12. Models of Set Theory

Modern set theory uses extensively construction of models to establish relative consistency of various axioms and conjectures. As the techniques often involve standard model-theoretic concepts, we assume familiarity with basic notions of models and satisfaction, submodels and embeddings, as well as Skolem functions, direct limit and ultraproducts. We shall review the basic notions, notation and terminology of model theory.

Review of Model Theory

A *language* is a set of symbols: relation symbols, function symbols, and constant symbols:

$$\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\}.$$

Each P is assumed to be an n-placed relation for some integer $n \geq 1$; each F is an m-placed function symbol for some $m \geq 1$.

Terms and *formulas* of a language $\mathcal{L}$ are certain finite sequences of symbols of $\mathcal{L}$, and of *logical symbols* (identity symbol, parentheses, variables, connectives, and quantifiers). The set of all terms and the set of all formulas are defined by recursion. If the language is countable (i.e., if $|\mathcal{L}| \leq \aleph_0$), then we may identify the symbols of $\mathcal{L}$, as well as the logical symbols, with some hereditarily finite sets (elements of V_ω); then formulas are also hereditarily finite.

A *model* for a given language $\mathcal{L}$ is a pair $\mathfrak{A} = (A, \mathcal{I})$, where A is the universe of $\mathfrak{A}$ and $\mathcal{I}$ is the *interpretation* function which maps the symbols of $\mathcal{L}$ to appropriate relations, functions, and constants in A. A model for $\mathcal{L}$ is usually written in displayed form as

$$\mathfrak{A} = (A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

By recursion on length of terms and formulas one defines the *value* of a term

$$t^{\mathfrak{A}}[a_1, \ldots, a_n]$$

and *satisfaction*

$$\mathfrak{A} \vDash \varphi[a_1, \ldots, a_n]$$

where t is a term, φ is a formula, and $\langle a_1, \ldots, a_n \rangle$ is a finite sequence in A.

Two models $\mathfrak{A} = (A, P, \dots, F, \dots, c, \dots)$ and $\mathfrak{A}' = (A', P', \dots, F', \dots, c', \dots)$ are *isomorphic* if there is an *isomorphism* between $\mathfrak{A}$ and $\mathfrak{A}'$, that is a one-to-one function f of A onto A' such that

(i) $P(x_1, \dots, x_n)$ if and only if $P'(f(x_1), \dots, f(x_n))$,
(ii) $f(F(x_1, \dots, x_n)) = F'(f(x_1), \dots, f(x_n))$,
(iii) $f(c) = c'$,

for all relations, functions, and constants of $\mathfrak{A}$. If f is an isomorphism, then $f(t^{\mathfrak{A}}[a_1, \dots, a_n]) = t^{\mathfrak{A}'}[f(a_1), \dots, f(a_n)]$ for each term, and

$$\mathfrak{A} \vDash \varphi[a_1, \dots, a_n] \quad \text{if and only if} \quad \mathfrak{A}' \vDash \varphi[f(a_1), \dots, f(a_n)]$$

for each formula φ and all $a_1, \dots, a_n \in A$.

A *submodel* of $\mathfrak{A}$ is a subset $B \subset A$ endowed with the relations $P^{\mathfrak{A}} \cap B^n$, $\dots$, functions $F^{\mathfrak{A}} \restriction B^m$, $\dots$, and constants $c^{\mathfrak{A}}$, $\dots$; all $c^{\mathfrak{A}}$ belong to B, and B is closed under all $F^{\mathfrak{A}}$ (if $(x_1, \dots, x_m) \in B^m$, then $F^{\mathfrak{A}}(x_1, \dots, x_m) \in B$).

An *embedding* of $\mathfrak{B}$ into $\mathfrak{A}$ is an isomorphism between $\mathfrak{B}$ and a submodel $\mathfrak{B}' \subset \mathfrak{A}$.

A submodel $\mathfrak{B} \subset \mathfrak{A}$ is an *elementary submodel*

$$\mathfrak{B} \prec \mathfrak{A}$$

if for every formula φ, and every $a_1, \dots, a_n \in B$,

$$\mathfrak{B} \vDash \varphi[a_1, \dots, a_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[a_1, \dots, a_n]. \tag{12.1}$$

Two models $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent* if they satisfy the same sentences.

The key lemma in construction of elementary submodels is this: A subset $B \subset A$ forms an elementary submodel of $\mathfrak{A}$ if and only if for every formula $\varphi(u, x_1, \dots, x_n)$, and every $a_1, \dots, a_n \in B$,

(12.2) if $\exists a \in A$ such that $\mathfrak{A} \vDash \varphi[a, a_1, \dots, a_n]$, then $\exists a \in B$ such that $\mathfrak{A} \vDash \varphi[a, a_1, \dots, a_n]$.

A function $h : A^n \to A$ is a *Skolem function* for φ if

$$(\exists a \in A)\, \mathfrak{A} \vDash \varphi[a, a_1, \dots, a_n] \quad \text{implies} \quad \mathfrak{A} \vDash \varphi[h(a_1, \dots, a_n), a_1, \dots, a_n]$$

for every a_1, $\dots$, a_n. Using the Axiom of Choice, one can construct a Skolem function for every φ. If a subset $B \subset A$ is closed under (some) Skolem functions for all formulas, then B satisfies (12.2) and hence forms an elementary submodel of $\mathfrak{A}$.

Given a set of Skolem functions, one for each formula of $\mathcal{L}$, the closure of a set $X \subset A$ is a *Skolem hull* of X. It is clear that the Skolem hull of X is an elementary submodel of $\mathfrak{A}$, and has cardinality at most $|X| \cdot |\mathcal{L}| \cdot \aleph_0$. In particular, we have the following:

Theorem 12.1 (Löwenheim-Skolem). *Every infinite model for a countable language has a countable elementary submodel.* □

An *elementary embedding* is an embedding whose range is an elementary submodel.

A set $X \subset A$ is *definable over* $\mathfrak{A}$ if there exist a formula φ and some $a_1, \ldots, a_n \in A$ such that

$$X = \{x \in A : \mathfrak{A} \vDash \varphi[x, a_1, \ldots, a_n]\}.$$

We say that X is *definable in* $\mathfrak{A}$ *from* $a_1, \ldots, a_n$. If φ is a formula of x only, without parameters $a_1, \ldots, a_n$, then X is *definable* in $\mathfrak{A}$. An element $a \in A$ is *definable* (*from* $a_1, \ldots, a_n$) if the set $\{a\}$ is definable (from $a_1, \ldots, a_n$).

