# 13. Constructible Sets

Constructible sets were introduced by Gödel in his proof of consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis. The class $L$ of all constructible sets (the *constructible universe*) is a transitive model of ZFC, and is the smallest transitive model of ZF that contains all ordinal numbers. In this chapter we study constructible sets and some related concepts.

## The Hierarchy of Constructible Sets

Recall that a set $X$ is *definable* over a model $(M, \in)$ (where $M$ is a set) if there exist a formula $\varphi \in Form$ (the set of all formulas of the language $\{\in\}$) and some $a_1, \ldots, a_n \in M$ such that $X = \{x \in M : (M, \in) \vDash \varphi[x, a_1, \ldots, a_n]\}$. Let

$$\operatorname{def}(M) = \{X \subset M : X \text{ is definable over } (M, \in)\}.$$

Clearly, $M \in \operatorname{def}(M)$ and $M \subset \operatorname{def}(M) \subset P(M)$.

**Definition 13.1.** We define by transfinite induction

(i) $L_0 = \emptyset$, $L_{\alpha+1} = \operatorname{def}(L_\alpha)$,
(ii) $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ if $\alpha$ is a limit ordinal, and
(iii) $L = \bigcup_{\alpha \in Ord} L_\alpha$.

The (definable) class $L$ is the class of *constructible* sets. The statement $V = L$, i.e., "every set is constructible," is the *Axiom of Constructibility.*

It follows from Definition 13.1 that $\langle L_\alpha : \alpha \in Ord\rangle$ is a *cumulative hierarchy* (see (12.30)); in particular, each $L_\alpha$ is transitive, $L_\alpha \subset L_\beta$ if $\alpha < \beta$, and $L$ is a transitive class.

**Lemma 13.2.** *For every $\alpha$, $\alpha \subset L_\alpha$ (and $L_\alpha \cap Ord = \alpha$).*

*Proof.* By induction on $\alpha$. At stage $\alpha + 1$, we need to show that $\alpha \in L_{\alpha+1}$, or that $\alpha$ is a definable subset of $L_\alpha$. Since $\alpha = \{x \in L_\alpha : x \text{ is an ordinal}\}$, and "$x$ is an ordinal" is a $\Delta_0$ formula, we have $\alpha = \{x \in L_\alpha : L_\alpha \vDash x \text{ is an ordinal}\}$. □

**Theorem 13.3.** *$L$ is a model of* ZF*.*

*Proof.* We show that $\sigma^L$ holds for every axiom $\sigma$ of ZF. Since $L$ is a transitive class, every $\Delta_0$ formula is absolute for $L$.

*Extensionality.* $L$ is transitive and therefore extensional.

*Pairing.* Given $a, b \in L$, let $c = \{a, b\}$. Let $\alpha$ be such that $a \in L_\alpha$ and $b \in L_\alpha$. Since $\{a, b\}$ is definable over $L_\alpha$, we have $c \in L_{\alpha+1}$, and since "$c = \{a, b\}$" is $\Delta_0$, the Pairing Axiom holds in $L$.

*Separation.* Let $\varphi$ be a formula. Given $X, p \in L$, we wish to show that the set $Y = \{u \in X : \varphi^L(u, p)\}$ is in $L$. By the Reflection Principle (applied to the cumulative hierarchy $L_\alpha$, cf. Exercise 12.6), there exists an $\alpha$ such that $X, p \in L_\alpha$ and $Y = \{u \in X : \varphi^{L_\alpha}(u, p)\}$. Thus $Y = \{u \in L_\alpha : L_\alpha \vDash u \in X \wedge \varphi(u, p)\}$ and so $Y \in L$.

*Union.* Given $X \in L$, let $Y = \bigcup X$. As $L$ is transitive, we have $Y \subset L$; let $\alpha$ be such that $X \in L_\alpha$ and $Y \subset L_\alpha$. $Y$ is definable over $L_\alpha$ by the $\Delta_0$ formula "$x \in \bigcup X$" and so $Y \in L$. Since "$Y = \bigcup X$" is $\Delta_0$, the Axiom of Union holds in $L$.

*Power Set.* Given $X \in L$, let $Y = P(X) \cap L$. Let $\alpha$ be such that $Y \subset L_\alpha$. $Y$ is definable over $L_\alpha$ by the $\Delta_0$ formula "$x \subset X$" and so $Y \in L$. We claim that $Y = P^L(X)$, i.e., that "$Y$ is the power set of $X$" holds in $L$. But "$x \in Y \leftrightarrow x \subset X$" is a $\Delta_0$ formula true for every $x \in L$.

*Infinity.* We can repeat the proof from Theorem 12.11 as $\omega \in L$.

*Replacement.* The easiest way to verify these axioms is to refer to Exercise 1.15, specifically to (1.10). If a class $F$ is a function in $L$ then for every $X \in L$ there exists an $\alpha$ such that $\{F(x) : x \in X\} \subset L_\alpha$. Since $L_\alpha \in L$, this suffices.

*Regularity.* If $S \in L$ is nonempty, let $x \in S$ be such that $x \cap S = \emptyset$. Then $x \in L$ and the $\Delta_0$ formula "$x \cap S = \emptyset$" holds in $L$. □

We will show that the model $L$ satisfies both the Axiom of Choice and the Generalized Continuum Hypothesis, thus establishing the consistency of AC and GCH (relative to ZF). This will be done by showing that $L$ is a model of the Axiom of Constructibility ($V = L$), and that $V = L$ implies both AC and GCH.

It is rather clear that $V = L$ implies AC: it is relatively straightforward to define a well-ordering of $L$ (by transfinite induction, using some enumeration of the set *Form* of all formulas).

It may appear that $L$ is trivially a model of "every set is constructible." However, to verify $V = L$ in $L$, we have to prove first that the property "$x$ is constructible" is absolute for $L$, i.e., that for every $x \in L$ we have $(x \text{ is constructible})^L$. We shall do this by analyzing the complexity of the property "constructible." While this can be done working directly with the model-theoretic concepts involved, we prefer to use an alternative approach (also due to Gödel).

## Gödel Operations

The Axiom Schema of Separation states that given a formula $\varphi(x)$, for every $X$ there exists a set $Y = \{u \in X : \varphi(u)\}$. It turns out that for $\Delta_0$ formulas, the construction of $Y$ from $X$ can be described by means of a finite number of elementary operations.

**Theorem 13.4 (Gödel's Normal Form Theorem).** *There exist operations $G_1, \ldots, G_{10}$ such that if $\varphi(u_1, \ldots, u_n)$ is a $\Delta_0$ formula, then there is a composition $G$ of $G_1, \ldots, G_{10}$ such that for all $X_1, \ldots, X_n$,*

(13.1)
$$G(X_1, \ldots, X_n) = \{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\}.$$

The operations $G_1, \ldots, G_{10}$ will be defined below. Compositions of $G_1, \ldots, G_{10}$ are called *Gödel operations.*

We call the following sentence an instance of $\Delta_0$-*Separation*:

$$\forall p_1 \ldots \forall p_n \, \forall X \, \exists Y \, \forall u \, (u \in Y \leftrightarrow u \in X \wedge \varphi(u, p_1, \ldots, p_n)) \tag{13.2}$$

where $\varphi$ is a $\Delta_0$ formula. We say that a transitive class $M$ satisfies $\Delta_0$-*Separation* if for every $\Delta_0$ formula $\varphi$, $M$ satisfies (13.2).

A class $C$ is *closed under an operation* $F$ if $F(x_1, \ldots, x_n) \in C$ whenever $x_1, \ldots, x_n \in C$. If a class $M$ is closed under the operations $G_1, \ldots, G_{10}$ then $M$ is closed under all Gödel operations.

**Corollary 13.5.** *If $M$ is a transitive class closed under Gödel operations then $M$ satisfies $\Delta_0$-Separation.*

*Proof.* Let $\varphi(u, p_1, \ldots, p_n)$ be a $\Delta_0$ formula, and let $X, p_1, \ldots, p_n \in M$. Let

$$Y = \{u \in X : \varphi(u, p_1, \ldots, p_n)\}.$$

By Lemma 12.9 it suffices to show that $Y \in M$, in order that $M$ satisfy (13.2). By Gödel's Normal Form Theorem, there is a Gödel operation $G$ such that

$$G(X, \{p_1\}, \ldots, \{p_n\}) = \{(u, p_1, \ldots, p_n) : u \in X \wedge \varphi(u, p_1, \ldots, p_n)\}.$$

It follows that

$$\begin{aligned} Y &= \{u : \exists u_1 \ldots \exists u_n \, (u, u_1, \ldots, u_n) \in G(X, \{p_1\}, \ldots, \{p_n\})\} \\ &= \underbrace{\operatorname{dom} \ldots \operatorname{dom}}_{n \text{ times}} G(X, \{p_1\}, \ldots, \{p_n\}). \end{aligned}$$

Since both $\{x, y\}$ and $\operatorname{dom}(x)$ are Gödel operations (see below) and since $M$ is closed under Gödel operations, we have $Y \in M$. $\square$

**Definition 13.6 (Gödel Operations).**

$$\begin{aligned}
G_1(X,Y) &= \{X,Y\},\\
G_2(X,Y) &= X \times Y,\\
G_3(X,Y) &= \varepsilon(X,Y) = \{(u,v) : u \in X \wedge v \in Y \wedge u \in v\},\\
G_4(X,Y) &= X - Y,\\
G_5(X,Y) &= X \cap Y,\\
G_6(X) &= \bigcup X,\\
G_7(X) &= \operatorname{dom}(X),\\
G_8(X) &= \{(u,v) : (v,u) \in X\},\\
G_9(X) &= \{(u,v,w) : (u,w,v) \in X\},\\
G_{10}(X) &= \{(u,v,w) : (v,w,u) \in X\}.
\end{aligned}$$

*Proof of Theorem* 13.4. The theorem is proved by induction on the complexity of $\Delta_0$ formulas. To simplify matters, we consider only formulas of this form:

(13.3)
- (i) the only logical symbols in $\varphi$ are $\neg$, $\wedge$, and restricted $\exists$;
- (ii) $=$ does not occur;
- (iii) the only occurrence of $\in$ is $u_i \in u_j$ where $i \neq j$;
- (iv) the only occurrence of $\exists$ is

$$(\exists u_{m+1} \in u_i)\, \psi(u_1, \ldots, u_{m+1})$$

where $i \le m$.

