10. Measurable Cardinals

The theory of large cardinals owes its origin to the basic problem of measure theory, the Measure Problem of H. Lebesgue.

The Measure Problem

Let S be an infinite set. A (*nontrivial σ-additive probabilistic*) *measure* on S is a real-valued function μ on $P(S)$ such that:

(10.1)
(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$;
(ii) if $X \subset Y$, then $\mu(X) \le \mu(Y)$;
(iii) $\mu(\{a\}) = 0$ for all $a \in S$ (nontriviality);
(iv) if X_n, $n = 0, 1, 2, \ldots$, are pairwise disjoint, then
$$\mu\Big(\bigcup_{n=0}^{\infty} X_n\Big) = \sum_{n=0}^{\infty} \mu(X_n) \qquad (\sigma\text{-additivity}).$$

It follows from (ii) that $\mu(X)$, the *measure of X*, is nonnegative for every $X \subset S$; in a special case of (iv) we get $\mu(X \cup Y) = \mu(X) + \mu(Y)$ whenever $X \cap Y = \emptyset$ (finite additivity).

More generally, let $\mathcal{A}$ be a σ-complete algebra of sets. A measure on $\mathcal{A}$ is a real-valued function μ on $\mathcal{A}$ satisfying (i)–(iv). Thus a measure on S is a measure on $P(S)$.

An example of a measure on a σ-complete algebra of sets is the Lebesgue measure on the algebra of all Lebesgue measurable subsets of the unit interval $[0, 1]$. The Lebesgue measure has, in addition to (i)–(iv), the following property:

(10.2) If X is congruent by translation to a measurable set Y, then X is measurable and $\mu(X) = \mu(Y)$.

It is well known that there exist sets of reals that are not Lebesgue measurable, and in fact that there is no measure on $[0, 1]$ with the property (10.2) (*translation invariant measure*); see Exercise 10.1.

The natural question to ask is whether the Lebesgue measure can be extended to some measure (not translation invariant) such that all subsets of $[0, 1]$ are measurable, or whether there exists any measure on $[0, 1]$. Or, whether there exists a measure on some set S.

The investigation of this problem has lead to important discoveries in set theory, opening up a new field, the theory of large cardinal numbers, which has far-reaching consequences both in pure set theory and in descriptive set theory.

A measure μ on S is *two-valued* if $\mu(X)$ is either 0 or 1 for all $X \subset S$. If μ is a two-valued measure on S, let

$$U = \{X \subset S : \mu(X) = 1\}. \tag{10.3}$$

It is easy to verify that U is an ultrafilter on S. (For instance, if $X \in U$ and $Y \in U$, then $X \cap Y \in U$. If $\mu(X) = \mu(Y) = 1$, then $X = (X - Y) \cup (X \cap Y)$ and $Y = (Y - X) \cup (X \cap Y)$. If $\mu(X \cap Y)$ were not 1, then $\mu(X - Y) = \mu(Y - X) = 1$, and we would have $\mu(X \cup Y) = 2$.)

Next we note that the ultrafilter U is σ-complete. This is so because μ is σ-additive, and an ultrafilter U on S is σ-complete if and only if there is no partition of S into countably many disjoint parts $S = \bigcup_{n=0}^{\infty} X_n$ such that $X_n \notin U$, for all n.

Thus if μ is a two-valued measure on S, U is a σ-complete ultrafilter on S. Conversely, if U is a σ-complete ultrafilter on S, then the following function is a two-valued measure on S:

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U, \\ 0 & \text{if } X \notin U. \end{cases} \tag{10.4}$$

Let μ be a measure on S. A set $A \subset S$ is an *atom* of μ if $\mu(A) > 0$ and if for every $X \subset A$, we have either $\mu(X) = 0$ or $\mu(X) = \mu(A)$.

If μ has an atom A, then

$$U = \{X \subset S : \mu(X \cap A) = \mu(A)\} \tag{10.5}$$

is again a σ-complete ultrafilter on S.

A measure μ on S is *atomless* if it has no atoms. Then every set $X \subset S$ of positive measure can be split into two disjoint sets of positive measure: $X = Y \cup Z$, and $\mu(Y) > 0$, $\mu(Z) > 0$.

We shall eventually prove various strong consequences of the existence of a nontrivial σ-additive measure and establish the relationship between the Measure Problem and large cardinals. Our starting point is the following theorem which shows that if a measure exists, then there exists at least a weakly inaccessible cardinal.

