11. Borel and Analytic Sets

Descriptive set theory deals with sets of reals that are described in some simple way: sets that have a simple topological structure (e.g., continuous images of closed sets) or are definable in a simple way. The main theme is that questions that are difficult to answer if asked for arbitrary sets of reals, become much easier when asked for sets that have a simple description. An example of that is the Cantor-Bendixson Theorem (Theorem 4.6): Every closed set of reals is either at most countable or has size $2^{\aleph_0}$.

Since properties of definable sets can usually be established effectively, without use of the Axiom of Choice, we shall work in set theory ZF without the Axiom of Choice. When some statement depends on the Axiom of Choice, we shall explicitly say so. However, we shall assume a weak form of the Axiom of Choice. The reason is that in descriptive set theory one frequently considers unions and intersections of countably many sets of reals, and we shall often use facts like "the union of countably many countable sets is countable." Thus we shall work, throughout this chapter, in set theory ZF + the Countable Axiom of Choice.

In this chapter we develop the basic theory of Borel and analytic sets in Polish spaces. A Polish space is a topological space that is homeomorphic to a complete separable metric space (Definition 4.12).

A canonical example of a Polish space is the Baire space $\mathcal{N}$. The following lemma shows that every Polish space is a continuous image of $\mathcal{N}$:

Lemma 11.1. *Let X be a Polish space. Then there exists a continuous mapping from $\mathcal{N}$ onto X.*

Proof. Let X be a complete separable metric space; we construct a mapping f of $\mathcal{N}$ onto X as follows: It is easy to construct, by induction on the length of $s \in Seq$, a collection $\{C_s : s \in Seq\}$ of closed balls such that $C_\emptyset = X$ and

(11.1)
(i) $\text{diameter}(C_s) \le 1/n$ where $n = \text{length}(s)$,
(ii) $C_s \subset \bigcup_{k=0}^{\infty} C_{s^\frown k}$ (all $s \in Seq$),
(iii) if $s \subset t$ then $\text{center}(C_t) \in C_s$.

For each $a \in \mathcal{N}$, let $f(a)$ be the unique point in $\bigcap\{C_s : s \subset a\}$; it is easily checked that f is continuous and that $X = f(\mathcal{N})$. □

Borel Sets

Let X be a Polish space. A set $A \subset X$ is a *Borel set* if it belongs to the smallest σ-algebra of subsets of X containing all closed sets. We shall now give a more explicit description of Borel sets. For each $\alpha < \omega_1$, let us define the collections $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of subsets of X:

$$
\begin{aligned}
\mathbf{\Sigma}^0_1 &= \text{the collection of all open sets;} \\
\mathbf{\Pi}^0_1 &= \text{the collection of all closed sets;} \\
\mathbf{\Sigma}^0_\alpha &= \text{the collection of all sets } A = \textstyle\bigcup_{n=0}^\infty A_n, \text{ where each } A_n \\
&\qquad \text{belongs to } \mathbf{\Pi}^0_\beta \text{ for some } \beta < \alpha; \\
\mathbf{\Pi}^0_\alpha &= \text{the collection of all complements of sets in } \mathbf{\Sigma}^0_\alpha \\
&= \text{the collection of all sets } A = \textstyle\bigcap_{n=0}^\infty A_n, \text{ where each } A_n \\
&\qquad \text{belongs to } \mathbf{\Sigma}^0_\beta \text{ for some } \beta < \alpha.
\end{aligned} \tag{11.2}
$$

It is clear (by induction on α) that the elements of each $\mathbf{\Sigma}^0_\alpha$ and each $\mathbf{\Pi}^0_\alpha$ are Borel sets. Since every open set is the union of countably many closed sets, we have $\mathbf{\Sigma}^0_1 \subset \mathbf{\Sigma}^0_2$, and consequently, if $\alpha < \beta$, then

$$\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Sigma}^0_\beta, \qquad \mathbf{\Sigma}^0_\alpha \subset \mathbf{\Pi}^0_\beta, \qquad \mathbf{\Pi}^0_\alpha \subset \mathbf{\Pi}^0_\beta, \qquad \mathbf{\Pi}^0_\alpha \subset \mathbf{\Sigma}^0_\beta.$$

Hence

$$\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha \tag{11.3}$$

and it is easy to verify that the collection (11.3) is a σ-algebra (here we use the Countable Axiom of Choice). Hence every Borel set is in some $\mathbf{\Sigma}^0_\alpha$, $\alpha < \omega_1$.

Note that each $\mathbf{\Sigma}^0_\alpha$ (and each $\mathbf{\Pi}^0_\alpha$) is closed under finite unions, finite intersections, and inverse images by continuous functions (i.e., if $A \in \mathbf{\Sigma}^0_\alpha$ in Y, then $f_{-1}(A) \in \mathbf{\Sigma}^0_\alpha$ in X whenever $f : X \to Y$ is a continuous function).

If the Polish space X is countable, then of course every $A \in X$ is a Borel set, in fact an F_σ set. Uncountable Polish spaces are more interesting: Not all sets are Borel, and the collections $\mathbf{\Sigma}^0_\alpha$ form a hierarchy. We show below that for each α, $\mathbf{\Sigma}^0_\alpha \not\subset \mathbf{\Pi}^0_\alpha$, and hence $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^0_{\alpha+1}$ for all $\alpha < \omega_1$.

While we prove the next lemma for the special case when X is the Baire space, the proof can be modified to prove the same result for any uncountable Polish space.

Lemma 11.2. *For each $\alpha \geq 1$ there exists a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^0_\alpha$ (in $\mathcal{N}^2$), and that for every $\mathbf{\Sigma}^0_\alpha$ set A in $\mathcal{N}$ there exists some $a \in \mathcal{N}$ such that*

$$A = \{x : (x, a) \in U\}. \tag{11.4}$$

U is a *universal* $\mathbf{\Sigma}^0_\alpha$ set.