Every $\Delta_0$ formula can be rewritten in this form: The use of logical symbols can be restricted to $\neg$, $\wedge$, and $\exists$; $x = y$ can be replaced by $(\forall u \in x)\, u \in y \wedge (\forall v \in y)\, v \in x$, $x \in x$ can be replaced by $(\exists u \in x)\, u = x$ and the bound variables in $\varphi(u_1, \ldots, u_n)$ can be renamed so that the variable with the highest index is quantified.

Note that we allow dummy variables, so that for instance $\varphi(u_1, \ldots, u_5) = u_3 \in u_2$ and $\varphi(u_1, \ldots, u_6) = u_3 \in u_2$ are considered separately.

Thus let $\varphi(u_1, \ldots, u_n)$ be a formula in the form (13.3) and let us assume that the theorem holds for all subformulas of $\varphi$.

*Case I.* $\varphi(u_1, \ldots, u_n)$ is an atomic formula $u_i \in u_j$ $(i \neq j)$. We prove this case by induction on $n$.

*Case Ia.* $n = 2$. Here we have

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_1 \in u_2\} = \varepsilon(X_1, X_2)$$

and

$$\{(u_1, u_2) : u_1 \in X_1 \wedge u_2 \in X_2 \wedge u_2 \in u_1\} = G_8(\varepsilon(X_2, X_1)).$$

*Case Ib.* $n > 2$ and $i, j \neq n$. By the induction hypothesis, there is a $G$ such that

$$\{(u_1, \ldots, u_{n-1}) : u_1 \in X_1, \ldots, u_{n-1} \in X_{n-1} \wedge u_i \in u_j\} = G(X_1, \ldots, X_{n-1}).$$

Obviously

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \wedge u_i \in u_j\} = G(X_1, \ldots, X_{n-1}) \times X_n.$$

*Case Ic.* $n > 2$ and $i, j \neq n-1$. By the induction hypothesis (Case Ib) there is a $G$ such that

$$\{(u_1, \ldots, u_{n-2}, u_n, u_{n-1}) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_i \in u_j\} \\ = G(X_1, \ldots, X_n).$$

Noting that

$$(u_1, \ldots, u_{n-2}, u_n, u_{n-1}) = ((u_1, \ldots, u_{n-2}), u_n, u_{n-1})$$

we get

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_i \in u_j\} = G_9(G(X_1, \ldots, X_n)).$$

*Case Id.* $i = n-1$, $j = n$. By Ia, we have

$$\{(u_{n-1}, u_n) : u_{n-1} \in X_{n-1} \wedge u_n \in X_n \wedge u_{n-1} \in u_n\} = \varepsilon(X_{n-1}, X_n)$$

and so

$$\{((u_{n-1}, u_n), (u_1, \ldots, u_{n-2})) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_{n-1} \in u_n\} \\ = \varepsilon(X_{n-1}, X_n) \times (X_1 \times \ldots \times X_{n-2}) = G(X_1, \ldots, X_n).$$

Now we note that

$$((u_{n-1}, u_n), (u_1, \ldots, u_{n-2})) = (u_{n-1}, u_n, (u_1, \ldots, u_{n-2}))$$

and

$$(u_1, \ldots, u_n) = ((u_1, \ldots, u_{n-2}), u_{n-1}, u_n)$$

and thus

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } u_{n-1} \in u_n\} = G_{10}(G(X_1, \ldots, X_n)).$$

*Case Ie.* $i = n$, $j = n-1$. Similar to Case Id.

*Case II.* $\varphi(u_1, \ldots, u_n)$ is a negation, $\neg\psi(u_1, \ldots, u_n)$. By the induction hypothesis, there is a $G$ such that

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \psi(u_1, \ldots, u_n)\} = G(X_1, \ldots, X_n).$$

Clearly,

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} \\ = X_1 \times \ldots \times X_n - G(X_1, \ldots, X_n).$$

*Case III.* $\varphi$ is a conjunction, $\psi_1 \wedge \psi_2$. By the induction hypothesis,

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \psi_i(u_1, \ldots, u_n)\} = G_{(i)}(X_1, \ldots, X_n)$$

$(i = 1, 2)$. Hence

$$\{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} \\ = G_{(1)}(X_1, \ldots, X_n) \cap G_{(2)}(X_1, \ldots, X_n).$$

*Case IV.* $\varphi(u_1, \ldots, u_n)$ is the formula $(\exists u_{n+1} \in u_i)\, \psi(u_1, \ldots, u_{n+1})$. Let $\chi(u_1, \ldots, u_{n+1})$ be the formula $\psi(u_1, \ldots, u_{n+1}) \wedge u_{n+1} \in u_i$. By the induction hypothesis (we consider $\chi$ less complex than $\varphi$), there is a $G$ such that

$$\{(u_1, \ldots, u_{n+1}) : u_1 \in X_1, \ldots, u_{n+1} \in X_{n+1} \text{ and } \chi(u_1, \ldots, u_{n+1})\} \\ = G(X_1, \ldots, X_{n+1})$$

for all $X_1, \ldots, X_{n+1}$. We claim that

$$\text{(13.4)} \quad \{(u_1, \ldots, u_n) : u_1 \in X_1, \ldots, u_n \in X_n \text{ and } \varphi(u_1, \ldots, u_n)\} \\ = (X_1 \times \ldots \times X_n) \cap \operatorname{dom}(G(X_1, \ldots, X_n, \bigcup X_i)).$$

Let us denote $u = (u_1, \ldots, u_n)$ and $X = X_1 \times \ldots \times X_n$. For all $u \in X$, we have

$$\begin{aligned} \varphi(u) &\leftrightarrow (\exists v \in u_i)\, \psi(u_i, v) \\ &\leftrightarrow \exists v\, (v \in u_i \wedge \psi(u, v) \wedge v \in \bigcup X_i) \\ &\leftrightarrow u \in \operatorname{dom}\{(u, v) \in X \times \bigcup X_i : \chi(u, v)\} \end{aligned}$$

and (13.4) follows. This completes the proof of Theorem 13.4. ☐

The following lemma shows that Gödel operations are absolute for transitive models.

**Lemma 13.7.** *If $G$ is a Gödel operation then the property $Z = G(X_1, \ldots, X_n)$ can be written as a $\Delta_0$ formula.*

*Proof.* We show, by induction on the complexity of $G$ (a composition of $G_1$, …, $G_{10}$):

(13.5) (i) $u \in G(X, \ldots)$ is $\Delta_0$.
(ii) If $\varphi$ is $\Delta_0$, then so are $\forall u \in G(X, \ldots)\, \varphi$ and $\exists u \in G(X, \ldots)\, \varphi$.
(iii) $Z = G(X, \ldots)$ is $\Delta_0$.
(iv) If $\varphi$ is $\Delta_0$, then so is $\varphi(G(X, \ldots))$.

We proved (iii) for most of the $G_1$, …, $G_{10}$ in Lemma 12.10; the rest of the $G_i$ are handled similarly, e.g.,

$$\begin{aligned} & Z = G_8(X) \\ & \leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists u \in \operatorname{ran} X)(\exists v \in \operatorname{dom} X)(x = (v, u) \wedge z = (u, v)) \\ & \quad \wedge (\forall x \in X)(\forall u \in \operatorname{ran} X)(\forall v \in \operatorname{dom} X)(\exists z \in Z)(x = (v, u) \rightarrow z = (u, v)). \end{aligned}$$

We shall prove (i) and (ii) only for a typical example and leave the full proof to the reader (see also (12.19)). In (i) consider the formula

$$u \in F(X, \ldots) \times G(X, \ldots).$$

This can be written as

$$\exists x \in F(X, \ldots)\, \exists y \in G(X, \ldots)\, u = (x, y).$$

In (ii), consider the formula

$$\forall u \in \{F(X, \ldots), G(X, \ldots)\}\, \varphi(u),$$

which can be written as

$$\varphi(F(X, \ldots)) \wedge \varphi(G(X, \ldots)).$$

(iii) follows from (i) and (ii):

$$Z = G(X, \ldots) \leftrightarrow (\forall u \in Z)\, u \in G(X, \ldots) \wedge \forall u \in G(X, \ldots)\, u \in Z.$$

To prove (iv), let $\varphi$ be a $\Delta_0$ formula. Then $G(X, \ldots)$ occurs in $\varphi(G(X, \ldots))$ in the form $u \in G(X, \ldots)$, $G(X, \ldots) \in u$, $Z = G(X, \ldots)$, $\forall u \in G(X, \ldots)$, or $\exists u \in G(X, \ldots)$. Since $G(X, \ldots) \in u$ can be replaced by $(\exists v \in u)\, v = G(X, \ldots)$, we use (i)–(iii) to show that $\varphi(G(X, \ldots))$ is a $\Delta_0$ property. □

If $\varphi$ is a formula then $\varphi^M$ is a $\Delta_0$ formula, and so by Theorem 13.4 there is a Gödel operation $G$ such that for every transitive set $M$ and all $a_1, \ldots, a_n$,

$$\begin{aligned} \{x \in M : M \vDash \varphi[x, a_1, \ldots, a_n]\} &= \{x \in M : \varphi^M(x, a_1, \ldots, a_n)\} \\ &= G(M, a_1, \ldots, a_n). \end{aligned}$$

The same argument, by induction on the complexity of $\varphi$, shows that for every $\varphi \in$ *Form*, the set $\{x \in M : M \vDash \varphi[x, a_1, \ldots, a_n]\}$ is in the closure of $M \cup \{M\}$ under $G_1, \ldots, G_{10}$.

