14. Forcing

The method of forcing was introduced by Paul Cohen in his proof of independence of the Continuum Hypothesis and of the Axiom of Choice. Forcing proved to be a remarkably general technique for producing a large number of models and consistency results.

The main idea of forcing is to extend a transitive model M of set theory (the *ground model*) by adjoining a new set G (a *generic set*) in order to obtain a larger transitive model of set theory $M[G]$ called a *generic extension*. The generic set is approximated by *forcing conditions* in the ground model, and a judicious choice of forcing conditions determines what is true in the generic extension.

Cohen's original approach was to start with a countable transitive model M of ZFC (and a particular set of forcing conditions in M). A generic set can easily be proved to exist, and the main result was to show that $M[G]$ is a model of ZFC, and moreover, that the Continuum Hypothesis fails in $M[G]$.

A minor difficulty with this approach is that a countable transitive model need not exist. Its existence is unprovable, by Gödel's Second Incompleteness Theorem. The modern approach to forcing is to let the ground model be the universe V, and pretend that V has a generic extension, i.e., to postulate the existence of a generic set G, for the given set of forcing conditions. As the properties of the generic extension can be described entirely withing the ground model, statements about $V[G]$ can be understood as statements in the ground model using the language of forcing. We shall elaborate on this in due course.

Forcing Conditions and Generic Sets

Let M be a transitive model of ZFC, the *ground model*. In M, let us consider a nonempty partially ordered set $(P, <)$. We call $(P, <)$ a *notion of forcing* and the elements of P *forcing conditions*. We say that p is *stronger* than q if $p < q$. If p and q are conditions and there exists r such that both $r \leq p$ and $r \leq q$, then p and q are *compatible*; otherwise they are *incompatible*. A set $W \subset P$ is an *antichain* if its elements are pairwise incompatible. A set $D \subset P$ is *dense* in P if for every $p \in P$ there is $q \in D$ such that $q \leq p$.

Definition 14.1. A set $F \subset P$ is a *filter* on P if

(14.1) (i) F is nonempty;
(ii) if $p \le q$ and $p \in F$, then $q \in F$;
(iii) if $p, q \in F$, then there exists $r \in F$ such that $r \le p$ and $r \le q$.

A set of conditions $G \subset P$ is *generic* over M if

(14.2) (i) G is a filter on P;
(ii) if D is dense in P and $D \in M$, then $G \cap D \ne \emptyset$.

We also say that G is M-generic, or P-generic (over M), or just *generic*.

Note how genericity depends on the ground model M: What matters is which dense subsets of P are in M. Thus if $\mathcal{D}$ is any collection of sets, let us say that a set $G \subset P$ is a *$\mathcal{D}$-generic filter* on P if it is a filter and if $G \cap D \ne \emptyset$ for every dense subset of P that is in $\mathcal{D}$. Then G is generic over M just in case it is $\mathcal{D}$-generic where $\mathcal{D}$ is the collection of all $D \in M$ dense in P.

Genericity can be described in several equivalent ways. A set $D \subset P$ is *open dense* if it is dense and in addition, $p \in D$ and $q \le p$ imply $q \in D$; D is *predense* if every $p \in P$ is compatible with some $q \in D$. If $p \in P$, then D is dense (open dense, predense, an antichain) *below* p if it is dense (open dense, predense, an antichain) in the set $\{q \in P : q \le p\}$.

If D is either dense or a maximal antichain then D is predense. In Definition 14.1, "dense" in (14.2)(ii) can be replaced by "open dense," "predense," or "a maximal antichain"—see Exercises 14.3, 14.4, and 14.5.

Example 14.2. Let P be the following notion of forcing: The elements of P are finite 0–1 sequences $\langle p(0), \ldots, p(n-1)\rangle$ and a condition p is stronger than q ($p < q$) if p extends q. Clearly, p and q are compatible if either $p \subset q$ or $q \subset p$. Let M be the ground model (note that $(P, <) \in M$), and let $G \subset P$ be generic over M. Let $f = \bigcup G$. Since G is a filter, f is a function. For every $n \in \omega$, the sets $D_n = \{p \in P : n \in \operatorname{dom}(p)\}$ is dense in P, hence it meets G, and so $\operatorname{dom}(f) = \omega$.