Proof. By induction on α. To construct a universal open set in $\mathcal{N}^2$, let $G_1, \ldots, G_k, \ldots$ be an enumeration of all basic open sets in $\mathcal{N}$, and let $G_0 = \emptyset$. Let

$$(x, y) \in U \text{ if and only if } x \in G_{y(n)} \text{ for some } n. \tag{11.5}$$

Since $U = \bigcup_{n=0}^\infty H_n$ where each $H_n = \{(x, y) : x \in G_{y(n)}\}$ is an open set in $\mathcal{N}^2$, we see that U is open. Now if G is an open set in $\mathcal{N}$, we let $a \in \mathcal{N}$ be such that $G = \bigcup_{n=0}^\infty G_{a(n)}$; then $G = \{x : (x, a) \in U\}$.

Next let U be a universal $\mathbf{\Sigma}^0_\alpha$ set, and let us construct a universal $\mathbf{\Sigma}^0_{\alpha+1}$ set V. Let us consider some continuous mapping of $\mathcal{N}$ onto the product space $\mathcal{N}^\omega$; for each $a \in \mathcal{N}$ and each n, let $a_{(n)}$ be the nth coordinate of the image of a. [For instance, let us define $a_{(n)}$ as follows: $a_{(n)}(k) = a(\Gamma(n, k))$, where Γ is the canonical one-to-one pairing function $\Gamma : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$.] Now let

$$(x, y) \in V \text{ if and only if for some } n,\ (x, y_{(n)}) \notin U. \tag{11.6}$$

Since $V = \bigcup_{n=0}^\infty H_n$ where each $H_n = \{(x, y) : (x, y_{(n)}) \notin U\}$ is a $\mathbf{\Pi}^0_\alpha$ set, we see that V is $\mathbf{\Sigma}^0_{\alpha+1}$. If A is a $\mathbf{\Sigma}^0_{\alpha+1}$ set in $\mathcal{N}$, then $A = \bigcup_{n=0}^\infty A_n$ where each A_n is $\mathbf{\Pi}^0_\alpha$. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x, a) \in V\}$.

Finally, let α be a limit ordinal, and let U_β, $1 \le \beta \le \alpha$, be universal $\mathbf{\Sigma}^0_\beta$ sets. Let $1 \le \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ be an increasing sequence of ordinals such that $\lim_{n\to\infty} \alpha_n = \alpha$. Let

$$(x, y) \in U \text{ if and only if for some } n,\ (x, y_{(n)}) \notin U_{\alpha_n} \tag{11.7}$$

(where $a_{(n)}$ has the same meaning as above). The set U is $\mathbf{\Sigma}^0_\alpha$. If A is a $\mathbf{\Sigma}^0_\alpha$ set in $\mathcal{N}$ then $A = \bigcup_{n=0}^\infty A_n$ where each A_n is $\mathbf{\Pi}^0_{\alpha_n}$. For each n, let a_n be such that $\mathcal{N} - A_n = \{x : (x, a_n) \in U_{\alpha_n}\}$, and let a be such that $a_{(n)} = a_n$ for all n. Then $A = \{x : (x, a) \in U\}$. □

Corollary 11.3. *For every* $\alpha \ge 1$, *there is a set* $A \subset \mathcal{N}$ *that is* $\mathbf{\Sigma}^0_\alpha$ *but not* $\mathbf{\Pi}^0_\alpha$.

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Let us consider the set

$$A = \{x : (x, x) \in U\}. \tag{11.8}$$

Clearly, A is a $\mathbf{\Sigma}^0_\alpha$ set. If A were $\mathbf{\Pi}^0_\alpha$, then its complement would be $\mathbf{\Sigma}^0_\alpha$ and there would be some a such that

$$A = \{x : (x, a) \notin U\}.$$

But this contradicts (11.8): Simply let $x = a$. □

Analytic Sets

While the collection of Borel sets of reals is closed under Boolean operations, and countable unions and intersections, it is not closed under continuous images: As we shall learn presently, the image of a Borel set by a continuous function need not be a Borel set. We shall now investigate the continuous images of Borel sets.

Definition 11.4. A subset of A of a Polish space X is *analytic* if there exists a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$.

Definition 11.5. The *projection* of a set $S \subset X \times Y$ (into X) is the set $P = \{x \in X : \exists y\, (x, y) \in S\}$.

The following lemma gives equivalent definitions of analytic sets.

Lemma 11.6. *The following are equivalent, for any set A in a Polish space X:*

(i) *A is the continuous image of $\mathcal{N}$.*
(ii) *A is the continuous image of a Borel set B (in some Polish space Y).*
(iii) *A is the projection of a Borel set in $X \times Y$, for some Polish space Y.*
(iv) *A is the projection of a closed set in $X \times \mathcal{N}$.*

Proof. We shall prove that every closed set (in any Polish space) is analytic and that every Borel set is the projection of a closed set in $X \times \mathcal{N}$. Then the lemma follows: Since the projection map $\pi : X \times Y \to X$ defined by $\pi(x, y) = x$ is continuous, it follows that every Borel set is analytic and that the continuous image of a Borel set is analytic. Conversely, if $A \subset X$ is an analytic set, $A = f(\mathcal{N})$, then A is the projection of the set $\{(f(x), x) : x \in \mathcal{N}\}$ which is a closed set in $X \times \mathcal{N}$.

In order to prove that every closed set is analytic, note that every closed set in a Polish space is itself a Polish space, and thus a continuous image of $\mathcal{N}$ by Lemma 11.1.

