kappa > \lambda$ then every $f : \lambda \to \kappa$ is bounded and we have $\kappa^\lambda = \lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf}\kappa \leq \lambda$ then by Lemma 5.19, $\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\operatorname{cf}\kappa} = \kappa^{\operatorname{cf}\kappa}$. □

Theorem 5.20 shows that all cardinal exponentiation can be defined in terms of the gimel function:

Corollary 5.21. *For every κ and λ, the value of κ^λ is either 2^λ, or κ, or $\gimel(\mu)$ for some μ such that* $\operatorname{cf}\mu \le \lambda < \mu$.

Proof. If $\kappa^\lambda > 2^\lambda \cdot \kappa$, let μ be the least cardinal such that $\mu^\lambda = \kappa^\lambda$, and by Theorem 5.20 (for μ and λ), $\mu^\lambda = \mu^{\operatorname{cf}\mu}$. $\square$

In the Exercises, we list some properties of the gimel function.

A cardinal κ is a *strong limit* cardinal if

$$2^\lambda < \kappa \quad \text{for every } \lambda < \kappa.$$

Obviously, every strong limit cardinal is a limit cardinal. If the GCH holds, then every limit cardinal is a strong limit.

It is easy to see that if κ is a strong limit cardinal, then

$$\lambda^\nu < \kappa \quad \text{for all } \lambda, \nu < \kappa.$$

An example of a strong limit cardinal is $\aleph_0$. Actually, the strong limit cardinals form a proper class: If α is an arbitrary cardinal, then the cardinal

$$\kappa = \sup\{\alpha, 2^\alpha, 2^{2^\alpha}, \ldots\}$$

(of cofinality ω) is a strong limit cardinal.

Another fact worth mentioning is:

(5.23) *If κ is a strong limit cardinal, then $2^\kappa = \kappa^{\operatorname{cf}\kappa}$.*

We recall that κ is weakly inaccessible if it is uncountable, regular, and limit. We say that a cardinal κ is *inaccessible* (strongly) if $\kappa > \aleph_0$, κ is regular, and κ is strong limit.

Every inaccessible cardinal is weakly inaccessible. If the GCH holds, then every weakly inaccessible cardinal κ is inaccessible.

The inaccessible cardinals owe their name to the fact that they cannot be obtained from smaller cardinals by the usual set-theoretical operations.

If κ is inaccessible and $|X| < \kappa$, then $|P(X)| < \kappa$. If $|S| < \kappa$ and if $|X| < \kappa$ for every $X \in S$, then $|\bigcup S| < \kappa$.

In fact, $\aleph_0$ has this property too. Thus we can say that in a sense an inaccessible cardinal is to smaller cardinals what $\aleph_0$ is to finite cardinals. This is one of the main themes of the theory of large cardinals.

The Singular Cardinal Hypothesis

The *Singular Cardinal Hypothesis* (SCH) is the statement: For every singular cardinal κ, if $2^{\operatorname{cf}\kappa} < \kappa$, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

Obviously, the Singular Cardinals Hypothesis follows from GCH. If $2^{\operatorname{cf}\kappa} \ge \kappa$ then $\kappa^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa}$. If $2^{\operatorname{cf}\kappa} < \kappa$, then κ^+ is the least possible value of $\kappa^{\operatorname{cf}\kappa}$.

We shall prove later in the book that if SCH fails then a large cardinal axiom holds. In fact, the failure of SCH is equiconsistent with the existence of a certain large cardinal.

Under the assumption of SCH, cardinal exponentiation is determined by the continuum function on regular cardinals:

Theorem 5.22. *Assume that* SCH *holds.*

(i) *If κ is a singular cardinal then*
  (a) $2^\kappa = 2^{<\kappa}$ *if the continuum function is eventually constant below κ,*
  (b) $2^\kappa = (2^{<\kappa})^+$ *otherwise.*

(ii) *If κ and λ are infinite cardinals, then*:
  (a) *If $\kappa \le 2^\lambda$ then $\kappa^\lambda = 2^\lambda$.*
  (b) *If $2^\lambda < \kappa$ and $\lambda < \operatorname{cf}\kappa$ then $\kappa^\lambda = \kappa$.*
  (c) *If $2^\lambda < \kappa$ and $\operatorname{cf}\kappa \le \lambda$ then $\kappa^\lambda = \kappa^+$.*

Proof. (i) If κ is a singular cardinal, then by Theorem 5.16, 2^κ is either λ or $\lambda^{\operatorname{cf}\kappa}$ where $\lambda = 2^{<\kappa}$. The latter occurs if 2^α is not eventually constant below κ. Then $\operatorname{cf}\lambda = \operatorname{cf}\kappa$, and since $2^{\operatorname{cf}\kappa} < 2^{<\kappa} = \lambda$, we have $\lambda^{\operatorname{cf}\lambda} = \lambda^+$ by the Singular Cardinals Hypothesis.