The 0–1 function f is the characteristic function of a set $A \subset \omega$. We claim that the function f (or the set A) is not in the ground model. For every 0–1 function g in M, let $D_g = \{p \in P : p \not\subset g\}$. The set D_g is dense, hence it meets G, and it follows that $f \ne g$. □

This example describes the simplest way of adjoining a new set of natural numbers to the ground model. A set $A \subset \omega$ obtained this way is called a *Cohen generic* real.

Except in trivial cases, a generic set does not belong to the ground model; see Exercise 14.6.

Example 14.3. In the ground model M, consider the following partially ordered set P. The elements of P are finite sequences $p = \langle \alpha_0, \ldots, \alpha_{n-1}\rangle$ of countable ordinals (in M), and a condition p is stronger than a condition q ($p < q$) if p extends q. Now if $G \subset P$ is generic over M, we let $f = \bigcup G$. As in Example 14.2, f is a function on ω, and since for every $\alpha < \omega_1^M$, the set $E_\alpha = \{p \in P : \alpha \in \operatorname{ran}(p)\}$ is dense, it follows that $\operatorname{ran}(f) = \omega_1^M$. Thus in any model $N \supset M$ that contains G, the ordinal ω_1^M is countable. □

This example describes the simplest way of *collapsing a cardinal*.

As these examples suggest, a generic set over a transitive model need not exist in general. However, if the ground model is countable, then generic sets do exist. If M is countable and $(P, <) \in M$, then the collection $\mathcal{D}$ of all $D \in M$ that are dense in P is countable and the following lemma applies:

Lemma 14.4. *If $(P, <)$ is a partially ordered set and $\mathcal{D}$ is a countable collection of dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P. In fact, for every $p \in P$ there exists a $\mathcal{D}$-generic filter G on P such that $p \in G$.*

Proof. Let D_1, D_2, ... be the sets in $\mathcal{D}$. Let $p_0 = p$, and for each n, let p_n be such that $p_n \le p_{n-1}$ and $p_n \in D_n$. The set

$$G = \{q \in P : q \ge p_n \text{ for some } n \in \boldsymbol{N}\}$$

is a $\mathcal{D}$-generic filter on P and $p \in G$. □

We shall now state the first of the three main theorems on generic models. We shall prove these theorems (14.5, 14.6, 14.7) later in this chapter.

Theorem 14.5 (The Generic Model Theorem). *Let M be a transitive model of* ZFC *and let $(P, <)$ be a notion of forcing in M. If $G \subset P$ is generic over P, then there exists a transitive model $M[G]$ such that*:

(i) *$M[G]$ is a model of* ZFC;
(ii) *$M \subset M[G]$ and $G \in M[G]$*;
(iii) *$Ord^{M[G]} = Ord^M$*;
(iv) *if N is a transitive model of* ZF *such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.*

The model $M[G]$ is called a *generic extension* of M. The sets in $M[G]$ will be definable from G and finitely many elements of M. Each element of $M[G]$ will have a *name* in M describing how it has been constructed. An important feature of forcing is that the generic model $M[G]$ can be described within the ground model. Associated with the notion of forcing $(P, <)$ is a *forcing language*. This forcing language as well as the *forcing relation* $\Vdash$ are defined in the ground model M. The forcing language contains a *name* for every element of $M[G]$, including a constant $\dot{G}$, the name for a generic set (it is customary to denote names by dotted letters $\dot{a}$). Once we select a generic set G, then every constant of the forcing language is interpreted as an element of the model $M[G]$.

The forcing relation is a relation between the forcing conditions and sentences of the forcing language:

$$p \Vdash \sigma$$

(*p forces* σ). The forcing relation, which is defined in M, is a generalization of the notion of satisfaction. For instance, if $p \Vdash \sigma$ and if σ' is a logical consequence of σ, then $p \Vdash \sigma'$.