Gödel's Theorems

The cornerstone of modern logic are Gödel's theorems: the Completeness Theorem and two incompleteness theorems.

A set Σ of sentences of a language $\mathcal{L}$ is *consistent* if there is no formal proof of contradiction from Σ. The Completeness Theorem states that every consistent set of sentences has a model.

The First Incompleteness Theorem shows that no consistent (recursive) extension of Peano Arithmetic is complete: there exists a statement that is undecidable in the theory. In particular, if ZFC is consistent (as we believe), no additional axioms can prove or refute every sentence in the language of set theory.

The Second Incompleteness Theorem proves that sufficiently strong mathematical theories such as Peano Arithmetic or ZF (if consistent) cannot prove its own consistency. Gödel's Second Incompleteness Theorem implies that it is unprovable in ZF that there exists a model of ZF. This fact is significant for the theory of large cardinals, and we shall return to it later in this chapter.

Direct Limits of Models

An often used construction in model theory is the direct limit of a directed system of models. A *directed set* is a partially ordered set $(D, <)$ such that for every $i, j \in D$ there is a $k \in D$ such that $i \leq k$ and $j \leq k$.

First consider a system of models $\{\mathfrak{A}_i : i \in D\}$, indexed by a directed set D, such that for all $i, j \in D$, if $i < j$ then $\mathfrak{A}_i \prec \mathfrak{A}_j$. Let $\mathfrak{A} = \bigcup_{i \in D} \mathfrak{A}_i$; i.e., the universe of $\mathfrak{A}$ is the union of the universes of the $\mathfrak{A}_i$, $P^{\mathfrak{A}} = \bigcup_{i \in D} P^{\mathfrak{A}_i}$, etc. It is easily proved by induction on the complexity of formulas that $\mathfrak{A}_i \prec \mathfrak{A}$ for all i.

In general, we consider a *directed system* of models which consists of models $\{\mathfrak{A}_i : i \in D\}$ together with elementary embeddings $e_{i,j} : \mathfrak{A}_i \to \mathfrak{A}_j$ such that $e_{i,k} = e_{j,k} \circ e_{i,j}$ for all $i < j < k$.

Lemma 12.2. *If* $\{\mathfrak{A}_i, e_{i,j} : i,j \in D\}$ *is a directed system of models, there exists a model* $\mathfrak{A}$, *unique up to isomorphism, and elementary embeddings* $e_i : \mathfrak{A}_i \to \mathfrak{A}$ *such that* $\mathfrak{A} = \bigcup_{i\in D} e_i(\mathfrak{A}_i)$ *and that* $e_i = e_j \circ e_{i,j}$ *for all* $i < j$.

The model $\mathfrak{A}$ is called the *direct limit* of $\{\mathfrak{A}_i, e_{i,j}\}_{i,j\in D}$.

Proof. Consider the set S of all pairs (i, a) such that $i \in D$ and $a \in A_i$, and define an equivalence relation on S by

$$(i,a) \equiv (j,b) \leftrightarrow \exists k\,(i \le k,\ j \le k \text{ and } e_{i,k}(a) = e_{j,k}(b)).$$

Let $A = S/\!\equiv$ be the set of all equivalence classes, and let $e_i(a) = [(i,a)]$ for all $i \in D$ and $a \in A_i$. The rest is routine. $\square$

In set theory, a frequent application of direct limits involves the case when D is an ordinal number (and $<$ is its well-ordering).

Reduced Products and Ultraproducts

An important method in model theory uses filters and ultrafilters. Let S be a nonempty set and let $\{\mathfrak{A}_x : x \in S\}$ be a system of models (for a language $\mathcal{L}$). Let F be a filter on S. Consider the set

$$A = \prod_{x\in S} A_x/\!=_F$$

where $=_F$ is the equivalence relation on $\prod_{x\in x} A_x$ defined as follows:

$$f =_F g \quad \text{if and only if} \quad \{x \in S : f(x) = g(x)\} \in F. \tag{12.3}$$

It follows easily that $=_F$ is an equivalence relation.

The model $\mathfrak{A}$ with universe A is obtained by interpreting the language as follows:

If $P(x_1, \ldots, x_n)$ is a predicate, let

$$P^{\mathfrak{A}}([f_1], \ldots, [f_n]) \text{ if and only if } \{x \in S : P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in F. \tag{12.4}$$

If $F(x_1, \ldots, x_n)$ is a function, let

(12.5) $F^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f]$ where $f(x) = F^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))$ for all $x \in S$.

If c is a constant, let

$$c^{\mathfrak{A}} = [f] \text{ where } f(x) = c^{\mathfrak{A}_x} \text{ for all } x \in S. \tag{12.6}$$

(Note that (12.4) and (12.5) does not depend on the choice of representatives from the equivalence classes $[f_1]$, ..., $[f_n]$).

The model $\mathfrak{A}$ is called a *reduced product* of $\{\mathfrak{A}_x : x \in S\}$ (by F).

Reduced products are particularly important in the case when the filter is an ultrafilter. If U is an ultrafilter on S then the reduced product defined in (12.3)–(12.6) is called the *ultraproduct* of $\{\mathfrak{A}_x : x \in S\}$ by U:

$$\mathfrak{A} = \mathrm{Ult}_U\{\mathfrak{A}_x : x \in S\}.$$

The importance of ultraproducts is due mainly to the following fundamental property.

Theorem 12.3 (Łoś). *Let U be an ultrafilter on S and let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}_x : x \in S\}$ by U.*

(i) *If φ is a formula, then for every $f_1, \ldots, f_n \in \prod_{x \in S} A_x$,*

$$\mathfrak{A} \vDash \varphi([f_1], \ldots, [f_n]) \quad \textit{if and only if} \quad \{x \in S : \mathfrak{A}_x \vDash \varphi[f_1(x), \ldots, f_n(x)]\} \in U.$$

(ii) *If σ is a sentence, then*

$$\mathfrak{A} \vDash \sigma \quad \textit{if and only if} \quad \{x \in S : \mathfrak{A}_x \vDash \sigma\} \in U.$$

Part (ii) is a consequence of (i). Note that by the theorem, the satisfaction of φ at $[f_1]$, ..., $[f_n]$ does not depend on the choice of representatives f_1, ..., f_n for the equivalence classes $[f_1]$, ..., $[f_n]$. Thus we may further abuse the notation and write

$$\mathfrak{A} \vDash \varphi[f_1, \ldots, f_n].$$

It will also be convenient to adopt a measure-theoretic terminology. If

$$\{x \in S : \mathfrak{A}_x \vDash \varphi[f_1(x), \ldots, f_n(x)]\} \in U$$

we say that $\mathfrak{A}_x$ satisfies $\varphi(f_1(x), \ldots, f_n(x))$ for *almost all* x, or that $\mathfrak{A}_x \vDash \varphi(f_1(x), \ldots, f_n(x))$ holds *almost everywhere*. In this terminology, Łoś's Theorem states that $\varphi(f_1, \ldots, f_n)$ holds in the ultraproduct if and only if for almost all x, $\varphi(f_1(x), \ldots, f_n(x))$ holds in $\mathfrak{A}_x$.