In order to prove that every Borel set in X is the projection of a closed set in $X \times \mathcal{N}$, it suffices to show that the family P of all subsets of X that are such projections contains all closed sets, all open sets, and is closed under countable unions and intersections.

Clearly, the family P contains all closed sets. Moreover, every open set is a countable union of closed sets; thus it suffices to show that P is closed under $\bigcup_{n=0}^\infty$ and $\bigcap_{n=0}^\infty$.

Recall the continuous mapping $a \mapsto \langle a_{(n)} : n \in \boldsymbol{N} \rangle$ of $\mathcal{N}$ onto $\mathcal{N}^\omega$ from Lemma 11.2, and also recall that the inverse image of a closed set under a continuous function is closed. Let A_n, $n < \omega$, be projections of closed sets in $X \times \mathcal{N}$; we shall show that $\bigcup_{n=0}^\infty A_n$ and $\bigcap_{n=0}^\infty A_n$ are projections of closed sets.

For each n, let $F_n \subset X \times \mathcal{N}$ be a closed set such that

$$A_n = \{x : \exists a\, (x, a) \in F_a\}.$$

Thus

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty} A_n &\leftrightarrow \exists n\, \exists a\, (x, a) \in F_n \\ &\leftrightarrow \exists a\, \exists b\, (x, a) \in F_{b(0)} \\ &\leftrightarrow \exists c\, (x, c_{(0)}) \in F_{c_{(1)}(0)}, \end{aligned}$$

and

$$\begin{aligned} x \in \bigcap_{n=0}^{\infty} A_n &\leftrightarrow \forall n\, \exists a\, (x, a) \in F_n \\ &\leftrightarrow \exists c\, \forall n\, (x, c_{(n)}) \in F_n \\ &\leftrightarrow \exists c\, (x, c) \in \bigcap_{n=0}^{\infty} \{(x, c) : (x, c_{(n)}) \in F_n\}. \end{aligned}$$

Hence $\bigcup_{n=0}^{\infty} A_n$ is the projection of the closed set

$$\{(x, c) : (x, c_{(0)}) \in F_{c_{(1)}(0)}\}$$

and $\bigcap_{n=0}^{\infty} A_n$ is the projection of an intersection of closed sets. □

The Suslin Operation $\mathcal{A}$

For each $a \in \omega^\omega$, $a \restriction n$ is the finite sequence $\langle a_k : k < n \rangle$. For each $s \in Seq$, $O(s)$ is the basic open set $\{a \in \mathcal{N} : a \restriction n = s\}$ of the Baire space. $O(s)$ is both open and closed. For every set A in a Polish space, $\overline{A}$ denotes the closure of A.

Let $\{A_s : s \in Seq\}$ be a collection of sets indexed by elements of Seq. We define

$$\mathcal{A}\{A_s : s \in Seq\} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} A_{a \restriction n} \tag{11.9}$$

Note that if $\{B_s : s \in Seq\}$ is arbitrary, then

$$\bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} B_{a \restriction n} = \bigcup_{a \in \omega^\omega} \bigcap_{n=0}^{\infty} (B_{a \restriction 0} \cap B_{a \restriction 1} \cap \ldots \cap B_{a \restriction n})$$

and hence $\mathcal{A}\{B_s : s \in Seq\} = \mathcal{A}\{A_s : s \in Seq\}$ where the sets A_s are finite intersections of the sets B_s and satisfy the following condition:

$$\text{if } s \subset t, \text{ then } A_s \supset A_t. \tag{11.10}$$

Thus we shall restrict our use of $\mathcal{A}$ to families that satisfy condition (11.10). The operation $\mathcal{A}$ is called the *Suslin operation.*

Lemma 11.7. *A set A in a Polish space is analytic if and only if A is the result of the operation $\mathcal{A}$ applied to a family of closed sets.*

Proof. First we show that if F_s, $s \in Seq$, are closed sets in a Polish space X, then $A = \mathcal{A}\{F_s : s \in Seq\}$ is analytic. We have

$$\begin{aligned} x \in A &\leftrightarrow \exists a \in \mathcal{N}\, x \in \bigcap_{n=0}^{\infty} F_{a\restriction n} \\ &\leftrightarrow \exists a\, (x, a) \in \bigcap_{n=0}^{\infty} B_n \end{aligned}$$

where $B_n = \{(x, a) : x \in F_{a\restriction n}\}$. Clearly, each B_n is a Borel set in $X \times \mathcal{N}$ and hence A is analytic.

Conversely, let $A \subset X$ be analytic. There is a continuous function $f : \mathcal{N} \to X$ such that $A = f(\mathcal{N})$. Notice that for every $a \in \mathcal{N}$,

$$\bigcap_{n=0}^{\infty} f(O(a\restriction n)) = \bigcap_{n=0}^{\infty} \overline{f(O(a\restriction n))} = \{f(a)\}. \tag{11.11}$$

Thus

$$A = f(\mathcal{N}) = \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} \overline{f(O(a\restriction n))},$$

and hence A is the result of the operation $\mathcal{A}$ applied to the closed sets $\overline{f(O(s))}$ (which satisfy the condition (11.10)). □

It follows from the preceding lemmas that the collection of all analytic sets in a Polish space is closed under countable unions and intersections, continuous images, and inverse images, and the Suslin operation (the last statement is proved like the first part of Lemma 11.7). It is however not the case that the complement of an analytic set is analytic (if X is an uncountable Polish space). In the next section we establish exactly that; we show that there exists an analytic set (in $\mathcal{N}$) whose complement is not analytic.