Theorem 10.1 (Ulam). *If there is a σ-additive nontrivial measure on S, then either there exists a two-valued measure on S and $|S|$ is greater than or equal to the least inaccessible cardinal, or there exists an atomless measure on $2^{\aleph_0}$ and $2^{\aleph_0}$ is greater than or equal to the least weakly inaccessible cardinal.*

Theorem 10.1 will be proved in a sequence of lemmas, which will also provide additional information on the Measure Problem and introduce basic notions and methods of the theory of large cardinals. First we make the following observation. Let κ be the least cardinal that carries a nontrivial σ-additive two-valued measure. Clearly, κ is uncountable and is also the least cardinal that has a nonprincipal countably complete ultrafilter. And we observe that such an ultrafilter is in fact κ-complete:

Lemma 10.2. *Let κ be the least cardinal with the property that there is a nonprincipal σ-complete ultrafilter on κ, and let U be such an ultrafilter. Then U is κ-complete.*

Proof. Let U be a σ-complete ultrafilter on κ, and let us assume that U is not κ-complete. Then there exists a partition $\{X_\alpha : \alpha < \gamma\}$ of κ such that $\gamma < \kappa$, and $X_\alpha \notin U$ for all $\alpha < \gamma$. We shall now use this partition to construct a nonprincipal σ-complete ultrafilter on γ, thus contradicting the choice of κ as the least cardinal that carries such an ultrafilter.

Let f be the mapping of κ onto γ defined as follows:

$$f(x) = \alpha \quad \text{if and only if} \quad x \in X_\alpha \qquad (x \in \kappa).$$

The mapping f induces a σ-complete ultrafilter on γ: we define $D \subset P(\gamma)$ by

$$Z \in D \quad \text{if and only if} \quad f_{-1}(Z) \in U. \tag{10.6}$$

The ultrafilter D is nonprincipal: Assume that $\{\alpha\} \in D$ for some $\alpha < \gamma$. Then $X_\alpha \in U$, contrary to our assumption on X_α. Thus γ carries a σ-complete nonprincipal ultrafilter. □

Measurable and Real-Valued Measurable Cardinals

We are now ready to define the central notion of this chapter.

Definition 10.3. An uncountable cardinal κ is *measurable* if there exists a κ-complete nonprincipal ultrafilter U on κ.

By Lemma 10.2, the least cardinal that carries a nontrivial two-valued σ-additive measure is measurable. Note that if U is a κ-complete nonprincipal ultrafilter on κ, then every set $X \in U$ has cardinality κ because every set of smaller size is the union of fewer than κ singletons. For similar reasons, κ is a regular cardinal because if κ is singular, then it is the union of fewer than κ small sets. The next lemma gives a first link of the Measure Problem with large cardinals.

Lemma 10.4. *Every measurable cardinal is inaccessible.*

Proof. We have just given an argument why a measurable cardinal is regular. Let us show that measurable cardinals are strong limit cardinals. Let κ be measurable, and let us assume that there exists $\lambda < \kappa$ such that $2^\lambda \geq \kappa$; we shall reach a contradiction.

Let S be a set of functions $f : \lambda \to \{0, 1\}$ such that $|S| = \kappa$, and let U be a κ-complete nonprincipal ultrafilter on S. For each $\alpha < \lambda$, let X_α be that one of the two sets $\{f \in S : f(\alpha) = 0\}$, $\{f \in S : f(\alpha) = 1\}$ which is in U, and let ε_α be 0 or 1 accordingly. Since U is κ-complete, the set $X = \bigcap_{\alpha<\lambda} X_\alpha$ is in U. However, X has at most one element, namely the function f that has the values $f(\alpha) = \varepsilon_\alpha$. A contradiction. $\square$

Let us now turn our attention to measures that are not necessarily two-valued. Let μ be a nontrivial σ-additive measure on a set S. In analogy with (10.3) we consider the ideal of all *null sets*:

$$I_\mu = \{X \subset S : \mu(X) = 0\}. \tag{10.7}$$

I_μ is a nonprincipal σ-complete ideal on S. Moreover, it has these properties:

(10.8) (i) $\{x\} \in I$ for every $x \in S$;
(ii) every family of pairwise disjoint sets $X \subset S$ that are not in I is at most countable.

To see that (ii) holds, note that if W is a disjoint family of set of positive measure, then for each integer $n > 0$, there are only finitely many sets $X \in W$ of measure $\geq 1/n$.

A σ-complete nonprincipal ideal I on S is called *σ-saturated* if it satisfies (10.8).

The following lemma is an analog of Lemma 10.2:

Lemma 10.5.

(i) *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure on κ. Then the ideal I_μ of null sets is κ-complete.*

(ii) *Let κ be the least cardinal with the property that there is a σ-complete σ-saturated ideal on κ, and let I be such an ideal. Then I is κ-complete.*

Proof. (i) Let us assume that I_μ is not κ-complete. There exists a collection of null sets $\{X_\alpha : \alpha < \gamma\}$ such that $\gamma < \kappa$ and that their union X has positive measure. We may assume without loss of generality that the sets X_α, $\alpha < \gamma$, are pairwise disjoint; let $m = \mu(X)$.