Proof. We shall prove (i) by induction on the complexity of formulas. We shall prove that (i) holds for atomic formulas, and then prove the induction step for $\neg$, $\wedge$, and $\exists$.

Atomic formulas. First we consider the formula $u = v$, and we have

$$\begin{aligned}
\mathfrak{A} \vDash [f] = [g] &\leftrightarrow [f] = [g] \\
&\leftrightarrow f =_U g \\
&\leftrightarrow \{x : f(x) = g(x)\} \in U \\
&\leftrightarrow \{x : \mathfrak{A}_x \vDash f(x) = g(x)\} \in U.
\end{aligned} \tag{12.7}$$

For a predicate $P(v_1, \ldots, v_n)$ we have

$$\begin{aligned} \mathfrak{A} \vDash P([f_1], \ldots, [f_n]) &\leftrightarrow P^{\mathfrak{A}}([f_1], \ldots, [f_n]) \\ &\leftrightarrow \{x : P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash P(f_1(x), \ldots, f_n(x))\} \in U. \end{aligned} \tag{12.8}$$

Both (12.7) and (12.8) remain true if variables are replaced by terms, and so (i) holds for all atomic formulas.

Logical connectives. First we assume that (i) holds for φ and show that it also holds for $\neg\varphi$ (here we use that $X \in U$ if and only if $S - X \notin U$).

$$\begin{aligned} \mathfrak{A} \vDash \neg\varphi[f] &\leftrightarrow \text{not } \mathfrak{A} \vDash \varphi[f] \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi[f(x)]\} \notin U \\ &\leftrightarrow \{x : \mathfrak{A}_x \nvDash \varphi[f(x)]\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \neg\varphi[f(x)]\} \in U. \end{aligned}$$

Similarly, if (i) is true for φ and ψ, we have

$$\begin{aligned} \mathfrak{A} \vDash \varphi \wedge \psi &\leftrightarrow \mathfrak{A} \vDash \varphi \text{ and } \mathfrak{A} \vDash \psi \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi\} \in U \text{ and } \{x : \mathfrak{A}_x \vDash \psi\} \in U \\ &\leftrightarrow \{x : \mathfrak{A}_x \vDash \varphi \wedge \psi\} \in U \end{aligned}$$

(The last equivalence uses this: $X \in U$ and $Y \in U$ if and only if $X \cap Y \in U$.)

Existential quantifier. We assume that (i) is true for $\varphi(u, v_1, \ldots, v_n)$ and show that it remains true for the formula $\exists u\, \varphi$. Let us assume first that

$$\mathfrak{A} \vDash \exists u\, \varphi[f_1, \ldots, f_n]. \tag{12.9}$$

Then there is $g \in \prod_{x \in S} A_x$ such that $\mathfrak{A} \vDash \varphi[g, f_1, \ldots, f_n]$, and therefore

$$\{x : \mathfrak{A}_x \vDash \varphi[g(x), f_1(x), \ldots, f_n(x)]\} \in U, \tag{12.10}$$

and it clearly follows that

$$\{x : \mathfrak{A}_x \vDash \exists u\, \varphi[u, f_1(x), \ldots, f_n(x)]\} \in U. \tag{12.11}$$

Now let us assume that (12.11) holds. For each $x \in S$, let $u_x \in A_x$ be such that $\mathfrak{A}_x \vDash [u_x, f_1(x), \ldots, f_n(x)]$ if such u_x exists, and arbitrary otherwise. If we define $g \in \prod_{x \in S} A_x$ by $g(x) = u_x$, then we have (12.10), and therefore

$$\mathfrak{A} \vDash \varphi[g, f_1, \ldots, f_n].$$

Now (12.9) follows. □

Let us consider now the special case of ultraproducts, when each $\mathfrak{A}_x$ is the same model $\mathfrak{A}$. Then the ultraproduct is called an *ultrapower* of $\mathfrak{A}$; denoted $\mathrm{Ult}_U\,\mathfrak{A}$.

Corollary 12.4. *An ultrapower of a model $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{A}$.*

Proof. By Theorem 12.3(ii) we have $\mathrm{Ult}_U\,\mathfrak{A} \vDash \sigma$ if and only if $\{x : \mathfrak{A} \vDash \sigma\}$ is either S or empty, according to whether $\mathfrak{A} \vDash \sigma$ or not. □

We shall now show that a model $\mathfrak{A}$ is elementarily embeddable in its ultrapower. If U is an ultrafilter on S, we define the *canonical embedding* $j : \mathfrak{A} \to \mathrm{Ult}_U\,\mathfrak{A}$ as follows: For each $a \in A$, let c_a be the *constant function* with value a:

$$c_a(x) = a \qquad (\text{for every } x \in S), \tag{12.12}$$

and let

$$j(a) = [c_a]. \tag{12.13}$$

Corollary 12.5. *The canonical embedding $j : \mathfrak{A} \to \mathrm{Ult}_U\,\mathfrak{A}$ is an elementary embedding.*

Proof. Let $a \in A$. By Łoś's Theorem, $\mathrm{Ult}_U\,\mathfrak{A} \vDash \varphi[j(a)]$ if and only if $\mathrm{Ult}_U\,\mathfrak{A} \vDash \varphi[c_a]$ if and only if $\mathfrak{A} \vDash \varphi[a]$ for almost all x if and only if $\mathfrak{A} \vDash \varphi[a]$. □

Models of Set Theory and Relativization

The language of set theory consists of one binary predicate symbol $\in$, and so models of set theory are given by its universe M and a binary relation E on M that interprets $\in$.

We shall also consider models of set theory that are proper classes. However, due to Gödel's Second Incompleteness Theorem, we have to be careful how the generalization is formulated.