The Hierarchy of Projective Sets

For each $n \geq 1$, we define the collections $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ of subsets of a Polish space X as follows:

$$\begin{aligned} \mathbf{\Sigma}^1_1 &= \text{the collection of all analytic sets}, \\ \mathbf{\Pi}^1_1 &= \text{the complements of analytic sets}, \\ \mathbf{\Sigma}^1_{n+1} &= \text{the collection of the projections of all } \mathbf{\Pi}^1_n \text{ sets in } X \times \mathcal{N}, \\ \mathbf{\Pi}^1_n &= \text{the complements of the } \mathbf{\Sigma}^1_n \text{ sets in } X, \\ \mathbf{\Delta}^1_n &= \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n. \end{aligned} \tag{11.12}$$

The sets belonging to one of the collections $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ are called *projective sets*. It is easily seen that for every n, $\mathbf{\Delta}^1_n \subset \mathbf{\Sigma}^1_n \subset \mathbf{\Delta}^1_{n+1}$ and $\mathbf{\Delta}^1_n \subset \mathbf{\Pi}^1_n \subset \mathbf{\Delta}^1_{n+1}$.

We shall show that for each n there is a $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}$ that is not $\mathbf{\Pi}^1_n$; thus the above inclusions are proper inclusions.

Lemma 11.8. *For each $n \geq 1$, there exists a universal $\mathbf{\Sigma}^1_n$ set in $\mathcal{N}^2$; i.e., a set $U \subset \mathcal{N}^2$ such that U is $\mathbf{\Sigma}^1_n$ and that for every $\mathbf{\Sigma}^1_n$ set A in $\mathcal{N}$ there exists some $v \in \mathcal{N}$ such that*

$$A = \{x : (x, v) \in U\}.$$

Proof. Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$. If $n = 1$, let V be a universal $\mathbf{\Sigma}^0_1$ set; if $n > 1$, let V be, by the induction hypothesis, a universal $\mathbf{\Sigma}^1_{n-1}$ set. Let

$$(x, y) \in U \quad \text{if and only if} \quad \exists a \in \mathcal{N}\, (h(x, a), y) \notin V. \tag{11.13}$$

Since the set $\{(x, y, a) : (h(x, a), y) \notin V\}$ is closed (if $n = 1$) or $\mathbf{\Pi}^1_{n-1}$ (if $n > 1$), U is $\mathbf{\Sigma}^1_n$.

If $A \subset \mathcal{N}$ is $\mathbf{\Sigma}^1_n$, there is a closed (or $\mathbf{\Pi}^1_{n-1}$) set B such that

$$x \in A \quad \text{if and only if} \quad \exists a \in \mathcal{N}\, (x, a) \in B. \tag{11.14}$$

The set $C = \mathcal{N} - h(B)$ is open (or $\mathbf{\Sigma}^1_{n-1}$) in $\mathcal{N}$ and since V is universal, there exists a v such that $C = \{u : (u, v) \in V\}$. Then by (11.13), we have

$$\begin{aligned} x \in A &\leftrightarrow (\exists a \in \mathcal{N})\, (x, a) \in B \leftrightarrow (\exists a \in \mathcal{N})\, h(x, a) \notin C \\ &\leftrightarrow (\exists a \in \mathcal{N})\, (h(x, a), v) \notin V \leftrightarrow (x, v) \in U. \end{aligned}$$

Hence U is a universal $\mathbf{\Sigma}^1_n$ set. □

Corollary 11.9. *For each $n \geq 1$, there is a set $A \subset \mathcal{N}$ that is $\mathbf{\Sigma}^1_n$ but not $\mathbf{\Pi}^1_n$.*

Proof. Let $U \subset \mathcal{N}^2$ be a universal $\mathbf{\Sigma}^1_n$ set and let

$$A = \{x : (x, x) \in U\}$$ □

The collection of all $\mathbf{\Delta}^1_1$ sets in a Polish space is a σ-algebra and contains all Borel sets. It turns out that $\mathbf{\Delta}^1_1$ is exactly the collection of all Borel sets.

Theorem 11.10 (Suslin). *Every analytic set whose complement is also analytic is a Borel set. Thus $\mathbf{\Delta}^1_1$ is the collection of all Borel sets.*

Let X be a Polish space and let A and B be two disjoint analytic sets in X. We say that A and B are *separated* by a Borel set if there exists a Borel set D such that $A \subset D$ and $B \subset X - D$.

Lemma 11.11. *Any two disjoint analytic sets are separated by a Borel set.*

This lemma is often called "the $\boldsymbol{\Sigma}^1_1$-Separation Principle." It clearly implies Suslin's Theorem since if A is an analytic set such that $B = X - A$ is also analytic, A and B are separated by a Borel set D and we clearly have $D = A$.

Proof. First we make the following observation: If $A = \bigcup_{n=0}^\infty A_n$ and $B = \bigcup_{m=0}^\infty B_m$ are such that for all n and m, A_n and B_m are separated, then A and B are separated. This is proved as follows: For each n and each m, let $D_{n,m}$ be a Borel set such that $A_n \subset D_{n,m} \subset X - B_m$. Then A and B are separated by the Borel set $D = \bigcup_{n=0}^\infty \bigcap_{m=0}^\infty D_{n,m}$.

Let A and B be two disjoint analytic sets in X. Let f and g be continuous functions such that $A = f(\mathcal{N})$ and $B = g(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$ and $B_s = g(O(s))$; the sets A_s and B_s are all analytic sets. For each s we have $A_s = \bigcup_{n=0}^\infty A_{s^\frown n}$ and $B_s = \bigcup_{m=0}^\infty B_{s^\frown m}$. If $a \in \omega^\omega$, then

$$\{f(a)\} = \bigcap_{n=0}^{\infty} f(O(a{\restriction}n)) = \bigcap_{n=0}^{\infty} A_{a{\restriction}n},$$

and similarly for the sets B_s.