Conversely, if $G$ is a composition of $G_1, \ldots, G_{10}$ then by Lemma 13.7 there is a $\Delta_0$ formula $\varphi$ such that for all $M$ and all $a_1, \ldots, a_n$, if $X = G(M, a_1, \ldots, a_n)$ then $X = \{x : \varphi(M, x, a_1, \ldots, a_n)\}$. If, moreover, $M$ is transitive and $X \subset M$, then $X = \{x \in M : M \vDash \psi[x, a_1, \ldots, a_n]\}$ (where $\psi$ is an obvious modification of $\varphi$, e.g., replacing $\exists u \in M$ by $\exists u$). Thus we have the following description of $\operatorname{def}(M)$:

**Corollary 13.8.** *For every transitive set $M$,*

$$\operatorname{def}(M) = \operatorname{cl}(M \cup \{M\}) \cap P(M),$$

*where* cl *denotes the closure under $G_1, \ldots, G_{10}$.* □

## Inner Models of ZF

An *inner model* of ZF is a transitive class that contains all ordinals and satisfies the axioms of ZF. The constructible universe $L$ is an inner model of ZF, and as we show later in this chapter, $L$ is the smallest inner model of ZF.

In Chapter 12 we proved that $\Delta_0$ formulas are absolute for all transitive models, i.e., $\varphi^M$ is equivalent to $\varphi$, for every transitive class $M$. One can extend the use of superscripts to concepts other than formulas, namely classes, operations and constants:

If $C$ is a class $\{x : \varphi(x)\}$ then $C^M$ denotes the class $\{x : \varphi^M(x)\}$. As an example, $Ord^M$ is either $Ord$ (if $M$ contains all ordinals), or is the least ordinal not in $M$.

If $F$ is an operation then $F^M$ is the corresponding operation in $M$ (if $x \in M$ then $F^M(x)$ is defined if $M$ satisfies the statement that $F(x)$ exists). If $F^M(x) = F(x)$ for all $x$ for which $F^M(x)$ is defined, we say that $F$ is *absolute* for $M$. By Lemma 13.7, all Gödel operations are absolute for transitive models. As an example, $P^M(X) = P(X) \cap M$, and $V_\alpha^M = V_\alpha \cap M$ (Exercise 13.6).

Similarly, if $c$ is a constant symbol then $c^M$, if it exists, is the corresponding constant in $M$. Thus $\emptyset^M = \emptyset$ (if $\emptyset \in M$), $\omega^M = \omega$ (if $\omega \in M$), etc.

The following theorem gives a necessary and sufficient condition for a transitive class to be an inner model of ZF:

**Theorem 13.9.** *A transitive class $M$ is an inner model of* ZF *if and only if it is closed under Gödel operations and is* almost universal, *i.e., every subset $X \subset M$ is included in some $Y \in M$.*

*Proof.* As Gödel operations are absolute for transitive models, an inner model is necessarily closed under $G_1, \ldots, G_{10}$. If $X$ is a subset of an inner model $M$, then $X \subset V_\alpha \cap M$ for some $\alpha$, and $V_\alpha \cap M$ is in $M$ because $\alpha \in M$ and $V_\alpha \cap M = V_\alpha^M$. Thus the condition is necessary.

Now let $M$ be a transitive almost universal class that is closed under Gödel operations. Except for the Separation Schema, the verification of the axioms of ZF in $M$ follows closely the proof of Theorem 13.3 (or of Theorem 12.11), but using almost universality. For example, if $X \in M$ then $P(X) \cap M$ is included in some $Y \in M$, verifying the weak version (1.9) of the Power Set Axiom. We leave the details to the reader.

*Separation.* We will show that for every $X \in M$ the set $Y = \{u \in X : \varphi^M(u)\}$ is in $M$. (For simplicity, we disregard the parameter in the formula $\varphi$.)

Let $\varphi(u_1, \ldots, u_n)$ be a formula with $k$ quantifiers. We let $\bar\varphi(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$ be the $\Delta_0$ formula obtained by replacing each $\exists x$ (or $\forall x$) in $\varphi$ by $\exists x \in Y_j$ (or $\forall x \in Y_j$) for $j = 1, \ldots, k$. We shall prove, by induction on $k$, that for every $\varphi(u_1, \ldots, u_n)$ with $k$ quantifiers, for every $X \in M$ there exist

$Y_1, \ldots, Y_k \in M$ such that

$$\varphi^M(u_1, \ldots, u_n) \text{ if and only if } \bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$$

for all $u_1, \ldots, u_n \in X$. Then it follows that $Y = \{u \in X : \bar{\varphi}(u, Y_1, \ldots, Y_k)\}$, and since $M$ satisfies $\Delta_0$-Separation (by Corollary 13.5), we have verified that $Y \in M$, completing the proof.

If $k = 0$ then $\bar{\varphi} = \varphi$. For the induction step, let $\varphi(u)$ be $\exists v\, \psi(u, v)$ where $\psi$ has $k$ quantifiers. Thus $\bar{\varphi}$ is $(\exists v \in Y_{k+1})\, \bar{\psi}(u, v, Y_1, \ldots, Y_k)$.

Let $X \in M$. We look for $Y_1, \ldots, Y_k, Y_{k+1} \in M$ such that for every $u \in X$,

$$(13.6) \qquad (\exists v\, \psi(u, v))^M \quad \text{if and only if} \quad (\exists v \in Y_{k+1})\, \bar{\psi}(u, v, Y_1, \ldots, Y_k).$$

By the Collection Principle (6.5) (applied to the formula $v \in M \wedge \psi^M(u, v)$), there exists a set $M_1$ such that $X \subset M_1 \subset M$ and that for every $u \in X$,

$$(13.7) \qquad (\exists v \in M)\, \psi^M(u, v) \quad \text{if and only if} \quad (\exists v \in M_1)\, \psi^M(u, v).$$

Since $M$ is almost universal, there exists a set $Y \in M$ such that $M_1 \subset Y$. It follows from (13.7) that for every $u \in X$,

$$(\exists v \in M)\, \psi^M(u, v) \quad \text{if and only if} \quad (\exists v \in Y)\, \psi^M(u, v).$$

By the induction hypothesis, given $Y \in M$, there exist $Y_1, \ldots, Y_k \in M$ such that for all $u, v \in Y$,

$$\psi^M(u, v) \quad \text{if and only if} \quad \bar{\psi}(u, v, Y_1, \ldots, Y_k).$$

Thus we let $Y_{k+1} = Y$, and since $X \subset Y$, we have for all $u \in X$,

$$\begin{aligned} (\exists v\, \psi(u, v))^M \quad & \text{if and only if} \quad (\exists v \in M)\, \psi^M(u, v) \\ & \text{if and only if} \quad (\exists v \in Y)\, \psi^M(u, v) \\ & \text{if and only if} \quad (\exists v \in Y)\, \bar{\psi}(u, v, Y_1, \ldots, Y_k). \qquad \square \end{aligned}$$

## The Lévy Hierarchy

Definable concepts can be classified by means of the following hierarchy of formulas, introduced by Azriel Lévy:

A formula is $\Sigma_0$ and $\Pi_0$ if its only quantifiers are bounded, i.e., a $\Delta_0$ formula. Inductively, a formula is $\Sigma_{n+1}$ if it is of the form $\exists x\, \varphi$ where $\varphi$ is $\Pi_n$, and $\Pi_{n+1}$ if its is of the form $\forall x\, \varphi$ where $\varphi$ is $\Sigma_n$.

We say that a property (class, relation) is $\Sigma_n$ (or $\Pi_n$) if it can be expressed by a $\Sigma_n$ (or $\Pi_n$) formula. A function $F$ is $\Sigma_n$ ($\Pi_n$) if the relation $y = F(x)$ is $\Sigma_n$ ($\Pi_n$).

This classification of definable concepts is not syntactical: To verify that a concept can be expressed in a certain way may need a proof (in ZF). To illustrate this, consider the proof of Lemma 13.10 bellow: To contract two like quantifiers into one uses an application of the Pairing Axiom.

Whenever we say that a property $P$ is $\Sigma_n$ we always mean $P$ can be expressed by a $\Sigma_n$ formula in ZF, unless we specifically state which axioms of ZF are assumed. Since every proof uses only finitely many axioms, every specific property requires a finite set $\Sigma$ of axioms of ZF for its classification in the hierarchy. This finite set is implicit in the use of the defining formula. When $M$ is a transitive model of $\Sigma$ then the relativization $P^M$ is unambiguous, namely the formula $\varphi^M$. We call such transitive models *adequate* for $P$. A property is $\Delta_n$ if it is both $\Sigma_n$ and $\Pi_n$.

**Lemma 13.10.** *Let $n \geq 1$.*

(i) *If $P$, $Q$ are $\Sigma_n$ properties, then so are $\exists x\, P$, $P \wedge Q$, $P \vee Q$, $(\exists u \in x)\, P$, $(\forall u \in x)\, P$.*
(ii) *If $P$, $Q$ are $\Pi_n$ properties, then so are $\forall x\, P$, $P \wedge Q$, $P \vee Q$, $(\forall u \in x)\, P$, $(\exists u \in x)\, P$.*
(iii) *If $P$ is $\Sigma_n$, then $\neg P$ is $\Pi_n$; if $P$ is $\Pi_n$, then $\neg P$ is $\Sigma_n$.*
(iv) *If $P$ is $\Pi_n$ and $Q$ is $\Sigma_n$, then $P \to Q$ is $\Sigma_n$; if $P$ is $\Sigma_n$ and $Q$ is $\Pi_n$, then $P \to Q$ is $\Pi_n$*
(v) *If $P$ and $Q$ are $\Delta_n$, then so are $\neg P$, $P \wedge Q$, $P \vee Q$, $P \to Q$, $P \leftrightarrow Q$, $(\forall u \in x)\, P$, $(\exists u \in x)\, P$.*
(vi) *If $F$ is a $\Sigma_n$ function, then* dom$(F)$ *is a $\Sigma_n$ class.*
(vii) *If $F$ is a $\Sigma_n$ function and* dom$(F)$ *is $\Delta_n$, then $F$ is $\Delta_n$.*
(viii) *If $F$ and $G$ are $\Sigma_n$ functions, then so is $F \circ G$.*
(ix) *If $F$ is a $\Sigma_n$ function and if $P$ is a $\Sigma_n$ property, then $P(F(x))$ is $\Sigma_n$.*

*Proof.* Let us prove the lemma for $n = 1$. The general case follows easily by induction.