Let f be the following mapping of X onto γ:

$$f(x) = \alpha \quad \text{if and only if} \quad x \in X_\alpha \qquad (x \in X).$$

The mapping f induces a measure ν on γ:

$$\nu(Z) = \frac{1}{m} \cdot \mu(f_{-1}(Z)). \tag{10.9}$$

The measure ν is σ-additive and is nontrivial since $\nu(\{\alpha\}) = \mu(X_\alpha) = 0$ for each $\alpha \in \gamma$. This contradicts the choice of κ as the least cardinal that carries a measure.

(ii) The proof is similar. We define an ideal J on γ by: $Z \in J$ if and only if $f_{-1}(Z) \in I$. The induced ideal J is σ-complete and σ-saturated. $\square$

Let $\{r_i : i \in I\}$ be a collection of nonnegative real numbers. We define

$$\sum_{i\in I} r_i = \sup\Big\{\sum_{i\in E} r_i : E \text{ is a finite subset of } I\Big\}. \tag{10.10}$$

Note that if the sum (10.10) is not ∞, then at most countably many r_i are not equal to 0.

Let κ be an uncountable cardinal. A measure μ on S is called *κ-additive* if for every $\gamma < \kappa$ and for every disjoint collection X_α, $\alpha < \gamma$, of subsets of S,

$$\mu\Big(\bigcup_{\alpha<\gamma} X_\alpha\Big) = \sum_{\alpha<\gamma} \mu(X_\alpha). \tag{10.11}$$

If μ is a κ-additive measure, then the ideal I_μ of null sets is κ-complete. The converse is also true and we get a better analog of Lemma 10.2 for real-valued measures:

Lemma 10.6. *Let μ be a measure on S, and let I_μ be the ideal of null sets. If I_μ is κ-complete, then μ is κ-additive.*

Proof. Let $\gamma < \kappa$, and let X_α, $\alpha < \gamma$, be disjoint subsets of S. Since the X_α are disjoint, at most countably many of them have positive measure. Thus let us write

$$\{X_\alpha : \alpha < \gamma\} = \{Y_n : n = 0, 1, 2, \ldots\} \cup \{Z_\alpha : \alpha < \gamma\},$$

where each Z_α has measure 0. Then we have

$$\mu\Big(\bigcup_{\alpha<\gamma} X_\alpha\Big) = \mu\Big(\bigcup_{n=0}^{\infty} Y_n\Big) + \mu\Big(\bigcup_{\alpha<\gamma} Z_\alpha\Big).$$

Now first μ is σ-additive, and we have

$$\mu\Big(\bigcup_{n=0}^{\infty} Y_n\Big) = \sum_{n=0}^{\infty} \mu(Y_n),$$

and secondly I_μ is κ-complete and

$$\mu\Big(\bigcup_{\alpha<\gamma} Z_\alpha\Big) = 0 = \sum_{\alpha<\gamma} \mu(Z_\alpha).$$

Thus $\mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha)$. $\square$

Corollary 10.7. *Let κ be the least cardinal that carries a nontrivial σ-additive measure and let μ be such a measure. Then μ is κ-additive.* □

Definition 10.8. An uncountable cardinal κ is *real-valued measurable* if there exists a nontrivial κ-additive measure μ on κ.

By Corollary 10.7, the least cardinal that carries a nontrivial σ-additive measure is real-valued measurable. We shall show that if a real-valued measurable cardinal κ is not measurable, then $\kappa \leq 2^{\aleph_0}$. Note that if μ is a nontrivial κ additive measure on κ, then every set of size $< \kappa$ has measure 0, and moreover κ cannot be the union of fewer than κ sets of size $< \kappa$. Thus a real-valued measurable cardinal is regular. We shall show that it is weakly inaccessible.

We shall first prove the first claim made in the preceding paragraph.

Lemma 10.9.

(i) *If there exists an atomless nontrivial σ-additive measure, then there exists a nontrivial σ-additive measure on some $\kappa \leq 2^{\aleph_0}$.*

(ii) *If I is a σ-complete σ-saturated ideal on S, then either there exists $Z \subset S$, such that $I{\restriction}Z = \{X \subset Z : X \in I\}$ is a prime ideal, or there exists a σ-complete σ-saturated ideal on some $\kappa \leq 2^{\aleph_0}$.*

Proof. (i) Let μ be such a measure on S. We construct a tree T of subsets of S, partially ordered by reverse inclusion. The 0th level of T is $\{S\}$. Each level of T consists of pairwise disjoint subsets of S of positive measure. Each $X \in T$ has two immediate successors: We choose two sets Y, Z of positive measure such that $Y \cup Z = X$ and $Y \cap Z = \emptyset$. If α is a limit ordinal, then the αth level consists of all intersections $X = \bigcap_{\xi<\alpha} X_\xi$ such that each X_ξ is on the ξth level of T and such that X has positive measure.