Let $a, b \in \omega^\omega$ be arbitrary. Since $f(\mathcal{N})$ and $g(\mathcal{N})$ are disjoint, we have $f(a) \neq g(b)$. Let G_a and G_b be two disjoint open neighbourhoods of $f(a)$ and $g(b)$, respectively. By the continuity of f and g there exists some n such that $A_{a{\restriction}n} \subset G_a$ and $B_{b{\restriction}n} \subset G_b$. It follows that the sets $A_{a{\restriction}n}$ and $B_{b{\restriction}n}$ are separated by a Borel set.

We shall now show, by contradiction, that the sets A and B are separated by a Borel set. If A and B are not separated, then because $A = \bigcup_{n=0}^\infty A_{\langle n\rangle}$ and $B = \bigcup_{m=0}^\infty B_{\langle m\rangle}$, there exist n_0 and m_0 such that the sets $A_{\langle n_0\rangle}$ and $B_{\langle m_0\rangle}$ are not separated. Then similarly there exist n_1 and m_1 such that the sets $A_{\langle n_0,n_1\rangle}$ and $B_{\langle m_0,m_1\rangle}$ are not separated, and so on. In other words, there exist $a = \langle n_0, n_1, n_2, \ldots\rangle$ and $b = \langle m_0, m_1, m_2, \ldots\rangle$ such that for every k, the sets $A_{\langle n_0,\ldots,n_k\rangle}$ and $B_{\langle m_0,\ldots,m_k\rangle}$ are not separated. This is a contradiction since in the preceding paragraph we proved exactly the opposite: There is k such that $A_{a{\restriction}k}$ and $B_{b{\restriction}k}$ are separated. □

Lebesgue Measure

We shall now review basic properties of Lebesgue measure on the n-dimensional Euclidean space.

The standard way of defining Lebesgue measure is to define first the *outer measure* $\mu^*(X)$ of a set $X \subset \boldsymbol{R}^n$ as the infimum of all possible sums $\sum\{v(I_k) : k \in \boldsymbol{N}\}$ where $\{I_k : k \in \boldsymbol{N}\}$ is a collection of n-dimensional intervals such that $X \subset \bigcup_{k=0}^\infty I_k$, and $v(I)$ denotes the volume of I. For each X, $\mu^*(X) \geq 0$ and possibly $= \infty$. A set X is *null* if $\mu^*(X) = 0$.

A set $A \subset \boldsymbol{R}^n$ is *Lebesgue measurable* if for each $X \subset \boldsymbol{R}^n$,

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X - A).$$

For a measurable set A, we write $\mu(A)$ instead of $\mu^*(A)$ and call $\mu(A)$ the *Lebesgue measure of A*.

The standard development of the theory of Lebesgue measure gives the following facts:

(11.15)

(i) Every interval is Lebesgue measurable, and its measure is equal to its volume.

(ii) The Lebesgue measurable sets form a σ-algebra; hence every Borel set is measurable.

(iii) μ is σ-additive: If A_n, $n < \omega$, are pairwise disjoint and measurable, then

$$\mu\Big(\bigcup_{n=0}^{\infty} A_n\Big) = \sum_{n=0}^{\infty} \mu(A_n).$$

(iv) μ is σ-finite: If A is measurable, then there exist measurable sets A_n, $n < \omega$, such that $A = \bigcup_{n=0}^{\infty} A_n$, and $\mu(A_n) < \infty$ for each n.

(v) Every null set is measurable. The null sets form a σ-ideal and contain all singletons.

(vi) If A is measurable, then

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact}\}.$$

(vii) If A is measurable, then there is an F_σ set F and a G_δ set G such that $F \subset A \subset G$ and $G - F$ is null.

This last property gives this characterization of Lebesgue measurable sets: A set $A \subset \boldsymbol{R}^n$ is measurable if and only if there is a Borel set B such that the symmetric difference $A \triangle B = (A - B) \cup (B - A)$ is null.

One consequence of this is that if we denote by $\mathcal{B}$ the σ-algebra of Borel sets and by $\mathcal{M}$ the σ-algebra of measurable sets, and if I_μ is the ideal of all null sets, then $\mathcal{B}/I_\mu = \mathcal{M}/I_\mu$. The Boolean algebra $\mathcal{B}/I_\mu$ is σ-complete; and since a familiar argument shows that I_μ is (as an ideal in $\mathcal{M}$) σ-saturated, we conclude that $\mathcal{B}/I_\mu$ is a complete Boolean algebra. We shall return to this in Part II.

Assuming the Axiom of Choice one can show that there exists a set of reals that is not Lebesgue measurable. One such example is the Vitali set in Exercise 10.1. As another example there exists a set $X \subset \boldsymbol{R}^n$ such that neither X nor its complement has a perfect subset (see Exercise 5.1 for a construction of such a set). The set X is not measurable: Otherwise, e.g., $\mu(X) > 0$ and by (11.15)(vi) there is a closed $K \subset X$ such that $\mu(K) > 0$; thus K is uncountable and hence contains a perfect subset, a contradiction.

However, we shall show in Part II that it is consistent (with ZF + DC) that all sets or reals are Lebesgue measurable.

We conclude this review of Lebesgue measurability with two lemmas. One is the well-known Fubini Theorem, and we state it here, without proof, for the sake of completeness. The other lemma will be used in the proof of Theorem 11.18 below.

If A is a subset of the plane $\boldsymbol{R}^2$ and $x \in \boldsymbol{R}$, let A_x denote the set $\{y : (x, y) \in A\}$.