Definition 12.6. Let M be a class, E a binary relation on M and let $\varphi(x_1, \ldots, x_n)$ be a formula of the language of set theory. The *relativization* of φ to M, E is the formula

$$\varphi^{M,E}(x_1, \ldots, x_n) \tag{12.14}$$

defined inductively as follows:

$$\begin{aligned} (x \in y)^{M,E} &\leftrightarrow x\ E\ y \\ (x = y)^{M,E} &\leftrightarrow x = y \\ (\neg\varphi)^{M,E} &\leftrightarrow \neg\varphi^{M,E} \\ (\varphi \wedge \psi)^{M,E} &\leftrightarrow \varphi^{M,E} \wedge \psi^{M,E} \\ (\exists x\, \varphi)^{M,E} &\leftrightarrow (\exists x \in M)\, \varphi^{M,E} \end{aligned} \tag{12.15}$$

and similarly for the other connectives and $\forall$.

When E is $\in$, we write φ^M instead of $\varphi^{M,\in}$.

When using relativization $\varphi^{M,E}(x_1,\ldots,x_n)$ it is implicitly assumed that the variables $x_1, \ldots, x_n$ range over M. We shall often write

$$(M,E) \vDash \varphi(x_1,\ldots,x_n)$$

instead of (12.14) and say that the *model* (M,E) satisfies φ. We point out however that while this is a legitimate statement in every particular case of φ, the general satisfaction relation is formally undefinable in ZF.

Let *Form* denote the set of all formulas of the language $\{\in\}$. As with any actual (metamathematical) natural number we can associate the corresponding element of $\boldsymbol{N}$, we can similarly associate with any given formula of set theory the corresponding element of the set *Form*. To make the distinction, if φ is a formula, let $\ulcorner\varphi\urcorner$ denote the corresponding element of *Form*.

If M is a set and E is a binary relation on M and if $a_1, \ldots, a_n$ are elements of M, then

$$\varphi^{M,E}(a_1,\ldots,a_n) \leftrightarrow (M,E) \vDash \ulcorner\varphi\urcorner[a_1,\ldots,a_n] \tag{12.16}$$

as can easily be verified. Thus in the case when M is a set and φ a particular (metamathematical) formula, we shall not make a distinction between the two meanings of the symbol $\vDash$. We note however that the left-hand side of (12.16) (relativization) is *not* defined for $\varphi \in$ *Form*, and the right-hand side (satisfaction) is *not* defined if M is a proper class.

Below we sketch a proof of a theorem of Tarski, closely related to Gödel's Second Incompleteness Theorem. The theorem states that there is no set-theoretical property $T(x)$ such that if σ is a sentence that $T(\ulcorner\sigma\urcorner)$ holds if and only if σ holds.

Let us arithmetize the syntax and consider some fixed effective enumeration of all expressions by natural numbers (*Gödel numbering*). In particular, if σ is a sentence, then $\#\sigma$ is the Gödel number of σ, a natural number. We say that $T(x)$ is a *truth definition* if:

(12.17) (i) $\forall x\,(T(x) \rightarrow x \in \omega)$;
(ii) if σ is a sentence, then $\sigma \leftrightarrow T(\#\sigma)$.

Theorem 12.7 (Tarski). *A truth definition does not exist.*

Proof. Let us assume that there is a formula $T(x)$ satisfying (12.17). Let

$$\varphi_0, \quad \varphi_1, \quad \varphi_2, \quad \ldots$$

be an enumeration of all formulas with one free variable. Let $\psi(x)$ be the formula

$$x \in \omega \wedge \neg T(\#(\varphi_x(x))).$$

There is a natural number k such that ψ is φ_k. Let σ be the sentence $\psi(k)$. Then we have

$$\sigma \leftrightarrow \psi(k) \leftrightarrow \neg T(\#(\varphi_k(k))) \leftrightarrow \neg T(\#\sigma)$$

which contradicts (12.17). □

Relative Consistency

By Gödel's Second Incompleteness Theorem it is impossible to show the consistency of ZF (or related theories) by means limited to ZF alone.

Once we assume that ZF (or ZFC) is consistent, we may ask whether the theory remains consistent if we add an additional axiom A.

Let T be a mathematical theory (in our case, T is either ZF or ZFC), and let A be an additional axiom. We say that T + A is *consistent relative to* T (or that A is *consistent with* T) if the following implication holds:

$$\text{if T is consistent, then so is T + A.}$$

If both A and ¬A are consistent with T, we say that A is *independent of* T.

The question whether A is consistent with T is equivalent to the question whether the negation of A is provable in T (provided T is consistent); this is because T + A is consistent if and only if ¬A is not provable in T.

The way to show that an axiom A is consistent with ZF (ZFC) is to use models. For assume that we have a model M (possibly a proper class) of ZF such that $M \vDash \mathrm{A}$. (More precisely, the relativizations σ^M hold for all axioms σ of ZF, as well as A^M.) Then A is consistent with ZF: If it were not, then ¬A would be provable in ZF, and since M is a model of ZF, M would satisfy ¬A. However, $(\neg \mathrm{A})^M$ contradicts A^M.

Transitive Models and Δ_0 Formulas

If M is a transitive class then the model $(M, \in)$ is called a *transitive model*. We note that transitive models satisfy the Axiom of Extensionality (see Exercise 12.4) and that every well-founded extensional model is isomorphic to a transitive model (Theorem 6.15).

Definition 12.8. A formula of set theory is a Δ_0-formula if

(i) it has no quantifiers, or
(ii) it is $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \rightarrow \psi$ or $\varphi \leftrightarrow \psi$ where φ and ψ are Δ_0-formulas, or
(iii) it is $(\exists x \in y)\,\varphi$ or $(\forall x \in y)\,\varphi$ where φ is a Δ_0-formula.

Lemma 12.9. *If M is a transitive class and φ is a Δ_0-formula, then for all $x_1, \ldots, x_n$,*

$$\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n). \tag{12.18}$$

If (12.18) holds, we say that the formula φ is *absolute* for the transitive model M.

Proof. If φ is an atomic formula, then (12.18) holds. If (12.18) holds for φ and ψ, then it holds for $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$.

Let φ be the formula $(\exists u \in x)\, \psi(u, x, \ldots)$ and assume that (12.18) is true for ψ. We show that (12.18) is true for φ (the proof for $\forall u \in x$ is similar).