(i) Let

$$P(x, \ldots) \leftrightarrow \exists z\, \varphi(z, x, \ldots),$$
$$Q(x, \ldots) \leftrightarrow \exists u\, \psi(u, x, \ldots)$$

where $\varphi$ and $\psi$ are $\Delta_0$ formulas. We have

$$\begin{aligned} (13.8) \quad \exists x\, P(x, \ldots) &\leftrightarrow \exists x\, \exists z\, \varphi(z, x, \ldots) \\ &\leftrightarrow \exists v\, \exists w \in v\, \exists x \in w\, \exists z \in w\, (v = (x, z) \wedge \varphi(z, x, \ldots)). \end{aligned}$$

The right-hand side of (13.8) is a $\Sigma_1$ formula. Furthermore,

$$\begin{aligned} P(x, \ldots) \wedge Q(x, \ldots) &\leftrightarrow \exists z\, \exists u\, (\varphi(z, x, \ldots) \wedge \psi(u, x, \ldots)), \\ P(x, \ldots) \vee Q(x, \ldots) &\leftrightarrow \exists z\, \exists u\, (\varphi(z, x, \ldots) \vee \psi(u, x, \ldots)), \\ (\exists u \in x)\, P(u, \ldots) &\leftrightarrow \exists z\, \exists u\, (u \in x \wedge \varphi(z, u, \ldots)). \end{aligned}$$

To show that $(\forall u \in x)\, P$ is a $\Sigma_1$ property, we use the Collection Principle:

$$\begin{aligned}(\forall u \in x)\, P(u, \ldots) &\leftrightarrow (\forall u \in x)\, \exists z\, \varphi(z, u, \ldots)\\ &\leftrightarrow \exists y\, (\forall u \in x)(\exists z \in y)\, \varphi(z, u, \ldots).\end{aligned}$$

(ii) follows from (i) and (iii).

(iii)

$$\begin{aligned}\neg\exists z\, \varphi(z, x, \ldots) &\leftrightarrow \forall z\, \neg\varphi(z, x, \ldots),\\ \neg\forall z\, \varphi(z, x, \ldots) &\leftrightarrow \exists z\, \neg\varphi(z, x, \ldots).\end{aligned}$$

(iv)

$$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q).$$

(v) follows from (i)–(iv).

(vi)

$$x \in \operatorname{dom}(F) \leftrightarrow \exists y\, y = F(x).$$

(vii) Since $F$ is a function, we have

$$y = F(x) \leftrightarrow x \in \operatorname{dom}(F) \wedge \forall z\, (z = F(x) \rightarrow y = z). \tag{13.9}$$

If $z = F(x)$ is $\Sigma_n$ and $x \in \operatorname{dom}(F)$ is $\Pi_n$, then the right-hand side of (13.9) is $\Pi_n$.

(viii)

$$y = F(G(x)) \leftrightarrow \exists z\, (z = G(x) \wedge y = F(z)).$$

(ix)

$$P(F(x)) \leftrightarrow \exists y\, (y = F(x) \wedge P(y)).$$ □

Since $\Delta_0$ properties are absolute for all transitive models, it is clear that $\Sigma_1$ properties are *upward absolute*: If $P(x)$ is $\Sigma_1$ and if $M$ is a transitive model (adequate for $P$) then for all $x \in M$, $P^M(x)$ implies $P(x)$. Similarly, $\Pi_1$ properties are *downward absolute*, and consequently, $\Delta_1$ properties are absolute for transitive models.

As an example of a $\Delta_1$ property we show

**Lemma 13.11.** *"$E$ is a well-founded relation on $P$" is a $\Delta_1$ property.*

*Proof.* The following is a $\Pi_1$ formula: $E$ is a relation on $P$ and $\forall X\, \varphi(E, P, X)$, where $\varphi(E, P, X)$ is the formula

$$\emptyset \neq X \subset P \rightarrow (\exists a \in X)\, a \text{ is } E\text{-minimal in } X.$$

(Both "$E$ is a relation on $P$" and $\varphi(E, P, X)$ are $\Delta_0$ formulas.)

On the other hand, $E$ is well-founded if and only if there exists a function $f$ from $P$ into *Ord* such that $f(x) < f(y)$ whenever $x\ E\ y$. Thus we have an equivalent $\Sigma_1$ formula: $E$ is a relation on $P$ and $\exists f\, (f$ is a function $\wedge\, (\forall u \in \operatorname{ran}(f))\, u$ is an ordinal $\wedge\, (\forall x, y \in P)(x\ E\ y \rightarrow f(x) < f(y)))$. □

Other examples of $\Delta_1$ concepts are given in the Exercises.

**Lemma 13.12.** *Let $n \geq 1$, let $G$ be a $\Sigma_n$ function (on $V$), and let $F$ be defined by induction:*

$$F(\alpha) = G(F{\restriction}\alpha).$$

*Then $F$ is a $\Sigma_n$ function on Ord.*

*Proof.* Since *Ord* is a $\Sigma_0$ class, it is enough to verify that the following expression is $\Sigma_n$:

$$(13.10)\quad y = F(\alpha) \text{ if and only if } \exists f\,(f \text{ is a function} \wedge \operatorname{dom}(f) = \alpha \wedge (\forall \xi < \alpha)\, f(\xi) = G(f{\restriction}\xi) \wedge y = G(f)).$$

All the properties and operations in (13.10) are $\Sigma_0$ and $G$ is $\Sigma_n$, and hence $y = F(\alpha)$ is $\Sigma_n$. □

The power set operation $P(X)$ is obviously $\Pi_1$; since it is not absolute as we shall see in Chapter 14, it is not $\Sigma_1$. Similarly, cardinal concepts are $\Pi_1$ but not $\Sigma_1$:

**Lemma 13.13.** *"$\alpha$ is a cardinal," "$\alpha$ is a regular cardinal," and "$\alpha$ is a limit cardinal" are $\Pi_1$.*

*Proof.* (a) $\neg\exists f\,(f$ is a function and $\operatorname{dom}(f) \in \alpha$ and $\operatorname{ran}(f) = \alpha)$.

(b) $\alpha > 0$ is a limit ordinal and

$$\neg\exists f\,(f \text{ is a function and } \operatorname{dom}(f) \in \alpha \text{ and } \textstyle\bigcup \operatorname{ran}(f) = \alpha).$$

(c) $(\forall \beta < \alpha)(\exists \gamma < \alpha)(\beta < \gamma$ and $\gamma$ is a cardinal). □

Consequently, if $M$ is an inner model of ZF, then every cardinal (regular cardinal, limit cardinal) is a cardinal (regular cardinal, limit cardinal) in $M$, and if $|X|^M = |Y|^M$ then $|X| = |Y|$.

In Chapter 12 we pointed out that the satisfaction relation $(V, \in) \vDash \varphi[a_1, \ldots, a_n]$ (for $\varphi \in Form$) is not formalizable in ZF; this follows from Theorem 12.7. For any particular $n$, the satisfaction relation $\vDash_n$ restricted to $\Sigma_n$ formulas is formalizable: For $n = 0$, we can use the absoluteness of $\Delta_0$ formulas for transitive models,

$$\vDash_0 \varphi[a_1, \ldots, a_k] \quad \text{if and only if}$$
$$\varphi \in Form,\ \varphi \text{ is } \Delta_0, \text{ and } \exists M\,(M \text{ is transitive and } (M, \in) \vDash \varphi[a_1, \ldots, a_k]);$$

then inductively

$$\vDash_{n+1} (\exists x\, \varphi)[a_1, \ldots, a_k] \quad \text{if and only if}$$
$$\varphi \in Form,\ \varphi \text{ is } \Pi_n, \text{ and } \exists a\, \neg \vDash_n (\neg\varphi)[a, a_1, \ldots, a_k].$$

Similarly, we can define $\vDash_n^M$ for any particular $n$ and any transitive class $M$. Even more generally, we can define $\vDash_n^{(M,\in)}$ for any class $M$ (transitive or not).

If $M \subset N$, we say that $(M, \in)$ is a $\Sigma_n$-*elementary submodel* of $(N, \in)$,

$$(M, \in) \prec_{\Sigma_n} (N, \in),$$

if for every $\Sigma_n$ formula $\varphi \in Form$ and all $a_1, \ldots, a_k \in M$, $\models_n^M \varphi[a_1, \ldots, a_k] \leftrightarrow \models_n^N \varphi[a_1, \ldots, a_k]$.

## Absoluteness of Constructibility

We prove in this section that the property "$x$ is constructible" is absolute for inner models of ZF.

**Lemma 13.14.** *The function $\alpha \mapsto L_\alpha$ is $\Delta_1$.*

*Proof.* The function $L_\alpha$ is defined by transfinite induction and so by Lemma 13.12 it suffices to show that the induction step is $\Sigma_1$. In view of Corollary 13.8 it suffices to verify that

$$Y = \operatorname{cl}(M) \tag{13.11}$$

(where cl denotes closure under Gödel operations) is $\Sigma_1$. But (13.11) is equivalent to

$$\begin{aligned}\exists W\, [W &\text{ is a function} \wedge \operatorname{dom}(W) = \omega \wedge Y = \textstyle\bigcup \operatorname{ran}(W) \wedge W(0) = M \\ &\wedge (\forall n \in \operatorname{dom}(W))(W(n+1) = W(n) \cup \{G_i(x, y) : x \in W(n),\, y \in W(n), \\ &\qquad\qquad i = 1, \ldots, 10\})].\end{aligned}$$

□

**Corollary 13.15.** *The property "$x$ is constructible" is absolute for inner models of* ZF.

*Proof.* Let $M$ be an inner model of ZF. Since $M \supset Ord$, we have for all $x \in M$

$$(x \text{ is constructible})^M \leftrightarrow \exists \alpha \in M\, x \in L_\alpha^M \leftrightarrow \exists \alpha\, x \in L_\alpha \leftrightarrow x \text{ is constructible}.$$

□

As an immediate consequence we have.

**Theorem 13.16 (Gödel).**

(i) *$L$ satisfies the Axiom of Constructibility* ($V = L$).
(ii) *$L$ is the smallest inner model of* ZF.

*Proof.* (i) For every $x \in L$, $(x \text{ is constructible})^L$ if and only if $x$ is constructible, and hence "every set is constructible" holds in $L$.