The second main theorem on generic models establishes the relation between forcing and truth in $M[G]$:

Theorem 14.6 (The Forcing Theorem). *Let $(P, <)$ be a notion of forcing in the ground model M. If σ is a sentence of the forcing language, then for every $G \subset P$ generic over M,*

$$M[G] \vDash \sigma \quad \textit{if and only if} \quad (\exists p \in G)\, p \Vdash \sigma. \tag{14.3}$$

[In the left-hand-side σ one interprets the constants of the forcing language according to G.]

The third main theorem lists the most important properties of the forcing relation.

Theorem 14.7 (Properties of Forcing). *Let $(P, <)$ be a notion of forcing in the ground model M, and let M^P be the class (in M) of all names.*

(i) (a) *If p forces φ and $q \le p$, then $q \Vdash \varphi$.*
(b) *No p forces both φ and $\neg\varphi$.*
(c) *For every p there is a $q \le p$ such that q* decides *φ, i.e., either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*

(ii) (a) *$p \Vdash \neg\varphi$ if and only if no $q \le p$ forces φ.*
(b) *$p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.*
$p \Vdash \forall x\, \varphi$ if and only if $p \Vdash \varphi(\dot a)$ for every $\dot a \in M^P$.
(c) *$p \Vdash \varphi \vee \psi$ if and only if $\forall q \le p\, \exists r \le q\, (r \Vdash \varphi$ or $r \Vdash \psi)$.*
$p \Vdash \exists x\, \varphi$ if and only if $\forall q \le p\, \exists r \le q\, \exists \dot a \in M^P\; r \Vdash \varphi(\dot a)$.

(iii) *If $p \Vdash \exists x\, \varphi$ then for some $\dot a \in M^P$, $p \Vdash \varphi(\dot a)$.*

Separative Quotients and Complete Boolean Algebras

While the forcing relation can be defined directly from the partial ordering $(P, <)$, it turns out that its properties, and the properties of the generic extension are determined by a certain complete Boolean algebra that can be associated with $(P, <)$. We shall therefore introduce the Boolean algebra $B(P)$ and then use it to define the class M^P (the P-names) and the forcing relation $\Vdash$.

Definition 14.8. A partially ordered set $(P, <)$ is *separative* if for all $p, q \in P$,

(14.4) if $p \not\le q$ then there exists an $r \le p$ that is incompatible with q.

The forcing notions in Examples 14.2 and 14.3 are separative. On the other hand, a linear ordering is not separative (if it has more than one element). Another example of a nonseparative partial order is the set of all infinite subsets of ω, ordered by inclusion.

If B is a Boolean algebra, then $(B^+, <)$ is a separative partial order. A more general statement is true. A set $D \subset P$ is *dense* in a partially ordered set $(P, <)$ if for every $p \in P$ there is a $d \in D$ such that $d \leq p$. A set $D \subset B^+$ is *dense* in a Boolean algebra B if it is dense in $(B^+, <)$. The following lemma is easy to verify:

Lemma 14.9. *If D is a dense subset of a Boolean algebra B, then $(D, <)$ is a separative partial order.* □

Conversely, every separative partial order can be embedded densely in a complete Boolean algebra:

Theorem 14.10. *Let $(P, <)$ be a separative partially ordered set. Then there is a complete algebra B such that*:

(i) *$P \subset B^+$ and $<$ agrees with the partial ordering of B.*
(ii) *P is dense in B.*

The algebra B is unique up to isomorphism.

Proof. The proof is exactly the same as the proof of Theorem 7.13. B is the set of all regular cuts in P and separativity implies that every U_p (where $p \in P$) is regular. □

When $(P, <)$ is not separative, we can replace it by a separative partial order that will produce the same generic extension. This is the consequence of the following lemma:

Lemma 14.11. *Let $(P, <)$ be a partially ordered set. There exists a separative partially ordered set $(Q, \prec)$ and a mapping h of P onto Q such that*

(14.5) (i) *$x \leq y$ implies $h(x) \preccurlyeq h(y)$;*
(ii) *x and y are compatible in P if and only if $h(x)$ and $h(y)$ are compatible in Q.*

Proof. Let us define the following equivalence relation on P:

$x \sim y$ if and only if $\forall z\,(z$ is compatible with x $\leftrightarrow$ z is compatible with y).