If φ^M holds then we have $(\exists u\, (u \in x \wedge \psi))^M$, i.e., $(\exists u \in M)(u \in x \wedge \psi^M)$. Since $\psi^M \leftrightarrow \psi$, it follows that $(\exists u \in x)\, \psi$. Conversely, if $(\exists u \in x)\, \psi$, then since M is transitive, u belongs to M, and since $\psi(u, x, \ldots) \leftrightarrow \psi^M(u, x, \ldots)$, we have $\exists u\, (u \in M \wedge u \in x \wedge \psi^M)$ and so $((\exists u \in x)\, \psi)^M$. $\square$

Lemma 12.10. *The following expressions can be written as Δ_0-formulas and thus are absolute for all transitive models.*

(i) $x = \{u, v\}$, $x = (u, v)$, *x is empty*, $x \subset y$, *x is transitive*, *x is an ordinal*, *x is a limit ordinal*, *x is a natural number*, $x = \omega$.
(ii) $Z = X \times Y$, $Z = X - Y$, $Z = X \cap Y$, $Z = \bigcup X$, $Z = \operatorname{dom} X$, $Z = \operatorname{ran} X$.
(iii) *X is a relation*, *f is a function*, $y = f(x)$, $g = f{\restriction}X$.

Proof.

(i) $x = \{u, v\} \leftrightarrow u \in x \wedge v \in x \wedge (\forall w \in x)(w = u \vee w = v)$.
$x = (u, v) \leftrightarrow (\exists w \in x)(\exists z \in x)(w = \{u\} \wedge z = \{u, v\})$
$\wedge\, (\forall w \in x)(w = \{u\} \vee w = \{u, v\})$.
x is empty $\leftrightarrow (\forall u \in x)\, u \neq u$.
$x \subset y \leftrightarrow (\forall u \in x)\, u \in y$.
x is transitive $\leftrightarrow (\forall u \in x)\, u \subset x$.
x is an ordinal $\leftrightarrow x$ is transitive $\wedge\, (\forall u \in x)(\forall v \in x)(u \in v \vee v \in u \vee u = v)$
$\wedge\, (\forall u \in x)(\forall v \in x)(\forall w \in x)(u \in v \in w \to u \in w)$.
x is a limit ordinal $\leftrightarrow x$ is an ordinal $\wedge\, (\forall u \in x)(\exists v \in x)\, u \in v$.
x is a natural number $\leftrightarrow x$ is an ordinal $\wedge$ (x is not a limit $\vee\, x = 0$)
$\wedge\, (\forall u \in x)(u = 0 \vee u$ is not a limit$)$.
$x = \omega \leftrightarrow x$ is a limit ordinal $\wedge\, x \neq 0 \wedge (\forall u \in x)\, x$ is a natural number.

(ii) $Z = X \times Y \leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists y \in Y)\, z = (x, y)$
$\wedge\, (\forall x \in X)(\forall y \in Y)(\exists z \in Z)\, z = (x, y)$.
$Z = X - Y \leftrightarrow (\forall z \in Z)(z \in X \wedge z \notin Y) \wedge (\forall z \in X)(z \notin Y \to z \in Z)$.
$Z = X \cap Y$ … similar.
$Z = \bigcup X \leftrightarrow (\forall z \in Z)(\exists x \in X)\, z \in x \wedge (\forall x \in X)(\forall z \in x)\, z \in Z$.
$Z = \operatorname{dom}(X) \leftrightarrow (\forall z \in Z)\, z \in \operatorname{dom} X \wedge (\forall z \in \operatorname{dom} X)\, z \in Z$,

and we show that:

(12.19) (a) $z \in \operatorname{dom} X$ is a Δ_0-formula;
(b) if φ is Δ_0, then $(\forall z \in \operatorname{dom} X)\, \varphi$ is Δ_0.

(a) $z \in \operatorname{dom} X \leftrightarrow (\exists x \in X)(\exists u \in X)(\exists v \in u)\, x = (z, v)$.
(b) $(\forall z \in \operatorname{dom} X)\, \varphi \leftrightarrow (\forall x \in X)(\forall u \in x)(\forall z, v \in u)(x = (z, v) \to \varphi)$.

An assertion similar to (12.19) holds for $\operatorname{ran}(X)$, and for $\exists$.

(iii) X is a relation $\leftrightarrow (\forall x \in X)(\exists u \in \operatorname{dom} X)(\exists v \in \operatorname{ran} X)\, x = (u, v)$.
f is a function $\leftrightarrow f$ is a relation $\wedge$
$$(\forall x \in \operatorname{dom} f)(\forall y, z \in \operatorname{ran} f)((x, y) \in f \wedge (x, z) \in f \to y = z)$$
where
$$(x, y) \in f \leftrightarrow (\exists u \in f)\, u = (x, y).$$
$$g = f{\restriction}X \leftrightarrow g \text{ is a function} \wedge g \subset f \wedge (\forall x \in \operatorname{dom} g)\, x \in X \\ \wedge (\forall x \in X)(x \in \operatorname{dom} f \to x \in \operatorname{dom} g). \quad \square$$

It should be emphasized that cardinal concepts are generally not absolute. In particular, the following expressions are known not to be absolute:

$$Y = P(X), \quad |Y| = |X|, \quad \alpha \text{ is a cardinal}, \quad \beta = \operatorname{cf}(\alpha), \quad \alpha \text{ is regular}.$$

Compare with Exercise 12.6.

Consistency of the Axiom of Regularity

As an application of the theory of transitive models we show that the Axiom of Regularity is consistent with the other axioms of ZF. In this section only we work in the theory ZF minus Regularity, i.e., axioms 1.1–1.7.

The cumulative hierarchy V_α is defined as in Chapter 6, and we denote (in the present section only) V not the universal class but the class $\bigcup_{\alpha \in Ord} V_\alpha$. We shall show that V is a transitive model of ZF. Thus the Axiom of Regularity is consistent relative to the theory 1.1–1.7.

Theorem 12.11. *In* ZF *minus Regularity, σ^V holds for every axiom σ of* ZF.

Proof. We use absoluteness of Δ_0-formulas and the fact that for every set x, if $x \subset V$, then $x \in V$.

Extensionality. The formula

$$((\forall u \in X)\, u \in Y \wedge (\forall u \in Y)\, u \in X) \to X = Y$$

is Δ_0.

Pairing. Given $a, b \in V$, let $c = \{a, b\}$. The set c is in V and since "$c = \{a, b\}$" is Δ_0 (see Lemma 12.10), the Pairing Axiom holds in V.

Separation. Let φ be a formula; we shall show that

$$V \vDash \forall X\, \forall p\, \exists Y\, \forall u\, (u \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$

Given $X, p \in V$, we let $Y = \{u \in X : \varphi^V(u, p)\}$. Since $Y \subset X$ and $X \in V$, we have $Y \in V$, and

$$V \vDash \forall u\, (y \in Y \leftrightarrow u \in X \wedge \varphi(u, p)).$$

Union. Given $X \in V$, let $Y = \bigcup X$. The set Y is in V and since "$Y = \bigcup X$" is Δ_0, the Axiom of Union holds in V.