(ii) If $M$ is an inner model then $L^M$ (the class of all constructible sets in $M$) is $L$ and so $L \subset M$. □

A detailed analysis of absoluteness of $L_\alpha$ for transitive models reveals that the following concept of adequacy suffices: Let us call a transitive set $M$ *adequate* if

(13.12) (i) $M$ is closed under $G_1, \ldots, G_{10}$,
(ii) for all $U \in M$, $\{G_i(x, y) : x, y \in U \text{ and } i = 1, \ldots, 10\} \in M$,
(iii) if $\alpha \in M$ then $\langle L_\beta : \beta < \alpha \rangle \in M$.

It follows that the $\Delta_1$ function $\alpha \mapsto L_\alpha$ is absolute for every adequate transitive set $M$. Also, we can verify that for every limit ordinal $\delta$, the transitive set $L_\delta$ is adequate. Moreover, adequacy can by formulated as follows: There is a sentence $\sigma$ such that for every transitive set $M$, $M$ is adequate if and only if $(M, \in) \models \sigma$. Therefore there exists a sentence $\sigma$ (which is $\Pi_2$) such that for every transitive set $M$

(13.13) $(M, \in) \models \sigma$ if and only if $M = L_\delta$ for some limit ordinal $\delta$.

This leads to the following:

**Lemma 13.17 (Gödel's Condensation Lemma).** *For every limit ordinal $\delta$, if $M \prec (L_\delta, \in)$ then the transitive collapse of $M$ is $L_\gamma$ for some $\gamma \le \delta$.* □

We wish to make two remarks at this point. First, it is enough to assume only $M \prec_{\Sigma_1} L_\delta$ for the Condensation Lemma to hold (as the sentence $\sigma$ in (13.13)) is $\Pi_2$. Secondly, the careful analysis of the definition of $L_\alpha$ makes it possible to find a $\Pi_2$ sentence $\sigma$ such that (13.13) holds even for (infinite) successor ordinals $\delta$. Thus Gödel's Condensation Lemma holds for all infinite ordinals $\delta$, a fact that is useful in some applications of $L$.

## Consistency of the Axiom of Choice

**Theorem 13.18 (Gödel).** *There exists a well-ordering of the class $L$. Thus $V = L$ implies the Axiom of Choice.*

Combining Theorems 13.16 and 13.18, we conclude that the Axiom of Choice holds in the model $L$, and so it is consistent with ZF.

*Proof.* We will show that $L$ has a definable well-ordering.

By induction, we construct for each $\alpha$ a well-ordering $<_\alpha$ of $L_\alpha$. We do it in such a way that if $\alpha < \beta$, then $<_\beta$ is an *end-extension* of $<_\alpha$, i.e.,

(13.14) (i) if $x <_\alpha y$, then $x <_\beta y$;
(ii) if $x \in L_\alpha$ and $y \in L_\beta - L_\alpha$, then $x <_\beta y$.

Notice that (13.14) implies that if $x \in y \in L_\alpha$, then $x <_\alpha y$.

First let us assume that $\alpha$ is a limit ordinal and that we have constructed $<_\beta$ for all $\beta < \alpha$ and that if $\beta_1 < \beta_2 < \alpha$, then $<_{\beta_2}$ is an end-extension of $<_{\beta_1}$. In this case we simply let

$$<_\alpha = \bigcup_{\beta<\alpha} <_\beta,$$

i.e., if $x, y \in L_\alpha$, we let

$$x <_\alpha y \quad \text{if and only if} \quad (\exists\beta < \alpha)\, x <_\beta y.$$

Thus assume that we have defined $<_\alpha$ and let us construct $<_{\alpha+1}$, a well-ordering of $L_{\alpha+1}$. We recall the definition of $L_{\alpha+1}$:

$$L_{\alpha+1} = P(L_\alpha) \cap \mathrm{cl}(L_\alpha \cup \{L_\alpha\}) = P(L_\alpha) \cap \bigcup_{n=0}^{\infty} W_n^\alpha,$$

where

$$W_0^\alpha = L_\alpha \cup \{L_\alpha\},$$
$$W_{n+1}^\alpha = \{G_i(X,Y) : X, Y \in W_n^\alpha,\ i = 1, \ldots, 10\}.$$

The idea of the construction of $<_{\alpha+1}$ is now as follows: First we take the elements of $L_\alpha$, then $L_\alpha$, then the remaining elements of $W_1^\alpha$, then the remaining elements of $W_2^\alpha$, etc. To order the elements of $W_{n+1}^\alpha$, we use the already defined well-ordering of $W_n^\alpha$ since every $x \in W_{n+1}^\alpha$ is equal to $G_i(u,v)$ for some $i = 1, \ldots, 10$ and some $u, v \in W_n^\alpha$. We let

(13.15) (i) $<_{\alpha+1}^0$ is the well-ordering of $L_\alpha \cup \{L_\alpha\}$ that extends $<_\alpha$ and such that $L_\alpha$ is the last element.

(ii) $<_{\alpha+1}^{n+1}$ is the following well-ordering of $W_{n+1}^\alpha$:
$x <_{\alpha+1}^{n+1} y$ if and only if either: $x <_{\alpha+1}^n y$,
or: $x \in W_n^\alpha$ and $y \notin W_n^\alpha$,
or: $x \notin W_n^\alpha$ and $y \notin W_n^\alpha$ and

(a) the least $i$ such that $\exists u, v \in W_n^\alpha\,(x = G_i(u,v))$ $<$ the least $j$ such that $\exists s, t \in W_n^\alpha\,(x = G_i(s,t))$, or

(b) the least $i$ = the least $j$ and [the $<_{\alpha+1}^n$-least $u \in W_n^\alpha$ such that $\exists v \in W_n^\alpha\,(x = G_i(u,v))$] $<_{\alpha+1}^n$ [the $<_{\alpha+1}^n$-least $s \in W_n^\alpha$ such that $\exists t \in W_n^\alpha\,(x = G_i(s,t))$], or

(c) the least $i$ = the least $j$ and the least $u$ = the least $s$ and [the $<_{\alpha+1}^n$-least $v \in W_n^\alpha$ such that $x = G_i(u,v)$] $<_{\alpha+1}^n$ [the $<_{\alpha+1}^n$-least $t \in W_n^\alpha$ such that $x = G_i(u,t)$].

Now we let

$$<_{\alpha+1} = \bigcup_{n=0}^{\infty} <_{\alpha+1}^n \cap\, (P(L_\alpha) \times P(L_\alpha)), \tag{13.16}$$

and it is clear that $<_{\alpha+1}$ is an end-extension of $<_\alpha$ and is a well-ordering of $L_{\alpha+1}$.

Having defined $<_\alpha$ for all $\alpha$, we let

$$x <_L y \quad \text{if and only if} \quad \exists \alpha\, x <_\alpha y.$$

The relation $<_L$ is a well-ordering of $L$. □

We call $<_L$ the *canonical well-ordering* of $L$.

The proof of Theorem 13.18 gives additional information about the complexity of the canonical well-ordering of $L$.

**Lemma 13.19.** *The relation $<_L$ is $\Sigma_1$ and moreover, for every limit ordinal $\delta$ and every $y \in L_\delta$, $x <_L y$ if and only if $x \in L_\delta$ and $(L_\delta, \in) \vDash x <_L y$.*

*Proof.* It suffices to prove that the function, $\alpha \mapsto <_\alpha$ which assigns to each $\alpha$ the canonical well-ordering of $L_\alpha$ is $\Sigma_1$.

The function $\alpha \mapsto <_\alpha$ is defined by induction and thus it suffices to show that the induction step is $\Sigma_1$. In fact, $<_{\alpha+1}$ is defined by induction from $<_\alpha$ (see (13.15) and (13.16)). It suffices to verify that $<_{\alpha+1}$ is obtained from $<_\alpha$ by means of a $\Delta_1$ operation (similar to the way in which $L_{\alpha+1}$ is obtained from $L_\alpha$ by $L_{\alpha+1} = \operatorname{def}(L_\alpha)$). The operation that yields $<_{\alpha+1}$ when applied to $<_\alpha$ is described in detail in (13.15). It can be written in a $\Sigma_1$ fashion in very much the same way as (13.11). The only potential difficulty might be the use of the words "the $<$-least," and that can be overcome as follows: For example, in (13.15)(ii)(c)

$$\text{the } <^n_{\alpha+1}\text{-least } v \in W^\alpha_n \text{ such that } x = G_i(u,v)$$
$$<^n_{\alpha+1} \text{ the } <^n_{\alpha+1}\text{-least } t \in W^\alpha_n \text{ such that } y = G_i(u,t)$$

can be written as

$$(\exists v \in W^\alpha_n)[x = G_i(u,v) \wedge (\forall t \in W^\alpha_n)(y = G_i(u,t) \to v <^n_{\alpha+1} t)].$$

The function $\alpha \mapsto <_\alpha$ is absolute for every adequate $M$ (see (13.12)) and therefore for every $L_\delta$ where $\delta$ is a limit ordinal. □

## Consistency of the Generalized Continuum Hypothesis

**Theorem 13.20 (Gödel).** *If $V = L$ then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every $\alpha$.*

*Proof.* We shall prove that if $X$ is a constructible subset of $\omega_\alpha$ then there exists a $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma$. Therefore $P^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$, and since $|L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$ (this is easy to show; see Exercise 13.19), we have $|P^L(\omega_\alpha)| \le \aleph_{\alpha+1}$.

Thus let $X \subset \omega_\alpha$. There exists a limit ordinal $\delta > \omega_\alpha$ such that $X \in L_\delta$. Let $M$ be an elementary submodel of $L_\delta$ such that $\omega_\alpha \subset M$ and $X \in M$, and that $|M| = \aleph_\alpha$. (As we can construct $M$ within $L$ which satisfies AC, this can be done even if AC does not hold in the universe.)

By the Condensation Lemma 13.17, the transitive collapse $N$ of $M$ is $L_\gamma$ for some $\gamma \le \delta$. Clearly, $\gamma$ is a limit ordinal, and $\gamma < \omega_{\alpha+1}$ because $|N| = |\gamma| = \aleph_\alpha$. As $\omega_\alpha \subset M$, the collapsing map $\pi$ is the identity on $\omega_\alpha$ and so $\pi(X) = X$. Hence $X \in L_\gamma$. □

The next theorem illustrates further the significance of Gödel's Condensation Lemma. The combinatorial principle $\diamondsuit$ was formulated by Ronald Jensen.