We observe that every branch of T has countable length: If $\{X_\xi : \xi < \alpha\}$ is a branch in T, then the set $\{Y_\xi : \xi < \alpha\}$, where $Y_\xi = X_\xi - X_{\xi+1}$, is a disjoint collection of sets of positive measure. Consequently, T has height at most ω_1. Similarly, each level of T is at most countable, and it follows that T has at most $2^{\aleph_0}$ branches.

Let $\{b_\alpha : \alpha < \kappa\}$, $\kappa \leq 2^{\aleph_0}$, be an enumeration of all branches $b = \{X_\xi : \xi < \gamma\}$ such that $\bigcap_{\xi<\gamma} X_\xi$ is nonempty; for each $\alpha < \kappa$, let $Z_\alpha = \bigcap\{X : X \in b_\alpha\}$. The collection $\{Z_\alpha : \alpha < \kappa\}$ is a partition of S into κ sets of measure 0.

We induce a measure ν on κ as follows: Let f be the mapping of S onto κ defined by

$$f(x) = \alpha \quad \text{if and only if} \quad x \in Z_\alpha \qquad (x \in S),$$

and let

$$\nu(Z) = \mu(f_{-1}(Z))$$

for all $Z \subset \kappa$. It follows that ν is a nontrivial σ-additive measure on κ.

(ii) The proof is similar. We define a tree T as above and then induce an ideal J on κ by letting $Z \in J$ if and only if $f_{-1}(Z) \in I$. □

The proof of Lemma 10.9 shows that if μ is atomless, then there is a partition of S into at most $2^{\aleph_0}$ null sets; in other words, μ is not $(2^{\aleph_0})^+$-additive. Hence if κ carries an atomless κ-additive measure, then $\kappa \le 2^{\aleph_0}$ and we have:

Corollary 10.10. *If κ is a real-valued measurable cardinal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$.*

More generally, if κ carries a κ-complete σ-saturated ideal, then either κ is measurable or $\kappa \le 2^{\aleph_0}$. □

The measure ν obtained in Lemma 10.9(i) is atomless; this follows from the fact that $\kappa \le 2^{\aleph_0}$ and Lemma 10.4. If there exists an atomless σ-additive measure, then there is one on some $\kappa \le 2^{\aleph_0}$. Clearly, such a measure can be extended to a measure on $2^{\aleph_0}$: For $X \subset 2^{\aleph_0}$, we let $\mu(X) = \mu(X \cap \kappa)$. Thus we conclude that there exists an atomless σ-additive measure on the set $\boldsymbol{R}$ of all reals. It turns out that using the same assumption, we can obtain a σ-additive measure on $\boldsymbol{R}$ that extends Lebesgue measure. This can be done by a slight modification of the proof of Lemma 10.9:

Using Exercise 10.3, we construct for each finite 0–1 sequence s, a set $X_s \subset S$ such that $X_\emptyset = S$, and for every $s \in Seq$, $X_{s^\frown 0} \cup X_{s^\frown 1} = X_s$, $X_{s^\frown 0} \cap X_{s^\frown 1} = \emptyset$, and $\mu(X_{s^\frown 0}) = \mu(X_{s^\frown 1}) = \frac{1}{2} \cdot \mu(X_{s^\frown 0})$. Then we define a measure ν_1 on 2^ω by

$$\nu_1(Z) = \mu(\bigcup\{X_f : f \in Z\}),$$

where $X_f = \bigcap_{n=0}^{\infty} X_{f\restriction n}$ for each $f \in 2^\omega$. Using the mapping $F : 2^\omega \to [0,1]$ defined by

$$F(f) = \sum_{n=0}^{\infty} f(n)/2^{n+1}$$

we obtain a nontrivial σ-additive measure ν on $[0,1]$. This measure agrees with the Lebesgue measure on all intervals $[k/2^n, (k+1)/2^n]$, and hence on all Borel sets. Every set of Lebesgue measure 0 is included in a Borel (in fact, G_δ) set of Lebesgue measure 0 and hence has ν-measure 0. Every Lebesgue measurable set X can be written as $X = (B - N_1) \cup N_2$, where N_1 and N_2 have Lebesgue measure 0, and hence the Lebesgue measure of X is equal to $\nu(X)$. Thus ν agrees with the Lebesgue measure on all Lebesgue measurable subsets of $[0,1]$.

We shall now show that a real-valued measurable cardinal is weakly inaccessible. The proof is by a combinatorial argument, using matrices of sets.

Definition 10.11. An *Ulam matrix* (more precisely, an Ulam $(\aleph_1, \aleph_0)$-matrix) is a collection $\{A_{\alpha,n} : \alpha < \omega_1,\ n < \omega\}$ of subsets of ω_1 such that:

(10.12) (i) if $\alpha \ne \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for every $n < \omega$;
(ii) for each α, the set $\omega_1 - \bigcup_{n=0}^{\infty} A_{\alpha,n}$ is at most countable.