Power Set. Given $X \in V$, let $Y = P(X)$. The set Y is in V, and we claim that $V \vDash \forall u\, \varphi(u)$ where $\varphi(u)$ is the formula $u \in Y \leftrightarrow u \subset X$. Since $\varphi(u)$ is Δ_0 and because $\varphi(u)$ holds for all u, we have $\varphi^V(u)$ for all $u \in V$, as claimed.

Infinity. We want to show that

$$V \vDash \exists S\, (\emptyset \in S \wedge (\forall x \in S)\, x \cup \{x\} \in S). \tag{12.20}$$

The formula in (12.20) contains defined notions, $\{\ \}$, $\cup$, and $\emptyset$; and strictly speaking, we should first eliminate these symbols and use a formula in which they are replaced by their definitions, using only $\in$ and $=$. However, we have already proved that both pairing and union are the same in the universe as in V, and similarly one shows that $X \in V$ is empty if and only if (X is empty)^V. In other words,

$$\{a, b\}^V = \{a, b\}, \qquad \bigcup\nolimits^V X = \bigcup X, \qquad \emptyset^V = \emptyset$$

where $\{a, b\}^V$, $\bigcup^V$, and $\emptyset^V$ denote pairing, union, and the empty set in the model V.

Since $\omega \in V$, we easily verify that (12.20) holds when $S = \omega$.

Replacement. Let φ be a formula; we shall show that

$$\begin{aligned} V \vDash {} & \forall x\, \forall y\, \forall z\, (\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z) \\ & \rightarrow \forall X\, \exists Y\, \forall y\, (y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p)). \end{aligned}$$

Given $p \in V$, assume that $V \vDash \forall x\, \forall y\, \forall z\, (\dots)$. Thus

$$F = \{(x, y) \in V : \varphi^V(x, y, p)\}$$

is a function, and we let $Y = F(X)$. Since $Y \subset V$, we have $Y \in V$, and one verifies that for every $y \in V$,

$$V \vDash y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p).$$

Regularity. We want to show that $V \vDash \forall S\, \varphi(S)$, where φ is the formula

$$S \neq \emptyset \rightarrow (\exists x \in S)\, S \cap x = \emptyset.$$

If $S \in V$ is nonempty, then let $x \in S$ be of least rank; then $S \cap x = \emptyset$. Hence $\varphi(S)$ is true for any S; moreover, $(S \cap x)^V = S \cap x$, and φ is Δ_0. Thus $V \vDash \forall S\, \varphi(S)$. $\square$

Inaccessibility of Inaccessible Cardinals

Theorem 12.12. *The existence of inaccessible cardinals is not provable in* ZFC. *Moreover, it cannot be shown that the existence of inaccessible cardinals is consistent with* ZFC.

We shall prove the first assertion and invoke Gödel's Second Incompleteness Theorem to obtain the second part.

First we prove (in ZFC):

Lemma 12.13. *If κ is an inaccessible cardinal, then V_κ is a model of* ZFC.

Proof. The proof of all axioms of ZFC except Replacement is as in the proof of consistency of the Axiom of Regularity (see Exercises 12.7 and 12.8). To show that $V_\kappa \vDash$ Replacement, it is enough to show:

(12.21) If F is a function from some $X \in V_\kappa$ into V_κ, then $F \in V_\kappa$.

Since κ is inaccessible, we have $|V_\kappa| = \kappa$ and $|X| < \kappa$ for every $X \in V_\kappa$. If F is a function from $X \in V_\kappa$ into V_κ, then $|F(X)| \leq |X| < \kappa$ and (since κ is regular) $F(X) \subset V_\alpha$ for some $\alpha < \kappa$. It follows that $F \in V_\kappa$. □

Proof of Theorem 12.12. If κ is an inaccessible cardinal, then not only is V_κ a model of ZFC, but in addition

$$(\alpha \text{ is an ordinal})^{V_\kappa} \leftrightarrow \alpha \text{ is an ordinal.}$$
$$(\alpha \text{ is a cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is a cardinal.}$$
$$(\alpha \text{ is a regular cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is a regular cardinal.}$$
$$(\alpha \text{ is an inaccessible cardinal})^{V_\kappa} \leftrightarrow \alpha \text{ is an inaccessible cardinal.}$$

We leave the details to the reader.

In particular, if κ is inaccessible cardinal, then

$$V_\kappa \vDash \text{there is no inaccessible cardinal.}$$

Thus we have a model of ZFC+ "there is no inaccessible cardinal" (if there is no inaccessible cardinal, we take the universe as the model). Hence it cannot be proved in ZFC that inaccessible cardinals exist.

To prove the second part, assume that it can be shown that the existence of inaccessible cardinals is consistent with ZFC; in other words, we assume

$$\text{if ZFC is consistent, then so is ZFC} + \text{I}$$

where I is the statement "there is an inaccessible cardinal."

We naturally assume that ZFC is consistent. Since I is consistent with ZFC, we conclude that ZFC + I is consistent. It is provable in ZFC + I that there is a model of ZFC (Lemma 12.13). Thus the sentence "ZFC is consistent" is provable in ZFC + I. However, we have assumed that "I is consistent with ZFC" is provable, and so "ZFC + I is consistent" is provable in ZFC + I. This contradicts Gödel's Second Incompleteness Theorem. □

The wording of the second part of Theorem 12.12 (and its proof) is somewhat vague; “it cannot be shown” means: It cannot be shown by methods formalizable in ZFC.

Reflection Principle

The theorem that we prove below is the analog of the Löwenheim-Skolem Theorem. While that theorem states that every model has a small elementary submodel, the Reflection Principle provides, for any *finite* number of formulas, a set M that is like an “elementary submodel” of the universe, with respect to the given formulas. The theorem is proved without the use of the Axiom of Choice, but using the Axiom of Choice, one can obtain countable model.

Theorem 12.14 (Reflection Principle).

(i) *Let $\varphi(x_1, \dots, x_n)$ be a formula. For each M_0 there exists a set $M \supset M_0$ such that*

$$\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \tag{12.22}$$

for every $x_1, \dots, x_n \in M$. (We say that M reflects φ.)

(ii) *Moreover, there is a transitive $M \supset M_0$ that reflects φ; moreover, there is a limit ordinal α such that $M_0 \subset V_\alpha$ and V_α reflects φ.*

(iii) *Assuming the Axiom of Choice, there is an $M \supset M_0$ such that M reflects φ and $|M| \le |M_0| \cdot \aleph_0$. In particular, there is a countable M that reflects φ.*

Remarks. 1. We may require either that M be transitive or that $|M| \le |M_0| \cdot \aleph_0$ but not both.