**Theorem 13.21 (Jensen).** *$V = L$ implies the Diamond Principle*:

($\diamondsuit$) *There exists a sequence of sets $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subset \alpha$, such that for every $X \subset \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a stationary subset of $\omega_1$.*

The sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ is called a $\diamondsuit$-*sequence*.

*Proof.* Assume $V = L$. By induction on $\alpha < \omega_1$, we define a sequence of pairs $(S_\alpha, C_\alpha)$, $\alpha < \omega_1$, such that $S_\alpha \subset \alpha$ and $C_\alpha$ is a closed unbounded subset of $\alpha$. We let $S_0 = C_0 = \emptyset$ and $S_{\alpha+1} = C_{\alpha+1} = \alpha + 1$ for all $\alpha$. If $\alpha$ is a limit ordinal, we define:

(13.17) $(S_\alpha, C_\alpha)$ is the $<_L$-least pair such that $S_\alpha \subset \alpha$, $C_\alpha$ is a closed unbounded subset of $\alpha$, and $S_\alpha \cap \xi \ne S_\xi$ for all $\xi \in C_\alpha$; if no such pair exists, let $S_\alpha = C_\alpha = \alpha$.

We are going to show that the sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence. Thus assume the contrary; then for some $X \subset \omega_1$, there exists a closed unbounded set $C$ such that

$$X \cap \alpha \ne S_\alpha \quad \text{for all } \alpha \in C. \tag{13.18}$$

Let $(X, C)$ be the $<_L$-least pair such that $X \subset \omega_1$, $C$ is a closed unbounded subset of $\omega_1$, and such that (13.18) holds.

Since $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1 \rangle$ is a $\omega_1$-sequence of pairs of subsets of $\omega_1$, it belongs to $L_{\omega_2}$, and moreover, it satisfies the same definition (13.17) in the model $(L_{\omega_2}, \in)$. Also, $(X, C) \in L_{\omega_2}$, and $(X, C)$ is, in $(L_{\omega_2}, \in)$, the $<_L$-least pair such that $X \subset \omega_1$, $C$ is a closed unbounded subset of $\omega_1$, and such that (13.18) holds.

Let $N$ be a countable elementary submodel of $(L_{\omega_2}, \in)$. Since $(X, C)$ and $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1 \rangle$ are definable in $(L_{\omega_2}, \in)$, they belong to $N$. The set $\omega_1 \cap N$ is an initial segment of $\omega_1$ (see Exercise 13.18), thus let $\delta = \omega_1 \cap N$.

The transitive collapse of $N$ is $L_\gamma$, for some $\gamma < \omega_1$, and let $\pi : N \to L_\gamma$ be the isomorphism. We have $\pi(\omega_1) = \delta$, $\pi(X) = X \cap \delta$, $\pi(C) = C \cap \delta$ and $\pi(\langle (S_\alpha, C_\alpha) : \alpha < \omega_1 \rangle) = \langle (S_\alpha, C_\alpha) : \alpha < \delta \rangle$.

Therefore $(L_\delta, \in)$ satisfies

(13.19) $(X \cap \delta, C \cap \delta)$ is the $<_L$-least pair $(Z, D)$ such that $Z \subset \delta$, $D \subset \delta$ is closed unbounded and $Z \cap \xi \neq S_\xi$ for all $\xi \in D$.

By absoluteness, (13.19) holds (in $L$, and $L = V$) and therefore, by (13.17), $X \cap \delta = S_\delta$. Since $C \cap \delta$ is unbounded in $\delta$, and $C$ is closed, it follows that $\delta \in C$. This contradicts (13.18). $\square$

## Relative Constructibility

Constructibility can be generalized by considering sets *constructible relative to* a given set $A$, resulting in an inner model $L[A]$. The idea is to relativize the hierarchy $L_\alpha$ by using the generalization

(13.20) $$\operatorname{def}_A(M) = \{X \subset M : X \text{ is definable over } (M, \in, A \cap M)\}$$

where $A \cap M$ is considered a unary predicate. A generalization of Corollary 13.8 provides an alternative description of $\operatorname{def}_A$: For every transitive set $M$,

(13.21) $$\operatorname{def}_A(M) = \operatorname{cl}(M \cup \{M\} \cup \{A \cap M\}) \cap P(M).$$

The class of all sets *constructible from* $A$ is defined as follows:

(13.22) $$\begin{aligned} &L_0[A] = \emptyset, \qquad L_{\alpha+1}[A] = \operatorname{def}_A(L_\alpha[A]),\\ &L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A] \quad \text{if } \alpha \text{ is a limit ordinal},\\ &L[A] = \bigcup_{\alpha \in Ord} L_\alpha[A]. \end{aligned}$$

The following theorem is the generalization of the relevant theorem on constructible sets:

**Theorem 13.22.** *Let $A$ be an arbitrary set.*

(i) *$L[A]$ is a model of* ZFC.
(ii) *$L[A]$ satisfies the axiom $\exists X\,(V = L[X])$.*
(iii) *If $M$ is an inner model of* ZF *such that $A \cap M \in M$, then $L[A] \subset M$.*
(iv) *There exists $\alpha_0$ such that for all $\alpha \geq \alpha_0$,*

$$L[A] \vDash 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

*Proof.* The proof follows closely the corresponding proofs for $L$, but some additional arguments are needed.

**Lemma 13.23.** *Let $\bar{A} = A \cap L[A]$. Then $L[\bar{A}] = L[A]$ and moreover $\bar{A} \in L[\bar{A}]$.*

*Proof.* We show by induction on $\alpha$ that $L_\alpha[\bar{A}] = L_\alpha[A]$. The induction step is obvious if $\alpha$ is a limit ordinal; thus assume that $L_\alpha[\bar{A}] = L_\alpha[A]$ and let us prove $L_{\alpha+1}[\bar{A}] = L_{\alpha+1}[A]$.

If we denote $U = L_\alpha[A]$, then we have

$$A \cap U = A \cap U \cap L[A] = \bar{A} \cap U,$$

and since $\operatorname{def}_A(U) = \operatorname{def}_{A\cap U}(U)$, we have

$$L_{\alpha+1}[A] = \operatorname{def}_A(U) = \operatorname{def}_{A\cap U}(U) = \operatorname{def}_{\bar{A}}(U) = L_{\alpha+1}[\bar{A}].$$

Thus $L[\bar{A}] = L[A]$. Moreover, there is $\alpha$ such that $A \cap L[A] = A \cap L_\alpha[A]$ and thus $\bar{A} \in L_{\alpha+1}[A]$. □

By Lemma 13.23 we may assume that $A \in L[A]$. In this case, $L[A]$ can be well-ordered by a relation that is definable from $A$.

In analogy with (13.13) there exists a $\Pi_2$ sentence (in the language $\{\in, A\}$ where $A$ is a unary predicate) such that for every transitive set $M$

(13.23) $(M, \in, A \cap M) \vDash \sigma$ if and only if $M = L_\delta$ for some limit ordinal $\delta$.

The Condensation Lemma is generalized as follows:

**Lemma 13.24.** *If $M \prec (L_\delta[A], \in, A \cap L_\delta[A])$ where $\delta$ is a limit ordinal, then the transitive collapse of $M$ is $L_\gamma[A]$ for some $\gamma \le \delta$.* □

Consequently, if $A \subset L_{\omega_\alpha}[A]$ then for every $X \subset \omega_\alpha$ in $L[A]$ there exists a $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma[A]$, completing the proof of Theorem 13.22. □

A consequence of Theorem 13.22(iv) is that if $V = L[A]$ and $A \subset \omega$, then the Generalized Continuum Hypothesis holds. For a slightly better result, see Exercise 13.26.

A different generalization yields for every set $A$ the smallest inner model $L(A)$ that contains $A$. (As an example, $L(\boldsymbol{R})$ is the smallest inner model that contains all reals.) The model $L(A)$ need not, however, satisfy the Axiom of Choice.

We define $L(A)$ as follows: Let $T = \operatorname{TC}(\{A\})$ be transitive closure of $A$ (to ensure that the resulting class $L(A)$ is transitive), and let

(13.24)
$$L_0(A) = T, \qquad L_{\alpha+1}(A) = \operatorname{def}(L_\alpha(A)),$$
$$L_\alpha(A) = \bigcup_{\beta<\alpha} L_\beta(A) \quad \text{if } \alpha \text{ is a limit ordinal, and}$$
$$L(A) = \bigcup_{\alpha\in Ord} L_\alpha(A).$$

The transitive class $L(A)$ is an inner model of ZF, contains $A$, and is the smallest such inner model.

## Ordinal-Definable Sets

A set $X$ is *ordinal-definable* if there is a formula $\varphi$ such that

$$X = \{u : \varphi(u, \alpha_1, \dots, \alpha_n)\} \tag{13.25}$$

for some ordinal numbers $\alpha_1, \dots, \alpha_n$.

It is not immediate clear that the property "ordinal-definable" is expressible in the language of set theory. Thus we give a different definition of ordinal definable sets and show that it is equivalent to (13.25).

We recall that $\text{cl}(M)$ denotes the closure of a set $M$ under Gödel operations. The class $OD$ of all *ordinal-definable sets* is define as follows:

$$OD = \bigcup_{\alpha \in Ord} \text{cl}\{V_\beta : \beta < \alpha\}. \tag{13.26}$$

In other words, $OD$ is the Gödel closure of $\{V_\alpha : \alpha \in Ord\}$, that is, ordinal definable sets are obtained from the $V_\alpha$ by applications of Gödel operations. We shall show that the elements of the class $OD$ are exactly the sets satisfying (13.25).

**Lemma 13.25.** *There exists a definable well-ordering of the class $OD$ (and a one-to-one definable mapping $F$ of $Ord$ onto $OD$).*

*Proof.* Earlier we described how to construct from a given well-ordering of a set $M$, a well-ordering of the set $\text{cl}(M)$. For every $\alpha$, the set $\{V_\beta : \beta < \alpha\}$ has an obvious well-ordering, which induces a well-ordering of $\text{cl}\{V_\beta : \beta < \alpha\}$. Thus we get a well-ordering of the class $OD$, and denote $F$ the corresponding (definable) one-to-one mapping of $Ord$ onto $OD$. □

Now it follows that every $X \in OD$ has the form (13.25). There exists $\alpha$ such that $X = \{u : \varphi(u, \alpha)\}$ where $\varphi(u, \alpha)$ is the formula $u \in F(\alpha)$.