Lemma 11.12. *Let $A \subset \boldsymbol{R}^2$ be a measurable set. Then A is null if and only if for almost all x, A_x is null (i.e., the set $\{x : A_x$ is not null$\}$ is null).* □

Lemma 11.13. *For any set $X \subset \boldsymbol{R}^n$ there exists a measurable set $A \supset X$ with the property that whenever $Z \subset A - X$ is measurable, then Z is null.*

Proof. If $\mu^*(X) < \infty$, then because $\mu^*(X) = \inf\{\mu(A) : A$ is measurable and $A \supset X\}$, there is a measurable $A \supset X$ such that $\mu(A) = \mu^*(X)$; clearly such an A will do. If $\mu^*(X) = \infty$, there exist pairwise disjoint X_n such that $X = \bigcup_{n=0}^{\infty} X_n$ and that for each n, $\mu^*(X_n) < \infty$. Let $A_n \supset X_n$, $n < \omega$, be measurable sets such that $\mu(A_n) = \mu^*(X_n)$, and let $A = \bigcup_{n=0}^{\infty} A_n$. □

It should be mentioned that the main results of descriptive set theory on Lebesgue measure can be proved in a more general context, namely for reasonable σ-additive measures on Polish spaces. An example of such a measure is the product measure in the Cantor space $\{0, 1\}^\omega$.

The Property of Baire

In Chapter 4 we proved the Baire Category Theorem (Theorem 4.8): The intersection of countably many dense open sets of reals is nonempty. It is fairly easy to see that the proof works not only for the real line $\boldsymbol{R}$ but for any Polish space.

Let us consider a Polish space X. Let us call a set $A \subset X$ *nowhere dense* if the complement of A contains a dense open set. Note that A is nowhere dense just in case for every nonempty open set G, there is a nonempty open set $H \subset G$ such that $A \cap H = \emptyset$. A set A is nowhere dense if and only if its closure $\overline{A}$ is nowhere dense.

A set $A \subset X$ is *meager* (or of *first category*) if A is the union of countably many nowhere dense sets. A nonmeager set is called a set of *second category.*

The Baire Category Theorem states in effect that in a Polish space every nonempty open set is of second category.

The meager sets form a σ-ideal. Moreover, in case of $\boldsymbol{R}^n$, $\mathcal{N}$, or the Cantor space, every singleton $\{x\}$ is nowhere dense and so the ideal of meager sets contains all countable sets.

Definition 11.14. A set A has the *Baire property* if there exists an open set G such that $A \triangle G$ is meager.

Clearly, every meager set has the Baire property. Note that if G is open, then $\overline{G} - G$ is nowhere dense. Hence if $A \triangle G$ is meager then $(X - A) \triangle (X - \overline{G}) = A \triangle \overline{G}$ is meager, and it follows that the complement of a set with the Baire property also has the Baire property. It is also easy to see that the union of countably many sets with the Baire property has the Baire property and we have:

Lemma 11.15. *The sets having the Baire property form a σ-algebra; hence every Borel set has the Baire property.* □

If $\mathcal{B}$ denotes the σ-algebra of Borel sets, and if we denote by $\mathcal{C}$ the σ-algebra of sets with the Baire property, and if I is the σ-ideal of meager sets, we have $\mathcal{B}/I = \mathcal{C}/I$. Note that the algebra $\mathcal{B}/I$ is σ-saturated: Let O be a countable topology base for X. For each nonmeager set X with the Baire property there exists $G \in O$ such that $G - X$ is meager. Thus the set $D = \{[G] : G \in O\}$ of equivalence classes is a dense set in $\mathcal{B}/I$. Hence $\mathcal{B}/I$ is σ-saturated and is a complete Boolean algebra.

The Axiom of Choice implies that sets without the Baire property exist. For instance, the Vitali set (Exercise 10.1) is such, see Exercise 11.7.

If $X \subset \boldsymbol{R}^n$ is such that neither X nor its complement has a perfect subset, then X does not have the Baire property: Otherwise, e.g., X is of second category and hence X contains a G_δ subset G of second category. Now G is uncountable, and this is a contradiction since as we shall prove in Theorem 11.18, every uncountable Borel set (even analytic) has a perfect subset.

The following two lemmas are analogs of Lemmas 11.12 and 11.13. The first one, although not very difficult to prove, is again stated without proof.

Lemma 11.16. *Let $A \subset \boldsymbol{R}^2$ have the property of Baire. Then A is meager if and only if A_x is meager for all x except a meager set.* □

Lemma 11.17. *For any set S in a Polish space X, there exists a set $A \supset S$ that has the Baire property and such that whenever $Z \subset A - S$ has the Baire property, then Z is meager.*

Proof. Let us consider a fixed countable topology basis O for X. Let $S \subset X$. Let

$$D(S) = \{x \in X : \text{for every } U \in O \text{ such that } x \in U,\ U \cap S \text{ is not meager}\}.$$

Note that the complement of $D(S)$ is the union of open sets and hence open; thus $D(S)$ is closed.

The set $S - D(S)$ is the union of all $S \cap U$ where $U \in O$ and $S \cap U$ is meager; since O is countable, $X - D(S)$ is meager. Let

$$A = S \cup D(S).$$

Since $A = (S - D(S)) \cup D(S)$ is the union of a meager and a closed set, A has the Baire property.

Let $Z \subset A - S$ have the Baire property; we shall show that Z is meager. Otherwise there is $U \in O$ such that $U - Z$ is meager; hence $U \cap S$ is meager. Since $U \cap Z \neq \emptyset$ and $Z \subset D(S)$, there is $x \in U$ such that $x \in D(S)$, and hence $U \cap S$ is not meager, a contradiction. □

Although both "null" and "meager" mean in a sense "negligible," see Exercise 11.8 that shows that the real line can be decomposed into a null set and a meager set.