2. The proof works for any finite number of formulas, not just one. Thus if $\varphi_1, \dots, \varphi_n$ are formulas, then there exists a set M that reflects each of $\varphi_1, \dots, \varphi_n$.

3. If σ is a true sentence, then the Reflection Principle yields a set M that is a model of σ; using the Axiom of Choice, one can get a countable transitive model of σ.

4. As a consequence of the Reflection Principle, and of Gödel’s Second Incompleteness Theorem, it follows that the theory ZF is not finitely axiomatizable: Any finite number of theorems of ZF have a model (a set) by the Reflection Principle, while the existence of a model of ZF is not provable. (By the same argument, no consistent extension of ZF is finitely axiomatizable.)

The key step in the proof of Theorem 12.14 is the following lemma, which we prove first.

Lemma 12.15.

(i) *Let* $\varphi(u_1, \ldots, u_n, x)$ *be a formula. For each set* M_0 *there exists a set* $M \supset M_0$ *such that*

$$\text{(12.23)} \qquad \textit{if } \exists x\, \varphi(u_1, \ldots, u_n, x) \quad \textit{then } (\exists x \in M)\, \varphi(u_1, \ldots, u_n, x)$$

for every $u_1, \ldots, u_n \in M$. *Assuming the Axiom of Choice, there is* $M' \supset M_0$ *such that* (12.23) *holds for* M' *and* $|M'| \le |M_0| \cdot \aleph_0$.

(ii) *If* $\varphi_1, \ldots, \varphi_k$ *are formulas, then for each* M_0 *there is an* $M \supset M_0$ *such that* (12.23) *holds for each* $\varphi_1, \ldots, \varphi_k$.

Proof. We shall give a detailed proof of (i). An obvious modification of the proof gives (ii); we leave that to the reader.

Note that the operation $H(u_1, \ldots, u_n)$ defined below plays the same role as Skolem functions in the Löwenheim-Skolem Theorem.

Let us recall the definition (6.4):

$$\text{(12.24)} \qquad \hat{C} = \{x \in C : (\forall z \in C)\ \operatorname{rank} x \le \operatorname{rank} z\}.$$

For every $u_1, \ldots, u_n$, let

$$\text{(12.25)} \qquad H(u_1, \ldots, u_n) = \hat{C}$$

where

$$\text{(12.26)} \qquad C = \{x : \varphi(u_1, \ldots, u_n, x)\}.$$

Thus $H(u_1, \ldots, u_n)$ is a set with the property

$$\text{(12.27)} \quad \text{if } \exists x\, \varphi(u_1, \ldots, u_n, x), \quad \text{then } (\exists x \in H(u_1, \ldots, u_n))\, \varphi(u_1, \ldots, u_n, x).$$

We construct the set M by induction. We let $M = \bigcup_{i=0}^{\infty} M_i$ where for each $i \in \boldsymbol{N}$,

$$\text{(12.28)} \qquad M_{i+1} = M_i \cup \bigcup \{H(u_1, \ldots, u_n) : u_1, \ldots, u_n \in M_i\}.$$

Now, if $u_1, \ldots, u_n \in M$, then there is an $i \in \boldsymbol{N}$ such that $u_1, \ldots, u_n \in M_i$ and if $\varphi(u_1, \ldots, u_n, x)$ holds for some x, then it holds for some $x \in M_{i+1}$, by (12.27) and (12.28).

Assuming the Axiom of Choice, let F be a choice function on $P(M)$. For every $u_1, \ldots, u_n \in M$, let $h(u_1, \ldots, u_n) = F(H(u_1, \ldots, u_n))$ (and let $h(u_1, \ldots, u_n)$ remain undefined if $H(u_1, \ldots, u_n)$ is empty). Let us define $M' = \bigcup_{i=0}^{\infty} M'_i$, where $M'_0 = M_0$ and for each $i \in \boldsymbol{N}$,

$$M'_{i+1} = M'_i \cup \{h(u_1, \ldots, u_n) : u_1, \ldots, u_n \in M'_i\}.$$

Condition (12.23) can be verified for M' in the same way as for M. Moreover, each M'_i has cardinality at most $|M_0| \cdot \aleph_0$, and so does M'. $\square$

Proof of Theorem 12.14. Let $\varphi(x_1, \ldots, x_n)$ be a formula. We may assume that the universal quantifier does not occur in φ ($\forall x \ldots$ can be replaced by $\neg\exists x \neg \ldots$). Let $\varphi_1, \ldots, \varphi_k$ be all the subformulas of the formula φ.

Given a set M_0, there exists, by Lemma 12.15(ii), a set $M \supset M_0$, such that

$$\exists x\, \varphi_j(u, \ldots, x) \to (\exists x \in M)\, \varphi_j(u, \ldots, x), \qquad j = 1, \ldots, k \tag{12.29}$$

for all $u, \ldots \in M$. We claim that M reflects each φ_j, $j = 1, \ldots, k$, and in particular M reflects φ. This is proved by induction on the complexity of φ_j.

It is easy to see that (every) M reflects atomic formulas, and that if M reflects formulas ψ and χ, then M reflects $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \to \chi$, and $\psi \leftrightarrow \chi$. Thus assume that M reflects $\varphi_j(u_1, \ldots, u_m, x)$ and let us prove that M reflects $\exists x\, \varphi_j$.

If $u_1, \ldots, u_m \in M$, then

$$\begin{aligned} M \vDash \exists x\, \varphi_j(u_1, \ldots, u_m, x) &\leftrightarrow (\exists x \in M)\, \varphi_j^M(u_1, \ldots, u_m, x) \\ &\leftrightarrow (\exists x \in M)\, \varphi_j(u_1, \ldots, u_m, x) \\ &\leftrightarrow \exists x\, \varphi_j(u_1, \ldots, u_m, x). \end{aligned}$$

The last equivalence holds by (12.29).

This proves part (i) of the theorem. Part (iii) is proved by taking M of size $\le |M_0| \cdot \aleph_0$. To prove (ii), one has to modify the proof of Lemma 12.15 so that the set M used in (12.29) is transitive (or $M = V_\alpha$). This is done as follows: In (12.28), we replace M_{i+1} by its transitive closure (or by the least $V_\gamma \supset M_{i+1}$). Then M is transitive (or $M = V_\alpha$). □

Exercises

12.1. Let U be a principal ultrafilter on S, such that $\{a\} \in U$. Show that the ultraproduct $\mathrm{Ult}_U\{\mathfrak{A}_x : x \in S\}$ is isomorphic to $\mathfrak{A}_a$.