(ii) We proceed by induction on κ, for a fixed λ. Let $\kappa > 2^\lambda$. If κ is a successor cardinal, $\kappa = \nu^+$, then $\nu^\lambda \le \kappa$ (by the induction hypothesis), and $\kappa^\lambda = (\nu^+)^\lambda = \nu^+ \cdot \nu^\lambda = \kappa$, by the Hausdorff formula.

If κ is a limit cardinal, then $\nu^\lambda < \kappa$ for all $\nu < \kappa$. By Theorem 5.20, $\kappa^\lambda = \kappa$ if $\lambda < \operatorname{cf}\kappa$, and $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$ if $\lambda \ge \operatorname{cf}\kappa$, In the latter case, $2^{\operatorname{cf}\kappa} \le 2^\lambda < \kappa$, and by the Singular Cardinals Hypothesis, $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. □

Exercises

5.1. There exists a set of reals of cardinality $2^{\aleph_0}$ without a perfect subset.

[Let $\langle P_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of all perfect sets of reals. Construct disjoint $A = \{a_\alpha : \alpha < 2^{\aleph_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\aleph_0}\}$ as follows: Pick a_α such that $a_\alpha \notin \{a_\xi : \xi < \alpha\} \cup \{b_\xi : \xi < \alpha\}$, and b_α such that $b_\alpha \in P_\alpha - \{a_\xi : \xi \le \alpha\}$. Then A is the set.]

5.2. If X is an infinite set and S is the set of all finite subsets of X, then $|S| = |X|$.

[Use $|X| = \aleph_\alpha$.]

5.3. Let $(P, <)$ be a linear ordering and let κ be a cardinal. If every initial segment of P has cardinality $< \kappa$, then $|P| \le \kappa$.

5.4. If A can be well-ordered then $P(A)$ can be linearly ordered.

[Let $X < Y$ if the least element of $X \bigtriangleup Y$ belongs to X.]

5.5. Prove the Axiom of Choice from Zorn's Lemma.

[Let S be a family of nonempty sets. To find a choice function on S, let $P = \{f : f$ is a choice function on some $Z \subset S\}$, and apply Zorn's Lemma to the partially ordered set $(P, \subset)$.]

5.6. The countable AC implies that every infinite set has a countable subset.

[If A is infinite, let $A_n = \{s : s$ is a one-to-one sequence in A of length $n\}$ for each n. Use a choice function for $S = \{A_n : n \in \boldsymbol{N}\}$ to obtain a countable subset of A.]

5.7. Use DC to prove the countable AC.

[Given $S = \{A_n : n \in \boldsymbol{N}\}$, consider the set A of all choice functions on some $S_n = \{A_i : i \le n\}$, with the binary relation $\supset$.]

5.8 (The Milner-Rado Paradox). For every ordinal $\alpha < \kappa^+$ there are sets $X_n \subset \alpha$ ($n \in \boldsymbol{N}$) such that $\alpha = \bigcup_n X_n$, and for each n the order-type of X_n is $\le \kappa^n$.

[By induction on α, choosing a sequence cofinal in α.]

5.9. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are two disjoint families such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\bigcup_{i\in I} X_i| = |\bigcup_{i\in I} Y_i|$.

[Use AC.]

5.10. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are such that $|X_i| = |Y_i|$ for each $i \in I$, then $|\prod_{i\in I} X_i| = |\prod_{i\in I} Y_i|$.

[Use AC.]

5.11. $\prod_{0<n<\omega} n = 2^{\aleph_0}$.

5.12. $\prod_{n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$.

5.13. $\prod_{\alpha<\omega+\omega} \aleph_\alpha = \aleph_{\omega+\omega}^{\aleph_0}$.

5.14. If GCH holds then

(i) $2^{<\kappa} = \kappa$ for all κ, and
(ii) $\kappa^{<\kappa} = \kappa$ for all regular κ.