An Ulam matrix has $\aleph_1$ rows and $\aleph_0$ columns. Each column consists of pairwise disjoint sets, and the union of each row contains all but countably many elements of ω_1.

Lemma 10.12. *An Ulam matrix exists.*

Proof. For each $\xi < \omega_1$, let f_ξ be a function on ω such that $\xi \subset \operatorname{ran}(f_\xi)$. Let us define $A_{\alpha,n}$ for $\alpha < \omega_1$ and $n < \omega$ by

$$\xi \in A_{\alpha,n} \quad \text{if and only if} \quad f_\xi(n) = \alpha. \tag{10.13}$$

If $n < \omega$, then for each $\xi \in \omega_1$ there is only one α such that $\xi \in A_{\alpha,n}$, namely $\alpha = f_\xi(n)$; and we have property (i) of (10.12). If $\alpha < \omega_1$, then for each $\xi > \alpha$ there is an n such that $f_\xi(n) = \alpha$ and hence $(\omega_1 - \bigcup_{n=0}^\infty A_{\alpha,n}) \subset \alpha + 1$; that verifies property (ii). □

Using an Ulam matrix, we can show that there is no measure on ω_1:

Lemma 10.13. *There is no nontrivial σ-additive measure on ω_1. More generally, there is no σ-complete σ-saturated ideal on ω_1.*

Proof. Let $\{A_{\alpha,n} : \alpha < \omega_1,\ n < \omega\}$ be an Ulam matrix. Assuming that we have a measure on ω_1, there is for each α some $n = n_\alpha$ such that $A_{\alpha,n}$ has positive measure (because of (10.12)(ii)). Hence there exist an uncountable set $W \subset \omega_1$ and some $n < \omega$ such that $n_\alpha = n$ for all $\alpha \in W$. Then $\{A_{\alpha,n} : \alpha \in W\}$ is an uncountable, pairwise disjoint (by (10.12)(i)) family of sets of positive measure; a contradiction. □

A straightforward generalization of Lemmas 10.12 and 10.13 gives the result mentioned above:

Lemma 10.14. *If $\kappa = \lambda^+$, then there is no κ-complete σ-saturated ideal on κ.*

Proof. For each $\xi < \lambda^+$, we let f_ξ be a function on λ such that $\xi \subset \operatorname{ran}(f_\xi)$, and let

$$\xi \in A_{\alpha,\eta} \quad \text{if and only if} \quad f_\xi(\eta) = \alpha.$$

Then $\{A_{\alpha,\eta} : \alpha < \lambda^+,\ \eta < \lambda\}$ is an Ulam (λ^+, λ)-matrix, that is a collection of subsets of λ^+ such that:

$$\begin{aligned} &\text{(i)}\ \ A_{\alpha,\eta} \cap A_{\beta,\eta} = \emptyset \text{ whenever } \alpha \neq \beta < \lambda^+, \text{ and } \eta < \lambda;\\ &\text{(ii)}\ \ |\lambda^+ - \textstyle\bigcup_{\eta<\lambda} A_{\alpha,\eta}| \le \lambda \text{ for each } \alpha < \lambda^+. \end{aligned} \tag{10.14}$$

The proof of Lemma 10.13 generalizes to show that there is no κ-complete σ-saturated ideal on κ. □

Corollary 10.15. *Every real-valued measurable cardinal is weakly inaccessible.* □

Lemma 10.14 completes the proof of Theorem 10.1: If there is a σ-additive nontrivial measure on S, then either the measure has an atom A and we can construct a two-valued measure on S via a σ-complete nonprincipal ultrafilter on A, and then $|S| \geq$ the least measurable cardinal, which is inaccessible; or the measure on S is atomless and we construct, as in Lemma 10.9, an atomless measure on $2^{\aleph_0}$, and then $2^{\aleph_0} \geq$ the least real-valued measurable cardinal, which is weakly inaccessible. □

Prior to Ulam's work, Banach and Kuratowski proved that if the Continuum Hypothesis holds then there exists no σ-additive measure on $\boldsymbol{R}$. We present their proof below; in fact, Lemma 10.16 gives a slightly more general result.

If f and g are functions from ω to ω, let $f < g$ mean that $f(n) < g(n)$ for all but finitely many $n \in \omega$. A κ-sequence of functions $\langle f_\alpha : \alpha < \kappa \rangle$ is called a *κ-scale* if $f_\alpha < f_\beta$ whenever $\alpha < \beta$, and if for every $g : \omega \to \omega$ there exists an α such that $g < f_\alpha$.

Lemma 10.16. *If there exists a κ-scale, then κ is not a real-valued measurable cardinal.*

Proof. Let f_α, $\alpha < \kappa$, be a κ-scale. We define an $(\aleph_0, \aleph_0)$-matrix of subsets of κ as follows: For $n, k < \omega$, let

$$\alpha \in A_{n,k} \quad \text{if and only if} \quad f_\alpha(n) = k \qquad (\alpha \in \kappa). \tag{10.15}$$

Since for each n and each α there is k such that $\alpha \in A_{n,k}$, we have

$$\bigcup_{k=0}^{\infty} A_{n,k} = \kappa$$

for every $n = 0, 1, 2, \ldots$.