Analytic Sets: Measure, Category, and the Perfect Set Property

Theorem 11.18.

(i) *Every analytic set of reals is Lebesgue measurable.*
(ii) *Every analytic set has the Baire property.*
(iii) *Every uncountable analytic set contains a perfect subset.*

Corollary 11.19. *Every* $\mathbf{\Pi}^1_1$ *set of reals is Lebesgue measurable and has the Baire property.* □

Corollary 11.20. *Every analytic* (*and in particular every Borel*) *set is either at most countable or has cardinality* $\mathfrak{c}$. □

We prove (ii) and (iii) for an arbitrary Polish space. The proof of (i) is general enough to work for other measures (in Polish spaces) as well.

Proof. The proof of (i) and (ii) is exactly the same and uses either Lemma 11.13 or Lemma 11.17 (and basic facts on Lebesgue measure and the Baire property). We give the proof of (i) and leave (ii) to the reader.

Let A be an analytic set of reals (or a subset of $\boldsymbol{R}^n$). Let $f : \mathcal{N} \to \boldsymbol{R}$ be a continuous function such that $A = f(\mathcal{N})$. For each $s \in Seq$, let $A_s = f(O(s))$. We have

$$A = \mathcal{A}\{A_s : s \in Seq\} = \mathcal{A}\{\overline{A_s} : s \in Seq\}, \tag{11.16}$$

and for every $s \in Seq$,

$$A_s = \bigcup_{n=0}^{\infty} A_{s^\frown n}. \tag{11.17}$$

By Lemma 11.13, there exists for each $s \in Seq$ a measurable set $B_s \supset A_s$ such that every measurable $Z \subset B_s - A_s$ is null. Since $\overline{A_s}$ is measurable, we may actually find B_s such that $A_s \subset B_s \subset \overline{A_s}$.

Let $B = B_\emptyset$. Since B is measurable, it suffices to show that $B - A$ is a null set. Notice that because $A_s \subset B_s \subset \overline{A_s}$, and because (11.16) holds, we have

$$A = \mathcal{A}\{B_s : s \in Seq\}.$$

Thus

$$B - A = B - \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a\restriction n}.$$

We claim that

$$B - \bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} B_{a\restriction n} \subset \bigcup_{s\in Seq} \Big(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}\Big). \tag{11.18}$$

To prove (11.18), assume that $x \in B$ is such that x is not a member of the right-hand side. Then for every s, if $x \in B_s$, then $x \in B_{s^\frown k}$ for some k. Hence there is k_0 such that $x \in B_{\langle k_0\rangle}$, then there is k_1 such that $x \in B_{\langle k_0,k_1\rangle}$, etc. Let $a = \langle k_0, k_1, k_2, \ldots\rangle$; we have $x \in \bigcap_{n=0}^\infty B_{a\restriction n}$ and hence x is not a member of the left-hand side.

Thus we have

$$B - A \subset \bigcup_{s\in Seq} \Big(B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k}\Big).$$

Since Seq is a countable set, it suffices to show that each $B_s - \bigcup_{k=0}^\infty B_{s^\frown k}$ is null. Let $s \in Seq$, and let $Z = B_s - \bigcup_{k=0}^\infty B_{s^\frown k}$. We have

$$Z = B_s - \bigcup_{k=0}^{\infty} B_{s^\frown k} \subset B_s - \bigcup_{k=0}^{\infty} A_{s^\frown k} = B_s - A_s.$$

Now because $Z \subset B_s - A_s$ and because Z is measurable, Z must be null.

(iii) The proof is a variant of the Cantor-Bendixson argument for closed sets in the Baire space. Recall that every closed set F in $\mathcal{N}$ is of the form $F = [T] = \{a : \forall n\ a\restriction n \in T\}$, where T is a tree, $T \subset Seq$. For each tree $T \subset Seq$ and each $s \in Seq$, let T_s denote the tree $\{t \in T : t \subset s \text{ or } s \subset t\}$; note that $[T_s] = [T] \cap O(s)$.

Let A be an analytic set (in a Polish space X), and let f be a continuous function such that $A = f(\mathcal{N})$. For each tree $T \subset Seq$, we define

$$T' = \{s \in T : f([T_s]) \text{ is uncountable}\}.$$

For each $\alpha < \omega_1$, we define $T^{(\alpha)}$ as follows:

$$T^{(0)} = Seq, \qquad T^{(\alpha+1)} = (T^{(\alpha)})',$$
$$T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Let $\alpha < \omega_1$ be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$. If $T^{(\alpha)} = \emptyset$, then

$$A = \bigcup_{\beta<\alpha} \{f([T_s^{(\beta)}]) : s \in T^{(\beta)} - T^{(\beta+1)}\},$$

and hence A is countable. Thus if A is uncountable, $T^{(\alpha)}$ is nonempty and for every $s \in T^{(\alpha)}$, $f([T_s^{(\alpha)}])$ is uncountable. In this case, we shall find a perfect subset of A.