Let $Q = P/{\sim}$ and let us define

$$[x] \preccurlyeq [y] \leftrightarrow (\forall z \leq x)[z \text{ and } y \text{ are compatible}].$$

The relation $\preccurlyeq$ on Q is a partial ordering, and it is easy to verify that $(Q, \prec)$ is separative. The mapping $h(x) = [x]$ satisfies (14.5). □

The partial order $(Q, \prec)$ is called the *separative quotient* of $(P, <)$ and is unique (up to isomorphism); see Exercise 14.9.

Corollary 14.12. *For every partially ordered set $(P, <)$ there is a complete Boolean algebra $B = B(P)$ and a mapping $e : P \to B^+$ such that:*

(14.6) (i) *if $p \le q$ then $e(p) \le e(q)$;*
(ii) *p and q are compatible if and only if $e(p) \cdot e(q) \ne 0$;*
(iii) *$\{e(p) : p \in P\}$ is dense in B.*

B is unique up to isomorphism. □

Our earlier statements about the generic extension being determined by $B(P)$ are based on the following facts:

Lemma 14.13. (i) *In the ground model M, let Q be the separative quotient of P and let h map P onto Q such that* (14.5) *holds. If $G \subset P$ is generic over M then $h(G) \subset Q$ is generic over M. Conversely, if $H \subset Q$ is generic over M then $h_{-1}(H) \subset P$ is generic over M.*

(ii) *In the ground model M, let P be a dense subset of a partially ordered set Q. If $G \subset Q$ is generic over M then $G \cap P \subset P$ is generic over M. Conversely, if $H \subset P$ is generic over M then $G = \{q \in Q : (\exists p \in G)\, p \le q\}$ is generic over M.*

Proof. The proof is an exercise in verifying definitions (Exercise 14.1 is useful here). □

As a consequence, if $e : P \to B(P)$ is as in Corollary 14.12 then $G \subset P$ and $H = \{u \in B : \exists p \in G\, e(p) \le u\}$ are definable from each other, and G is generic if and only if H is, and $M[G] = M[H]$. Thus P and $B(P)$ produce the same generic extension.

In the ground model M, let B be a complete Boolean algebra. Outside M, B is still a Boolean algebra, though not necessarily complete. An ultrafilter G on B is called *generic* (over M) if

$$\prod X \in G \text{ whenever } X \in M \text{ and } X \subset G. \tag{14.7}$$

A routine verification (see Exercise 14.10) shows that G is a generic ultrafilter if and only if G is a generic filter on B^+.

Boolean-Valued Models

Let B be a complete Boolean algebra. A *Boolean-valued model* (of the language of set theory) $\mathfrak{A}$ consists of a *Boolean universe* A and functions of two variables with values in B,

$$\|x = y\|, \qquad \|x \in y\| \tag{14.8}$$

(the *Boolean values* of $=$ and $\in$), that satisfy the following:

$$\text{(14.9)}\qquad \begin{aligned} &\text{(i)} && \|x = x\| = 1,\\ &\text{(ii)} && \|x = y\| = \|y = x\|,\\ &\text{(iii)} && \|x = y\| \cdot \|y = z\| \le \|x = z\|,\\ &\text{(iv)} && \|x \in y\| \cdot \|v = x\| \cdot \|w = y\| \le \|v \in w\|. \end{aligned}$$

For every formula $\varphi(x_1, \ldots, x_n)$, we define the *Boolean value* of φ

$$\|\varphi(a_1, \ldots, a_n)\| \qquad (a_1, \ldots, a_n \in A)$$

as follows:

(a) For atomic formulas, we have (14.8).