Now assume that μ is a nontrivial κ-additive measure on κ. For each n, let k_n be such that

$$\mu(A_{n,0} \cup A_{n,1} \cup \ldots \cup A_{n,k_n}) \geq 1 - (1/2^{n+2}),$$

and let $B_n = A_{n,0} \cup \ldots \cup A_{n,k_n}$. If we let $B = \bigcap_{n=0}^{\infty} B_n$, then we clearly have $\mu(B) \geq 1/2$.

Let $g : \omega \to \omega$ be the function $g(n) = k_n$. If $\alpha \in B$, then by the definition of B and by (10.15), we have

$$f_\alpha(n) \leq g(n)$$

for all $n = 0, 1, 2, \ldots$; hence $g \not< f_\alpha$. However, since B has positive measure, B has size κ, and therefore we have $g \not< f_\alpha$ for cofinally many $\alpha < \kappa$. This contradicts the assumption that the f_α form a scale. □

Corollary 10.17. *If there is a measure on $2^{\aleph_0}$, then $2^{\aleph_0} > \aleph_1$.*

Proof. If $2^{\aleph_0} = \aleph_1$, then there exists an ω_1-scale; a scale $\langle f_\alpha : \alpha < \omega_1 \rangle$ is constructed by transfinite induction to ω_1:

Let $\{g_\alpha : \alpha < \omega_1\}$ enumerate all functions from ω to ω. At stage α, we construct, by diagonalization, a function f_α such that for all $\beta < \alpha$, $f_\alpha > f_\beta$ and $f_\alpha > g_\beta$. Then $\langle f_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-scale. □

Measurable Cardinals

By Lemma 10.4, every measurable cardinal is inaccessible. While we shall investigate measurable cardinals extensively in Part II, we now present a few basic results that establish the relationship of measurable cardinals and the large cardinals introduced in Chapter 9.

We recall that by Lemma 9.26, a cardinal κ is weakly compact if and only if it is inaccessible and has the tree property.

Lemma 10.18. *Every measurable cardinal is weakly compact.*

Proof. Let κ be a measurable cardinal. To show that κ is weakly compact, it suffices to prove the tree property. Let $(T, <)$ be a tree of height κ with levels of size $< \kappa$. We consider a nonprincipal κ-complete ultrafilter U on T. Let B be the set of all $x \in T$ such that the set of all successors of x is in U. It is clear that B is a branch in T and it is easy to verify that each level of T has one element in B; thus B is a branch of size κ. □

Normal Measures

In Chapter 8 we defined the notion of a normal κ-complete filter, namely a filter closed under diagonal intersections (8.7).

Thus we call a normal κ-complete nonprincipal ultrafilter a *normal measure* on κ. Note that by Exercise 8.8, a measure is normal if and only if every regressive function on a set of measure one is constant on a set of measure one.

Lemma 10.19. *If D is a normal measure on κ, then every set in D is stationary.*

Proof. By Lemma 8.11, every closed unbounded set is in D, and the lemma follows. □

Theorem 10.20 below shows that if κ is measurable cardinal then a normal measure exists.

Theorem 10.20. *Every measurable cardinal carries a normal measure. If U is a nonprincipal κ-complete ultrafilter on κ then there exists a function $f : \kappa \to \kappa$ such that $f_*(U) = \{X \subset \kappa : f_{-1}(X) \in U\}$ is a normal measure.*

Proof. Let U be a nonprincipal κ-complete ultrafilter on κ. For f and g in κ^κ, let

$$f \equiv g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U.$$

It is easily seen that $\equiv$ is an equivalence relation on κ^κ. Let $[f]$ denote the equivalence class of $f \in \kappa^\kappa$. Furthermore, if we let

$$f < g \quad \text{if and only if} \quad \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U,$$

then $<$ is a linear ordering of (the equivalence classes of) κ^κ.

There exists no infinite descending sequence $f_0 > f_1 > \ldots > f_n > \ldots$: Otherwise, let $X_n = \{\alpha : f_n(\alpha) > f_{n+1}(\alpha)\}$, and let $X = \bigcap_{n=0}^\infty X_n$. X is nonempty, and if $\alpha \in X$, we would have $f_0(\alpha) > f_1(\alpha) > \ldots > f_n(\alpha) > \ldots$, a contradiction.

Thus $<$ is a well-ordering of $\kappa^\kappa/{\equiv}$.

Now let $f : \kappa \to \kappa$ be the least function (in this well-ordering) with the property that for all $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$. Such functions exist: for instance, the *diagonal function* $d(\alpha) = \alpha$ has this property.