12.2. If U is a principal ultrafilter, then the canonical embedding j is an isomorphism between $\mathfrak{A}$ and $\mathrm{Ult}_U\,\mathfrak{A}$.

12.3. Let κ be a measurable cardinal and let U be an ultrafilter on κ. Let $(A, <^*)$ be the ultrapower of $(\kappa, <)$ by U, and let $j : \kappa \to A$ be the canonical embedding.

(i) $(A, <^*)$ is a linear ordering.
(ii) If U is σ-complete then $(A, <^*)$ is a well-ordering; $(A, <^*)$ is isomorphic, and can be identified with, $(\gamma, <)$, where γ is an ordinal.
(iii) If U is κ-complete then $j(\alpha) = \alpha$ for all $\alpha < \kappa$
(iv) If d is the diagonal function, $[d] \ge \kappa$. The measure U is normal if and only if $[d] = \kappa$.
[Compare with Exercise 10.5.]

12.4. A class M is extensional if and only if σ^M holds where σ is the Axiom of Extensionality.

12.5. The following can be written as Δ_0-formulas: x is an ordered pair, x is a partial (linear) ordering of y, x and y are disjoint, $z = x \cup y$, $y = x \cup \{x\}$, x is an inductive set, f is a one-to-one function of X into (onto) Y, f is an increasing ordinal function, f is a normal function.

12.6. Let M be a transitive class.

(i) If $M \vDash |X| \le |Y|$, then $|X| \le |Y|$.
(ii) If $\alpha \in M$ and if α is a cardinal, then $M \vDash \alpha$ is a cardinal.

[$|X| \le |Y| \leftrightarrow \exists f\, \varphi(f, X, Y)$; α is a cardinal $\leftrightarrow \neg\exists f\, (\exists \beta \in \alpha)\, \psi(\alpha, \beta, f)$, where φ and ψ are Δ_0-formulas.]

12.7. If α is a limit ordinal, then V_α is a model of Extensionality, Pairing, Separation, Union, Power Set, and Regularity. If AC holds, then V_α is a model of AC.

12.8. If $\alpha > \omega$, then V_α is a model of Infinity.

12.9. V_ω, the set of all hereditarily finite sets, is a model of ZFC minus Infinity.

12.10. The existence of an infinite set is not provable in ZFC minus Infinity. Moreover, it cannot be shown that the existence of an infinite set is consistent with ZFC minus Infinity.

12.11. If κ is an inaccessible cardinal then $V_\kappa \vDash$ there is a countable model of ZFC.

[Since $\langle V_\kappa, \in \rangle$ is a model of ZFC, there is a countable model (by the Löwenheim-Skolem Theorem). Thus there is $E \subset \omega \times \omega$ such that $\mathfrak{A} = (\omega, E)$ is a model of ZFC. Verify that $V_\kappa \vDash$ ($\mathfrak{A}$ is a countable model of ZFC).]

12.12. If κ is an inaccessible cardinal, then there is $\alpha < \kappa$ such that $\langle V_\alpha, \in \rangle \prec \langle V_\kappa, \in \rangle$ Moreover, the set $\{\alpha < \kappa : \langle V_\alpha, \in \rangle \prec \langle V_\kappa, \in \rangle\}$ is closed unbounded.

[Construct Skolem functions h for V_κ, and let $\alpha = \lim_n \alpha_n$, where $\alpha_{n+1} < \kappa$ is such that $h(V_{\alpha_n}) \subset V_{\alpha_{n+1}}$ for each h.]

For every infinite regular cardinal κ let H_κ be the set of all x such that $|\operatorname{TC}(x)| < \kappa$. The sets in H_ω are hereditarily finite sets. The sets in H_{ω_1} are *hereditarily countable* sets. Each H_κ is transitive and $H_\kappa \subset V_\kappa$.

12.13. If κ is a regular uncountable cardinal then H_κ is a model of ZFC minus the Power Set Axiom.

12.14. For every formula φ, there is a closed unbounded class C_φ of ordinals such that for each $\alpha \in C_\varphi$, V_α reflects φ.

[$C_{\varphi \wedge \psi} = C_\varphi \cap C_\psi$, $C_{\exists x\, \varphi} = C_\varphi \cap K_\varphi$, where K_φ is the closed unbounded class $\{\alpha \in \mathit{Ord} : \forall x_1, \ldots, x_n \in V_\alpha\, (\exists x\, \varphi(x, x_1, \ldots, x_n) \to (\exists x \in V_\alpha)\, \varphi(x, x_1, \ldots, x_n))\}$.]

12.15. Let M be a transitive class and let φ be a formula. For each $M_0 \subset M$ there exists a set $M_1 \supset M_0$ such that $M_1 \subset M$ and that $\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^{M_1}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in M_1$.

A transfinite sequence $\langle W_\alpha : \alpha \in \mathit{Ord} \rangle$ is called a *cumulative hierarchy* if $W_0 = \emptyset$ and

(12.30) (i) $W_\alpha \subset W_{\alpha+1} \subset P(W_\alpha)$,
(ii) if α is limit, then $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$.

Each W_α is transitive and $W_\alpha \subset V_\alpha$.

12.16. Let $\langle W_\alpha : \alpha \in \mathit{Ord} \rangle$ be a cumulative hierarchy, and let $W = \bigcup_{\alpha \in \mathit{Ord}} W_\alpha$. Let φ be a formula. Show that there are arbitrary large limit ordinals α such that $\varphi^W(x_1, \ldots, x_n) \leftrightarrow \varphi^{W_\alpha}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in W_\alpha$.

Historical Notes

For concepts of model theory, the history of the subject and for model-theoretical terminology, I refer the reader to Chang and Keisler's book [1973].

Reduced products were first investigated by Łoś in [1955], who also proved Theorem 12.3 on ultraproducts.

For Tarski's Theorem 12.7, see Tarski [1939].

The impossibility of a consistency proof of the existence of inaccessible cardinals follows from Gödel's Theorem [1931]. An argument that more or less establishes the consistency of the Axiom of Regularity appeared in Skolem's work in 1923 (see Skolem [1970], pp. 137–152).

The study of transitive models of set theory originated with Gödel's work on constructible sets. The Reflection Principle was introduced by Montague; see [1961] and Lévy [1960b].

Exercise 12.12: Montague and Vaught [1959].

Exercise 12.14: Galvin.