(b) If φ is a negation, conjunction, etc.,

$$\begin{aligned} \|\neg\psi(a_1, \ldots, a_n)\| &= -\|\psi(a_1, \ldots, a_n)\|,\\ \|(\psi \wedge \chi)(a_1, \ldots, a_n)\| &= \|\psi(a_1, \ldots, a_n)\| \cdot \|\chi(a_1, \ldots, a_n)\|,\\ \|(\psi \vee \chi)(a_1, \ldots, a_n)\| &= \|\psi(a_1, \ldots, a_n)\| + \|\chi(a_1, \ldots, a_n)\|,\\ \|(\psi \to \chi)(a_1, \ldots, a_n)\| &= \|(\neg\psi \vee \chi)(a_1, \ldots, a_n)\|,\\ \|(\psi \leftrightarrow \chi)(a_1, \ldots, a_n)\| &= \|((\psi \to \chi) \wedge (\chi \to \psi))(a_1, \ldots, a_n)\|. \end{aligned}$$

(c) If φ is $\exists x\, \psi$ or $\forall x\, \psi$,

$$\begin{aligned} \|\exists x\, \psi(x, a_1, \ldots, a_n)\| &= \sum_{a \in A} \|\psi(a, a_1, \ldots, a_n)\|,\\ \|\forall x\, \psi(x, a_1, \ldots, a_n)\| &= \prod_{a \in A} \|\psi(a, a_1, \ldots, a_n)\|. \end{aligned}$$

Note how the notion of a Boolean-valued model generalizes the notion of a model; the Boolean value of φ is a generalization of the satisfaction predicate $\vDash$. If B is the trivial algebra $\{0, 1\}$, then a Boolean-valued model is just a (two-valued) model; i.e., consider $A/{\equiv}$ where $x \equiv y$ if and only if $\|x = y\| = 1$.

We say that $\varphi(a_1, \ldots, a_n)$ is *valid in* $\mathfrak{A}$, if $\|\varphi(a_1, \ldots, a_n)\| = 1$. An implication $\varphi \to \psi$ is valid if $\|\varphi\| \le \|\psi\|$. Hence it is postulated in (14.9) that the axioms for the equality predicate $=$ are valid in a Boolean-valued model. It can be easily verified that all the other axioms of predicate calculus are valid, and that the rules of inference applied to valid sentences result in valid sentences. Thus every sentence provable in predicate calculus has Boolean value 1, and if two formulas φ, ψ are provably equivalent, we have $\|\varphi\| = \|\psi\|$. For example, we have

$$\|x = y\| \cdot \|\varphi(x)\| \le \|\varphi(y)\|.$$

Boolean-valued models can therefore be used in consistency proofs in much the same way as two-valued models. Let $\mathfrak{A}$ be a Boolean-valued model such that all the axioms of ZFC are valid in $\mathfrak{A}$. (We say that $\mathfrak{A}$ is a Boolean-valued model of ZFC.) Let σ be a set-theoretical statement and assume that $\|\sigma\| \neq 0$. Then we can conclude that σ is consistent relative to ZFC; otherwise, $\neg\sigma$ would be provable in ZFC and therefore valid in $\mathfrak{A}$: $\|\neg\sigma\| = -\|\sigma\| = 1$.

There is an important special case of Boolean-valued models, and in this special case, the Boolean-valued model can be transformed into a two-valued model.

We say that a Boolean-valued model $\mathfrak{A}$ is *full* if for any formula $\varphi(x, x_1, \ldots, x_n)$ the following holds: For all $a_1, \ldots, a_n \in A$, there exists an $a \in A$ such that

$$\|\varphi(a, a_1, \ldots, a_n)\| = \|\exists x\, \varphi(x, a_1, \ldots, a_n)\|. \tag{14.10}$$

Let F be an ultrafilter on B. We define an equivalence relation on A by

$$x \equiv y \quad \text{if and only if} \quad \|x = y\| \in F, \tag{14.11}$$

and a binary relation E on $A/{\equiv}$ by

$$[x]\ E\ [y] \quad \text{if and only if} \quad \|x \in y\| \in F. \tag{14.12}$$

That $\equiv$ is an equivalence relation, and that (14.12) does not depend on the choice of representatives are easy consequences of (14.9) and the fact that F is a filter. Thus $\mathfrak{A}/F = (A/{\equiv}, E)$ is a model. Moreover, we have the following relationship between the Boolean-valued model $\mathfrak{A}$ and the model $\mathfrak{A}/F$:

Lemma 14.14. *Let $\mathfrak{A}$ be full. For any formula $\varphi(x_1, \ldots, x_n)$,*

$$\mathfrak{A}/F \vDash \varphi([a_1], \ldots, [a_n]) \quad \textit{if and only if} \quad \|\varphi(a_1, \ldots, a_n)\| \in F, \tag{14.13}$$

for all $a_1, \ldots, a_n \in A$.