We shall show that on the other hand, if $\varphi$ is a formula and $X$ is the set in (13.25), then $X \in OD$. By the Reflection Principle, let $\beta$ be such that $X \subset V_\beta$, $\alpha_1, \dots, \alpha_n < \beta$ and that $V_\beta$ reflects $\varphi$. Then we have

$$X = \{u \in V_\beta : \varphi^{V_\beta}(u, \alpha_1, \dots, \alpha_n)\}.$$

Since $\varphi^{V_\beta}$ is a $\Delta_0$ formula, we apply the normal form theorem and find a Gödel operation $G$ such that $X = G(V_\beta, \alpha_1, \dots, \alpha_n)$. Since every $\alpha$ is obtained (uniformly) from $V_\alpha$ by a Gödel operation (because $\alpha = \{x \in V_\alpha : x \text{ is an ordinal}\}$), there exists a Gödel operation $H$ such that $X = H(V_{\alpha_1}, \dots, V_{\alpha_n}, V_\beta)$ and therefore $X \in OD$.

Thus let $HOD$ denote the class of *hereditarily ordinal-definable* sets

$$HOD = \{x : \text{TC}(\{x\}) \subset OD\}.$$

The class $HOD$ is transitive and contains all ordinals.

**Theorem 13.26.** *The class HOD is a transitive model of* ZFC.

*Proof.* The class $HOD$ is transitive, and it is easy to see that it is closed under Gödel operations. Thus to show that $HOD$ is a model of ZF, it suffices to show that $HOD$ is almost universal. For that, it is enough to verify that $V_\alpha \cap HOD \in HOD$, for all $\alpha$. For any $\alpha$, the set $V_\alpha \cap HOD$ is a subset of $HOD$, and so it is sufficient to prove that $V_\alpha \cap HOD$ is ordinal-definable. This is indeed true because $V_\alpha \cap HOD$ is the set of all $u$ satisfying the formula

$$u \in V_\alpha \wedge (\forall z \in \mathrm{TC}(\{u\}))\, \exists \beta\, [z \in \mathrm{cl}\{V_\gamma : \gamma < \beta\}]$$

and thus $V_\alpha \cap HOD \in OD$.

It remains to prove that $HOD$ satisfies the Axiom of Choice. We shall show that for each $\alpha$ there exists a one-to-one function $g \in HOD$ of $V_\alpha \cap HOD$ into the ordinals. Since every such function is a subset of $HOD$, it suffices to find $g \in OD$.

By Lemma 13.25, there is a definable one-to-one mapping $G$ of the class $OD$ onto the ordinals. If we let $g$ be the restriction of $G$ to the ordinal-definable set $V_\alpha \cap HOD$, then $g$ is ordinal-definable. □

A set $X$ is *ordinal-definable from* $A$, $X \in OD[A]$, if there is a formula $\varphi$ such that

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A)\} \tag{13.27}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$.

As above, this notion is expressible in the language of set theory:

$$OD[A] = \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\}). \tag{13.28}$$

The class $OD[A]$ has a well-ordering definable from $A$ and thus every set in $OD[A]$ is of the form (13.27). Conversely (using the Reflection Principle), every set $X$ in (13.27) belongs to $OD[A]$.

The proof of Theorem 13.26 generalizes easily to the case of $HOD[A]$. Thus $HOD[A]$, the class of all sets hereditarily ordinal-definable from $A$, is a transitive model of ZFC.

As a further generalization, we call $X$ *ordinal-definable over* $A$, $X \in OD(A)$, if it belongs to the Gödel closure of $\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup A$. If $X \in OD(A)$, then $X \in \mathrm{cl}(\{V_\alpha : \alpha \in Ord\} \cup \{A\} \cup E)$, where $E = \{x_0, \ldots, x_k\}$ is a finite subset of $A$. Hence there is a finite sequence $s = \langle x_0, \ldots, x_k \rangle$ in $A$ such that $X$ is ordinal-definable from $A$ and $s$. On the other hand, if $s$ is a finite sequence in $A$, then obviously $s \in OD(A)$ and thus we have

$$OD(A) = \{X : X \in OD[A, s] \text{ for some finite sequence } s \text{ in } A\}.$$

In other words, $X \in OD(A)$ if and only if there is a formula $\varphi$ such that

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A, \langle x_0, \ldots, x_k \rangle)\}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$ and a finite sequence $\langle x_0, \ldots, x_k \rangle$ in $A$.

The class $HOD(A)$ of all sets hereditarily ordinal-definable over $A$ is a transitive model of ZF. To show that $HOD(A)$ is almost universal, it suffices to verify that $V_\alpha \cap HOD(A) \in OD(A)$. In fact, $V_\alpha \cap HOD(A)$ is ordinal-definable from $A$: It is the set

$$\{u \in V_\alpha : (\forall z \in \mathrm{TC}(\{u\}))\, z \in \mathrm{cl}(\{V_\beta : \beta \in Ord\} \cup \{A\} \cup A)\}.$$

## More on Inner Models

We conclude this chapter with some comments on inner models of ZF.

As we remarked earlier, cardinal concepts are generally not absolute. The following theorem summarizes the relations between some of the concepts and their relativizations (see also Lemma 13.13):

**Theorem 13.27.** *Let $M$ be an inner model of* ZF*. Then*

(i) $P^M(X) = P(X) \cap M$, $V_\alpha^M = V_\alpha \cap M$.
(ii) *If* $|X|^M = |Y|^M$ *then* $|X| = |Y|$.
(iii) *If $\alpha$ is a cardinal then $\alpha$ is a cardinal in $M$; if $\alpha$ is a limit cardinal, then $\alpha$ is a limit cardinal in $M$.*
(iv) $|\alpha| \le |\alpha|^M$, $\mathrm{cf}(\alpha) \le \mathrm{cf}^M(\alpha)$.
(v) *If $\alpha$ is a regular cardinal, then $\alpha$ is a regular cardinal in $M$; if $\alpha$ is weakly inaccessible, then $\alpha$ is weakly inaccessible in $M$.*
(vi) *If $M$ is a model of* ZFC *and $\kappa$ is inaccessible, then $\kappa$ is inaccessible in $M$.* □

Concerning (vi), if $\alpha < \kappa$, then since $M \vDash$ AC, we must have either $(2^\alpha)^M < \kappa$ or $(2^\alpha)^M \ge \kappa$ and the latter is impossible since $2^\alpha < \kappa$.

If $M$ is a transitive model of ZFC, then the Axiom of Choice in $M$ enables us to code all sets in $M$ by sets of ordinals and the model is determined by its sets or ordinals. The precise statement of this fact is: If $M$ and $N$ are two transitive models of ZFC with the same sets of ordinals, then $M = N$. In fact, a slightly stronger assertion is true. (On the other hand, one cannot prove that $M = N$ if neither model satisfies AC.)

**Theorem 13.28.** *Let $M$ and $N$ be transitive models of* ZF *and assume that the Axiom of Choice holds in $M$. If $M$ and $N$ have the same sets of ordinals, i.e., $P^M(Ord^M) = P^N(Ord^N)$, then $M = N$.*

*Proof.* We start with a rather trivial remark: $M$ and $N$ have the same sets of pairs of ordinals. To see this, use the absolute canonical one-to-one function $\Gamma : Ord \times Ord \to Ord$. If $X \subset Ord^2$ and $X \in M$, then $\Gamma(X)$ is both in $M$ and in $N$, and we have $X = \Gamma_{-1}(\Gamma(X)) \in N$.

First we prove that $M \subset N$. Let $X \in M$. Since $M$ satisfies AC, there is a one-to-one mapping $f \in M$ of some ordinal $\theta$ onto $\mathrm{TC}(\{X\})$. Let $E \in M$ be the following relation on $\theta$:

$$\alpha \mathrel{E} \beta \quad \text{if and only if} \quad f(\alpha) \in f(\beta).$$

$E$ is a set of pairs of ordinals and thus we have $E \in N$. In $M$, $E$ is well-founded and extensional. However, these properties are absolute and so $E$ is well-founded and extensional in $N$. Applying the Collapsing Theorem (in $N$), we get a transitive set $T \in N$ such that $(T, \in)$ is isomorphic to $(\theta, E)$. Hence $T$ is isomorphic to $\mathrm{TC}(\{X\})$ and since both are transitive, we have $T = \mathrm{TC}(\{X\})$. It follows that $\mathrm{TC}(\{X\}) \in N$ and so $X \in N$.

Now we prove $M = N$ by $\in$-induction. Let $X \in N$ and assume that $X \subset M$; we prove that $X \in M$. Let $Y \in M$ be such that $X \subset Y$ (for instance let $Y = V_\alpha^M$ where $\alpha = \mathrm{rank}(X)$; the rank function is absolute). Let $f \in M$ be a one-to-one function of $Y$ into the ordinals. Since $M \subset N$, $f$ is in $N$ and so $f(X) \in N$. Since $M \subset N$, $f$ is in $N$ and so $f(X) \in N$. However, $f(X)$ is a set of ordinals and so $f(X) \in M$, and we have $X = f_{-1}(f(X)) \in M$. □

## Exercises

**13.1.** If $M$ is a transitive set then its closure under Gödel operations is transitive.

**13.2.** If $M$ is closed under Gödel operations and extensional and if $X \in M$ is finite, then $X \subset M$. In particular, if $(x, y) \in M$, then $x \in M$ and $y \in M$.

**13.3.** If $M$ is closed under Gödel operations and extensional, and $\pi$ is the transitive collapse of $M$, then $\pi(G_i(X, Y)) = G_i(\pi X, \pi Y)$, $(i = 1, \ldots, 10)$ for all $X, Y \in M$.
[Use the Normal Form Theorem.]

**13.4.** The operations $G_5$ and $G_8$ are compositions of the remaining $G_i$.
[$G_8(X) = \mathrm{dom}(G_{10}(G_{10}(G_9(G_{10}(X \times X)))))$.]