Let $D = f_*(U) = \{X \subset \kappa : f_{-1}(X) \in U\}$. We claim that D is a normal measure.

It is easy to verify that D is a κ-complete ultrafilter. For every $\gamma < \kappa$, we have $f_{-1}(\{\gamma\}) \notin U$, and so $\{\gamma\} \notin D$, and so D is nonprincipal.

In order to show that D is normal, let h be a regressive function on a set $X \in D$. We shall show that h is constant on a set in D. Let g be the function defined by $g(\alpha) = h(f(\alpha))$. As $g(\alpha) < f(\alpha)$ for all $\alpha \in f_{-1}(X)$, we have $g < f$, and it follows by the minimality of f that g is constant on some $Y \in U$. Hence h is constant on $f(Y)$ and $f(Y) \in D$. □

As an application of normal measures we show that every measurable cardinal is a Mahlo cardinal, and improve Lemma 10.18 by showing that every measurable cardinal is a Ramsey cardinal.

Lemma 10.21. *Every measurable cardinal is a Mahlo cardinal.*

Proof. Let κ be a measurable cardinal. We shall show that the set of all inaccessible cardinals $\alpha < \kappa$ is stationary. As κ is strong limit, the set of all strong limit cardinals $\alpha < \kappa$ is closed unbounded, and it suffices to show that the set of all regular cardinals $\alpha < \kappa$ is stationary.

Let D be a normal measure on κ. We claim that $\{\alpha < \kappa : \alpha \text{ is regular}\} \in D$; this will complete the proof, since every set in D is stationary, by Lemma 10.19.

Toward a contradiction, assume that $\{\alpha : \operatorname{cf}\alpha < \alpha\} \in D$. By normality, there is some $\lambda < \kappa$ such that $E_\lambda = \{\alpha : \operatorname{cf}\alpha = \lambda\} \in D$. For each $\alpha \in E_\lambda$, let $\langle x_{\alpha,\xi} : \xi < \lambda\rangle$ be an increasing sequence with limit α. For each $\xi < \lambda$ there exist y_ξ and $A_\xi \in D$ such that $x_{\alpha,\xi} = y_\xi$ for all $\alpha \in A_\xi$. Let $A = \bigcap_{\xi<\lambda} A_\xi$. Then $A \in D$, but A contains only one element, namely $\lim_{\xi\to\lambda} y_\xi$; a contradiction. □

Theorem 10.22. *Let κ be a measurable cardinal, let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into less than κ pieces. Then there exists a set $H \in D$ homogeneous for F. Hence every measurable cardinal is a Ramsey cardinal.*

Proof. Let D be a normal measure on κ, and let F be a partition of $[\kappa]^{<\omega}$ into fewer than κ pieces. It suffices to show that for each $n = 1, 2, \ldots,$ there is $H_n \in D$ such that F is constant on $[H_n]^n$; then $H = \bigcap_{n=1}^{\infty} H_n$ is homogeneous for F.

We prove, by induction on n, that every partition of $[\kappa]^n$ into fewer than κ pieces is constant on $[H]^n$ for some $H \in D$. The assertion is trivial for $n = 1$, so we assume that it is true for n and prove that it holds also for $n+1$. Let $F : [\kappa]^{n+1} \to I$, where $|I| < \kappa$. For each $\alpha < \kappa$, we define F_α on $[\kappa - \{\alpha\}]^n$ by $F_\alpha(x) = F(\{\alpha\} \cup x)$.

By the induction hypothesis, there exists for each $\alpha < \kappa$ a set $X_\alpha \in D$ such that F_α is constant on $[X_\alpha]^n$; let i_a be its constant value. Let X be the diagonal intersection $X = \{\alpha < \kappa : \alpha \in \bigcap_{\gamma<\alpha} X_\gamma\}$. We have $X \in D$ since D is normal; also, if $\gamma < \alpha_1 < \ldots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$ and so $F(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = F_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma$. Now, there exist $i \in I$ and $H \subset X$ in D such that $i_\gamma = i$ for all $\gamma \in H$. It follows that $F(x) = i$ for all $x \in [H]^{n+1}$. □

Strongly Compact and Supercompact Cardinals

Among the various large cardinals that we shall investigate in more detail in Part II there are two that are immediate generalizations of measurable cardinals.

Definition 10.23. An uncountable cardinal κ is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Clearly, every strongly compact cardinal is measurable.

Let A be a set of size at least κ, and let us consider the filter F on $P_\kappa(A)$ generated by the sets $\hat{P} = \{Q \in P_\kappa(A) : P \subset Q\}$. F is a κ-complete filter and if κ is strongly compact, F can be extended to a κ-complete ultrafilter U. A κ-complete ultrafilter U on $P_\kappa(A)$ that extends F is called a *fine measure.* In Part II we prove that if a fine measure on $P_\kappa(A)$ exists for every A, then κ is strongly compact.