Proof. (a) If φ is atomic, then (14.13) is true by definition.

(b) If φ is a negation, conjunction, etc., we use the basic properties of an ultrafilter, and the definition of $\|\ \|$; e.g., we use

$$\begin{aligned} \|\neg\psi\| \in F \quad &\text{if and only if} \quad \|\psi\| \notin F, \\ \|\psi \wedge \chi\| \in F \quad &\text{if and only if} \quad \|\psi\| \in F \text{ and } \|\chi\| \in F. \end{aligned}$$

(c) If φ is $\exists x\, \psi(x, \ldots)$, we use the fullness of $\mathfrak{A}$ to prove (14.13), assuming it holds for ψ. By (14.10), we pick some $a \in A$ such that $\|\varphi(a, \ldots)\| = \|\exists x\, \varphi(x, \ldots)\|$ and then we have

$$\|\exists x\, \varphi(x, \ldots)\| \in F \quad \text{if and only if} \quad (\exists a \in A)\, \|\varphi(a, \ldots)\| \in F,$$

which enables us to do the induction step in this case. $\square$

The Boolean-Valued Model V^B

We now define the Boolean-valued model V^B. Let B be a complete Boolean algebra.

Our intention is to define a Boolean-valued model in which all the axioms of ZFC are valid. In particular, we want V^B to be *extensional*, i.e., the Axiom of Extensionality to be valid in V^B:

$$\|\forall u\,(u \in X \leftrightarrow u \in Y)\| \le \|X = Y\|. \tag{14.14}$$

We shall define V^B as a generalization of V: Instead of (two-valued) sets, we consider "Boolean-valued" sets, i.e., functions that assign Boolean values to its "elements." Thus we define V^B as follows:

(14.15)
(i) $V_0^B = \emptyset$,
(ii) $V_{\alpha+1}^B =$ the set of all functions x with $\operatorname{dom}(x) \subset V_\alpha^B$ and values in B,
$V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$ if α is a limit ordinal, and
(iii) $V^B = \bigcup_{\alpha \in Ord} V_\alpha^B$.

The definition of $\|x \in y\|$ and $\|x = y\|$ is motivated by (14.14), and the requirement that $x(t) \le \|t \in x\|$. We define Boolean values by induction. Each $x \in V^B$ is assigned the rank in V^B,

$$\rho(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}^B.$$

The forthcoming definition is by induction on pairs $(\rho(x), \rho(y))$, under the canonical well-ordering.

To make the notation more suggestive, we introduce the following Boolean operation that corresponds to the implication:

$$u \Rightarrow v = -u + v$$

Let

(14.16)
(i) $\|x \in y\| = \sum_{t \in \operatorname{dom} y} (\|x = t\| \cdot y(t))$,
(ii) $\|x \subset y\| = \prod_{t \in \operatorname{dom} x} (x(t) \Rightarrow \|t \in y\|)$, and
(iii) $\|x = y\| = \|x \subset y\| \cdot \|y \subset x\|$.

We are going to show that V^B is a Boolean-valued model. To do that, we have to verify (14.9). Clause (ii) in (14.9) is trivially satisfied since the definition of $\|x = y\|$ is symmetric in x and y.

Lemma 14.15. $\|x = x\| = 1$ *for all* $x \in V^B$.

Proof. By induction on $\rho(x)$. Clearly, it suffices to show that $\|x \subset x\| = 1$, i.e., we wish to show that $x(t) \Rightarrow \|t \in x\| = 1$ for all $t \in \operatorname{dom}(x)$, or equivalently, that $x(t) \le \|t \in x\|$. If $t \in \operatorname{dom}(x)$, then by the induction hypothesis we have $\|t = t\| = 1$ and hence, by definition of $\|t \in x\|$, $x(t) = \|t = t\| \cdot x(t) \le \|t \in x\|$. □

Now we prove (14.9)(iii) and (iv), simultaneously by induction:

Lemma 14.16. *For all $x, y, z \in V^B$,*

(i) $\|x = y\| \cdot \|y = z\| \le \|x = z\|$,
(ii) $\|x \in y\| \cdot \|x = z\| \le \|z \in y\|$,
(iii) $\|y \in x\| \cdot \|x = z\| \le \|y \in z\|$.