Let $s \in T^{(\alpha)}$ be arbitrary. Since $f([T_s^{(\alpha)}])$ has at least two elements, there exist $s_{\langle 0\rangle} \supset s$ and $s_{\langle 1\rangle} \supset s$ (in $T^{(\alpha)}$) such that $f([T^{(\alpha)}_{s_{\langle 0\rangle}}])$ and $f([T^{(\alpha)}_{s_{\langle 1\rangle}}])$ are disjoint. Then there are $s_{\langle 0,0\rangle} \supset s_{\langle 0\rangle}$ and $s_{\langle 0,1\rangle} \supset s_{\langle 0\rangle}$, and $s_{\langle 1,0\rangle} \supset s_{\langle 1\rangle}$, $s_{\langle 1,1\rangle} \supset s_{\langle 1\rangle}$ such that the four sets $f([T^{(\alpha)}_{s_{\langle i,j\rangle}}])$, $i, j = 0, 1$ are pairwise disjoint. In this fashion we construct $s_t \in T^{(\alpha)}$ for each finite 0–1 sequence t. These elements s_t generate a subtree $U = \{s : s \subset s_t \text{ for some } t\}$ of $T^{(\alpha)}$ such that (1) U is perfect, (2) every s has at most two immediate successors in U (hence $[U]$ is a compact set in $\mathcal{N}$), and (3) f is one-to-one on $[U]$.

Let P be the image of $[U]$ under the function f. Since $[U]$ is compact and f is continuous, P is also compact, and hence closed. Moreover, P has no isolated points because $[U]$ is perfect and f is continuous. Thus P is a perfect subset of A. $\square$

Exercises

11.1. The operations $\bigcup_{n=0}^{\infty}$ and $\bigcap_{n=0}^{\infty}$ are special cases of the operation $\mathcal{A}$.

11.2. Let A_s, $s \in Seq$, be Borel sets satisfying (11.10) and the additional condition: For each $s \in Seq$ and all $n \neq m$, $A_{s^\frown n} \cap A_{s^\frown m} = \emptyset$. Then $\mathcal{A}\{A_s : s \in Seq\}$ is a Borel set.

[$\bigcup_{a\in\omega^\omega} \bigcap_{n=0}^{\infty} A_{a\restriction n} = \bigcap_{n=0}^{\infty} \bigcup\{A_s : \text{length}(s) = n\}$.]

11.3. Let A_n, $n = 0, 1, 2, \ldots$, be pairwise disjoint analytic sets. Then there exist pairwise disjoint Borel sets D_n such that $A_n \subset D_n$ for all n.

[Modify the proof of Lemma 11.11.]

11.4. If A is a null set and $a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots$ is a sequence of positive numbers with $\lim_n a_n = 0$, then there exists a sequence G_n, $n = 0, 1, \ldots$, of finite unions of open intervals such that $A \subset \bigcup_{n=0}^{\infty} G_n$ and $\mu(G_n) < a_n$ for each n. Moreover, the intervals can be required to have rational endpoints.

[First find a sequence of open intervals I_k such that $A \subset \bigcup_{k=0}^{\infty} I_k$ and $\sum_{k=0}^{\infty} \mu(I_k) \leq a_0$.]

11.5. For every set A with the Baire property, there exist a G_δ set G and an F_σ set F such that $G \subset A \subset F$ and such that $F - G$ is meager.

[Note that every meager set is included in a meager F_σ set.]

11.6. For every set A with the Baire property, there exists a unique regular open set U such that $A \triangle U$ is meager.

[An open set U is *regular* if $U = \text{int}(\overline{U})$.]

11.7. The Vitali set M from Exercise 10.1 does not have the Baire property.

["Meager" and "Baire property" are invariant under translation. If M has the Baire property, then there is an interval (a, b) such that $(a, b) - M$ is meager. Then $(a, b) \cap M_q$ is meager for all rational $q \neq 0$, hence each $M \cap (a - q, b - q)$ is meager, hence M is meager, hence each M_q is meager; a contradiction since $\boldsymbol{R} = \bigcup_{q\in\boldsymbol{Q}} M_q$.]

11.8. There is a null set of reals whose complement is meager.

[Let $q_1, q_2, \ldots$ be an enumeration of the rationals. For each $n \geq 1$ and $k \geq 1$, let $I_{n,k}$ be the open interval with center q_n and length $1/(k \cdot 2^n)$. Let $D_k = \bigcup_{n=1}^{\infty} I_{n,k}$, and $A = \bigcap_{k=1}^{\infty} D_k$. Each D_k is open and dense, and $\mu(D_k) \leq 1/k$. Hence A is null and $\boldsymbol{R} - A$ is meager.]

Historical Notes

Borel sets were introduced by Borel in [1905]. Lebesgue in [1905] proved in effect Lemma 11.2. Suslin's discovery of an error in a proof in Lebesgue's article led to a construction of an analytic non-Borel set and introduction of the operation $\mathcal{A}$. The basic results on analytic sets as well as Theorem 11.10 appeared in Suslin's article [1917].

Projective sets were introduced by Luzin [1925] and [1927a], and Sierpiński [1925] and [1927]. The present notation ($\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$) appeared first in the paper [1959] of Addison who noticed the analogy between Luzin's hierarchy of projective sets and Kleene's hierarchy of analytic predicates [1955].

Lemma 11.8: Luzin [1930].

Lemma 11.11: Luzin [1927b].

For detailed treatment of Lebesgue measure, we refer the reader to Halmos' book [1950]; Lebesgue introduced his measure and integral in his thesis [1902]. Sets of first and second category were introduced by Baire [1899].

Lemmas 11.13 and 11.17: Marczewski [1930a].

Lemma 11.16: Kuratowski and Ulam [1932].

Theorem 11.18(i) (measurability of analytic sets) is due to Luzin [1917]. Theorem 11.18(ii) (Baire property) is due to Luzin and Sierpiński [1923] and Theorem 11.18(iii) (perfect subsets) is due to Suslin; cf. Luzin [1930]. The present proof of (i) and (ii) follows Marczewski [1930a]. Prior to Suslin (and following the Cantor-Bendixson Theorem for closed sets) Young proved in [1906] the perfect subset result for G_δ and F_σ sets; and Hausdorff [1916] and Aleksandrov [1916] proved the same for Borel sets.