5.15. If β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for every α, then $\beta < \omega$.

[Let $\beta \ge \omega$. Let α be least such that $\alpha + \beta > \beta$. We have $0 < \alpha \le \beta$, and α is limit. Let $\kappa = \aleph_{\alpha+\alpha}$; since $\operatorname{cf}\kappa = \operatorname{cf}\alpha \le \alpha < \kappa$, κ is singular. For each $\xi < \alpha$, $\xi + \beta = \beta$, and so $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\xi+\beta} = \aleph_{\alpha+\beta}$. By Corollary 5.17, $2^\kappa = \aleph_{\alpha+\beta}$, a contradiction, since $\aleph_{\alpha+\beta} < \aleph_{\alpha+\alpha+\beta}$.]

5.16. $\prod_{\alpha<\omega_1+\omega} \aleph_\alpha = \aleph_{\omega_1+\omega}^{\aleph_1}$.

[$\aleph_{\omega_1+\omega}^{\aleph_1} \le (\prod_{n=0}^\infty \aleph_{\omega_1+n})^{\aleph_1} = \prod_n \aleph_{\omega_1+n}^{\aleph_1} = \prod_n (\aleph_{\omega_1}^{\aleph_1} \cdot \aleph_{\omega_1+n}) = \aleph_{\omega_1}^{\aleph_1} \cdot \prod_n \aleph_{\omega_1+n} = \prod_{\alpha<\omega_1+\omega} \aleph_\alpha$.]

5.17. If κ is a limit cardinal and $\lambda < \operatorname{cf}\kappa$, then $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$.

5.18. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

5.19. If $\alpha < \omega_1$, then $\aleph_\alpha^{\aleph_1} = \aleph_\alpha^{\aleph_0} \cdot 2^{\aleph_1}$.

5.20. If $\alpha < \omega_2$, then $\aleph_\alpha^{\aleph_2} = \aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2}$.

5.21. If κ is regular and limit, then $\kappa^{<\kappa} = 2^{<\kappa}$. If κ is regular and strong limit then $\kappa^{<\kappa} = \kappa$.

5.22. If κ is singular and is not strong limit, then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.

5.23. If κ is singular and strong limit, then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\operatorname{cf}\kappa}$.

5.24. If $2^{\aleph_0} > \aleph_\omega$, then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$.

5.25. If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.

5.26. If $2^{\aleph_0} \geq \aleph_{\omega_1}$, then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

5.27. If $2^{\aleph_1} = \aleph_2$, then $\aleph_\omega^{\aleph_0} \neq \aleph_{\omega_1}$.

5.28. If κ is a singular cardinal and if $\kappa < \gimel(\lambda)$ for some $\lambda < \kappa$ such that $\operatorname{cf}\kappa \leq \operatorname{cf}\lambda$ then $\gimel(\kappa) \leq \gimel(\lambda)$.

5.29. If κ is a singular cardinal such that $2^{\operatorname{cf}\kappa} < \kappa \leq \lambda^{\operatorname{cf}\kappa}$ for some $\lambda < \kappa$, then $\gimel(\kappa) = \gimel(\lambda)$ where λ is the least λ such that $\kappa \leq \lambda^{\operatorname{cf}\kappa}$.

Historical Notes

The Axiom of Choice was formulated by Zermelo, who used it to prove the Well-Ordering Theorem in [1904]. Zorn's Lemma is as in Zorn [1935]; for a related principle, see Kuratowski [1922]. (Hausdorff in [1914], pp. 140–141, proved that every partially ordered set has a maximal linearly ordered subset.) The Principle of Dependent Choices was formulated by Bernays in [1942].

König's Theorem 5.10 appeared in J. König [1905]. Corollary 5.17 was found independently by Bukovský [1965] and Hechler. The discovery that cardinal exponentiation is determined by the gimel function was made by Bukovský; cf. [1965]. The inductive computation of κ^λ in Theorem 5.20 is as in Jech [1973a].

The Hausdorff formula (5.22): Hausdorff [1904].

Inaccessible cardinals were introduced in the paper by Sierpiński and Tarski [1930]; see Tarski [1938] for more details.

Exercise 5.1: Felix Bernstein.

Exercise 5.8: Milner and Rado [1965].

Exercise 5.15: L. Patai.

Exercise 5.17: Tarski [1925b].

Exercises 5.28–5.29: Jech [1973a].