A fine measure U on $P_{<\kappa}(A)$ is *normal* if whenever $f : P_\kappa(A) \to A$ is such that $f(P) \in P$ for all P in a set in U, then f is constant on a set in U. Equivalently, U is normal if it is closed under diagonal intersections $\triangle_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}$.

Definition 10.24. An uncountable cardinal κ is *supercompact* if for every A such that $|A| \geq \kappa$ there exists a normal measure on $P_\kappa(A)$.

We return to the subject of strongly compact and supercompact cardinals in Part II.

Exercises

10.1 (Vitali). Let M be maximal (under $\subset$) subset of $[0,1]$ with the property that $x - y$ is not a rational number, for any pair of distinct $x, y \in M$. Show that M is not Lebesgue measurable.

[Consider the sets $M_q = \{x + q : x \in M\}$ where q is rational. They are pairwise disjoint and $[0,1] \subset \bigcup\{M_q : q \in \boldsymbol{Q} \cap [-1,1]\} \subset [-1,2]$.]

10.2. Prove directly that the measure ν defined in the proof of Lemma 10.9(i) is atomless.

[Assume that Z is an atom of ν, and let $Y = f_{-1}(Z)$. If $X \in T$ is such that $\mu(Y \cap X) \neq 0$ and if X_1, X_2 are the two immediate successors of X, then either $\mu(Y \cap X_1) = 0$ or $\mu(Y \cap X_2) = 0$. Prove by induction that on each level of T there is a unique X such that $\mu(Y \cap X) \neq 0$, and that these X's constitute a branch in T of length ω_1; a contradiction.]

10.3. If μ is an atomless measure on S, there exists $Z \subset S$ such that $\mu(Z) = 1/2$. More generally, given $Z_0 \subset S$, there exists $Z \subset Z_0$ such that $\mu(Z) = (1/2) \cdot \mu(Z_0)$.

[Construct a sequence $S = S_0 \supset S_1 \supset \ldots \supset S_\alpha \supset \ldots$, $\alpha < \omega_1$, such that $\mu(S_\alpha) \geq 1/2$, and if $\mu(S_\alpha) > 1/2$, then $1/2 \leq \mu(S_{\alpha+1}) < \mu(S_\alpha)$; if α is a limit ordinal, let $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$. There exists $\alpha < \omega_1$ such that $\mu(S_\alpha) = 1/2$.]

10.4. Let μ be a two-valued measure and U the ultrafilter of all sets of measure one. Then μ is κ-additive if and only if U is κ-complete.

10.5. A measure U on κ is normal if and only if the diagonal function $d(\alpha) = \alpha$ is the least function f with the property that for all $\gamma < \kappa$, $\{\alpha : f(\alpha) > \gamma\} \in U$.

10.6. Let D be a normal measure on κ and let $f : [\kappa]^{<\omega} \to \kappa$ be such that $f(x) = 0$ or $f(x) < \min x$ for all $x \in [\kappa]^{<\omega}$. Then there is $H \in D$ such that for each n, f is constant on $[H]^n$.

[By induction, as in Theorem 10.22. Given f on $[\kappa]^{n+1}$, let $f_\alpha(s) = f(\{\alpha\} \cup s)$ for $\alpha < \min s$; f_α is constant on $[X_\alpha]^n$ with value $\gamma_\alpha < \alpha$. Let X be the diagonal intersection of X_α, $\alpha < \kappa$, and let γ and $H \subset X$ be such that $H \in D$ and $\gamma_\alpha = \gamma$ for all $\alpha \in H$.]

10.7. If κ is measurable then there exists a normal measure on $P_\kappa(\kappa)$.

Historical Notes

The study of measurable cardinals originated around 1930 with the work of Banach, Kuratowski, Tarski, and Ulam. Ulam showed in [1930] that measurable cardinals are large, that the least measurable cardinal is at least as large as the least inaccessible cardinal.

The main result on measurable and real-valued measurable cardinals (Theorem 10.1) is due to Ulam [1930]. The fact that a measurable cardinal is inaccessible (Lemma 10.4) was discovered by Ulam and Tarski (cf. Ulam [1930]). Prior to Ulam, Banach and Kuratowski proved in [1929] that if $2^{\aleph_0} = \aleph_1$, then there is no measure on the continuum; their proof is as in Lemma 10.16. Real-valued measurable cardinals were introduced by Banach in [1930].

Lemma 10.18: Erdős and Tarski [1943]. Hanf [1963/64a] proved that the least inaccessible cardinal is not measurable. That every measurable cardinal is a Ramsey cardinal was proved by Erdős and Hajnal [1962]; the stronger version (Theorem 10.22) is due to Rowbottom [1971].

Strongly compact cardinals were introduced by Keisler and Tarski in [1963/64]; supercompact cardinals were defined by Reinhardt and Solovay, cf. Solovay *et al.* [1978].

Exercise 10.1: Vitali [1905].