**13.5.** The Axioms of Comprehension in the Bernays-Gödel set theory can be proved from a finite number of axioms of the form

$$\forall X \, \forall Y \, \exists Z \; Z = G(X, Y)$$

where the $G$'s are operations analogous to $G_1, \ldots, G_{10}$. Thus the theory BG is finitely axiomatizable.
[Formulate and prove an analog of the Normal Form Theorem.]

**13.6.** Prove that for every transitive $M$, $V_\alpha^M = V_\alpha \cap M$ (for all $\alpha \in M$).

**13.7.** Show that "$X$ is finite" is $\Delta_1$.
[To get a $\Pi_1$ formulation, use $T$-finiteness from Chapter 1.]

**13.8.** The functions $\alpha + \beta$ and $\alpha \cdot \beta$ are $\Delta_1$.

**13.9.** The canonical well-ordering of *Ord* × *Ord* is a $\Delta_0$ relation. The function $\Gamma$ is $\Delta_1$.

**13.10.** The function $S \mapsto \mathrm{TC}(S)$ is $\Delta_1$.

**13.11.** The function $x \mapsto \mathrm{rank}(x)$ is $\Delta_1$.

**13.12.** "$X$ is countable" is $\Sigma_1$.

**13.13.** $|X| \le |Y|$, $|X| = |Y|$ are $\Sigma_1$.

**13.14.** The relation $\models_0$ is $\Sigma_1$; for each $n \ge 1$, $\models_n$ is $\Sigma_n$.

**13.15.** $M \prec_{\Sigma_0} V$ holds for every transitive set $M$.

**13.16.** Let $n$ be a natural number. For every $M_0$ there exists a set $M \supset M_0$ such that $M \prec_{\Sigma_n} V$.

[Use the Reflection Principle.]

**13.17.** If $M \prec (L_{\omega_1}, \in)$, then $M = L_\alpha$ for some $\alpha$.

[Show that $M$ is transitive. Let $X \in M$. Let $f$ be the $<$-least mapping of $\omega$ onto $X$. Since $f$ is definable in $(L_{\omega_1}, \in)$ from $X$, $f$ is in $M$. Hence $f(n) \in M$ for each $n$ and we get $X \subset M$.]

**13.18.** If $M \prec (L_{\omega_2}, \in)$, then $\omega_1 \cap M = \alpha$ for some $\alpha \le \omega_1$.

[Same argument as in Exercise 13.17: If $\gamma < \omega_1$ and $\gamma \in M$, then $\gamma \subset M$.]

**13.19.** For all $\alpha \ge \omega$, $|L_\alpha| = |\alpha|$.

**13.20.** If $\alpha \ge \omega$ and $X$ is a constructible subset of $\alpha$, then $X \in L_\beta$, where $\beta$ is the least cardinal in $L$ greater than $\alpha$.

**13.21.** The canonical well-ordering of $L$, restricted to the set $\boldsymbol{R}^L = \boldsymbol{R} \cap L$ of all constructible reals, has order-type $\omega_1^L$.

[$\boldsymbol{R} \cap L \subset L_{\omega_1^L}$.]

**13.22.** If $\kappa$ is a regular uncountable cardinal in $L$, then $L_\kappa$ is a model of $\mathrm{ZF}^-$ (Zermelo-Fraenkel without the Power Set Axiom).

[Prove it in $L$. Replacement: (i) If $X \in L_\kappa$, then $|X| < \kappa$; (ii) if $Y \subset L_\kappa$ and $|Y| < \kappa$, then $Y \in L_\kappa$.]

**13.23.** If $\kappa$ is inaccessible in $L$, then $L_\kappa = V_\kappa^L = V_\kappa \cap L$ and $L_\kappa$ is a model of $\mathrm{ZFC} + (V = L)$.

**13.24.** If $\delta$ is a limit ordinal, then the model $(L_\delta, \in)$ has definable Skolem functions. Therefore, for every $X \subset L_\delta$, there exists a smallest $M \prec (L_\delta, \in)$ such that $X \subset M$.

[The well-ordering $<_\delta$ is definable in $(L_\delta, \in)$. Let $h_\varphi(x) =$ the $<_\delta$-least $y$ such that $(L_\delta, \in) \models \varphi[x, y]$.]

**13.25.** If $\diamondsuit$ holds, then there exists a family $\mathcal{F}$ of stationary subsets of $\omega_1$ such that $|\mathcal{F}| = 2^{\aleph_1}$ and $|S_1 \cap S_2| \le \aleph_0$ whenever $S_1$ and $S_2$ are distinct elements of $\mathcal{F}$.

[Let $\mathcal{F} = \{S_X : X \subset \omega_1\}$, where $S_X = \{\alpha : X \cap \alpha = S_\alpha\}$.]

**13.26.** If $V = L[A]$ where $A \subset \omega_1$, then $2^{\aleph_0} = \aleph_1$. (Consequently, GCH holds.)

[Show that if $X \subset \omega$, then $X \in L_\alpha[A \cap \xi]$ for some $\alpha < \omega_1$ and $\xi < \omega_1$. It follows that $|P(\omega)| = \aleph_1$.]

**13.27.** For every $X$ there is a set of ordinals $A$ such that $L[X] = L[A]$.

[Let $\bar{X} = X \cap L[X]$, and let $(\theta, E)$ be isomorphic to $\text{TC}(\{\bar{X}\})$ (in $L[X]$). Let $A = \Gamma(E)$ where $\Gamma$ is the canonical mapping of $Ord^2$ onto $Ord$. Then $A \in L[X]$ and $X \in L[A]$, and hence $L[A] = L[X]$.]

**13.28.** Let $\alpha \geq \omega$ be a countable ordinal. There exists $A \subset \omega$ such that $\alpha$ is countable in $L[A]$.

[Let $W \subset \omega \times \omega$ be a well-ordering of $\omega$ of order-type $\alpha$; let $A \subset \omega$ be such that $L[A] = L[W]$.]

**13.29.** If $\omega_1$ (in $V$) is not a limit cardinal in $L$, then there exists $A \subset \omega$ such that $\omega_1 = \omega_1^{L[A]}$.

[There exists $\alpha < \omega_1$ such that in $L$, $\omega_1$ is the successor of $\alpha$. Let $A$ be such that $\alpha$ is countable in $L[A]$.]

**13.30 (ZFC).** There exists $A \subset \omega_1$ such that $\omega_1 = \omega_1^{L[A]}$.

[For each $\alpha < \omega_1$, choose $A_\alpha \subset \omega$ such that $\alpha$ is countable in $L[A_\alpha]$. Let $A \subset \omega_1 \times \omega_1$ be such that $A_\alpha = \{\xi : (\alpha, \xi) \in A\}$ for all $\alpha$; then $\omega_1^{L[A]} = \omega_1$.]

**13.31 (ZFC).** If $\omega_2$ is not inaccessible in $L$, then there exists $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$.

If $A$ is a class, let us define $L[A]$ as in (13.22) where $\text{def}_A(M)$ is defined as in (13.20).

**13.32.** $L[A] = L[\bar{A}]$, where $\bar{A} = A \cap L[A]$, and $L[A]$ is a model of ZFC. Moreover, $L[A]$ is the smallest inner model $M$ such that $V_\alpha^M \cap A \in M$ for all $\alpha$.

**13.33.** Assume that there exists a choice function $F$ on $V$. Then there is a class $A \subset Ord$ such that $V = L[A]$.

**13.34.** Let $M$ be a transitive model of ZF, $M \supset Ord$, and let $X$ be a subset of $M$. Then there is a least model $M[X]$ of ZF such that $M \subset M[X]$ and $X \in M[X]$. If $M \vDash \text{AC}$, then $M[x] \vDash \text{AC}$.

[Modify the construction in (13.24).]

**13.35.** If $X \in OD$, then there exists $\gamma$ such that $X$ is a definable subset of $(V_\gamma, \in)$ (without parameters). Hence $OD$ is the class of all $X$ definable in some $V_\gamma$.

[If $X = \{u \in V_\beta : \varphi^{V_\beta}(u, \alpha)\}$, consider $\gamma = \Gamma(\alpha, \beta)$.]

**13.36.** If $F$ is a definable function on $Ord$, then $\text{ran}(F) \subset OD$. Thus: $OD$ is the largest class for which there exists a definable one-to-one correspondence with the class of all ordinals.

**13.37.** *HOD* is the largest transitive model of ZF for which there exists a definable one-to-one correspondence with the class of all ordinals.

## Historical Notes

The main results, namely consistency of the Axiom of Choice and the Generalized Continuum Hypothesis, are due to Kurt Gödel, as is the concept of constructible sets. The results were announced in [1938], and an outline of proof appeared in [1939]. Gödel's monograph [1940] contains a detailed construction of $L$, and the proof that $L$ satisfies AC and GCH.

In [1939] Gödel defined constructible sets using $L_{\alpha+1}$ = the set of all subsets of $L_\alpha$ definable over $L_\alpha$; in [1940] he used finitely many operations (and worked in the system BG).

The investigation of transitive models of set theory was of course motivated by Gödel's construction of the model $L$. The first systematic study of transitive models was done by Shepherdson in [1951, 1952, 1953]. Bernays in [1937], employed a finite number of operations on classes to give a finite axiomatization of BG. Theorem 13.9 is explicitly stated by Hajnal in [1956].

The $\Sigma_n$ hierarchy was introduced by Lévy in [1965a]. Another result of Lévy [1965b] is that the truth predicate $\models_{n+1}$ is $\Sigma_{n+1}$

Karp's paper [1967] investigates $\Sigma_1$ relations and gives a detailed computation verifying that constructibility is $\Sigma_1$. The characterization of the sets $L_\alpha$ as transitive models of a single sentence $\sigma$ is a result of Boolos [1970].

The Diamond Principle was introduced by Jensen in [1972].

Relative constructibility was investigated by Hajnal [1956], Shoenfield [1959] and most generally by Lévy [1957] and [1960a].

The concept of ordinal definability was suggested by Gödel in his talk in 1946, cf. [1965]; the theory was developed independently by Myhill and Scott in [1971] and by Vopěnka, Balcar, and Hájek in [1968].

Theorem 13.28 is due to Vopěnka and Balcar [1967].