Proof. By induction on triples $\{\rho(x), \rho(y), \rho(z)\}$.

(i) It suffices to prove that $\|x \subset y\| \cdot \|y = z\| \le \|x \subset z\|$. Let $t \in \operatorname{dom}(x)$ be arbitrary; we wish to show that

$$\|y = z\| \cdot (x(t) \Rightarrow \|t \in y\|) \le x(t) \Rightarrow \|t \in z\| \tag{14.17}$$

(using the definition of $\|x \subset z\|$). By the induction hypothesis, we have $\|t \in y\| \cdot \|y = z\| \le \|t \in z\|$. Thus $\|y = z\| \cdot (-x(t) + \|t \in y\|) = (\|y = z\| - x(t)) + (\|y = z\| \cdot \|t \in y\| \le -x(t) + \|t \in z\|$, and (14.17) follows.

(ii) Let $t \in \operatorname{dom}(y)$ be arbitrary. By the induction hypothesis we have $\|x = z\| \cdot \|x = t\| \le \|z = t\|$ and so

$$\|x = z\| \cdot \|x = t\| \cdot y(t) \le \|z = t\| \cdot y(t). \tag{14.18}$$

Taking the sum of (14.18) over all $t \in \operatorname{dom}(y)$, we get

$$\|x = z\| \cdot \sum_{t \in \operatorname{dom} y} (\|x = t\| \cdot y(t)) \le \sum_{t \in \operatorname{dom} y} (\|z = t\| \cdot y(t)),$$

that is, $\|x = z\| \cdot \|x \in y\| \le \|z \in y\|$.

(iii) Let $t \in \operatorname{dom}(x)$. By the definition of $\|x = z\|$ we have $x(t) \cdot \|x = z\| \le \|t \in z\|$ and so

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \le \|y = t\| \cdot \|t \in z\|.$$

By the induction hypothesis, $\|y = t\| \cdot \|t \in z\| \le \|y \in z\|$, and therefore

$$\|y = t\| \cdot x(t) \cdot \|x = z\| \le \|y \in z\|. \tag{14.19}$$

Taking the sum of the left-hand side of (14.19) over all $t \in \operatorname{dom}(x)$, we get

$$\sum_{t \in \operatorname{dom} x} (\|y = t\| \cdot x(t)) \cdot \|x = z\| \le \|y \in z\|,$$

that is, $\|y \in x\| \cdot \|x = z\| \le \|y \in z\|$. □

Thus V^B is a Boolean-valued model. We will show that all axioms of ZFC are valid in V^B. First we show that V^B is extensional, and full.

Lemma 14.17. *V^B is extensional.* $\square$

Proof. Let $X, Y \in V^B$. By the definition of $a \Rightarrow b$ we observe that if $a \le a'$, then $(a' \Rightarrow b) \le (a \Rightarrow b)$. Thus for any $u \in V^B$ we have $(\|u \in X\| \Rightarrow \|u \in Y\|) \le (X(u) \Rightarrow \|u \in Y\|)$ and therefore

$$\prod_{u \in V^B} (\|u \in X\| \Rightarrow \|u \in Y\|) \le \prod_{u \in V^B} (X(u) \Rightarrow \|u \in Y\|). \tag{14.20}$$

While the left-hand side of (14.20) is equal to $\|\forall u\, (u \in X \to u \in Y)\|$, the right-hand side is easily seen to equal $\|X \subset Y\|$. Consequently,

$$\|\forall u\, (u \in X \leftrightarrow u \in Y)\| \le \|X = Y\|. \qquad \square$$

Lemma 14.18. *If W is a set of pairwise disjoint elements of B and if a_u, $u \in W$, are elements of V^B, then there exists some $a \in V^B$ such that $u \le \|a = a_u\|$ for all $u \in W$.*

Proof. Let $D = \bigcup_{u \in W} \operatorname{dom}(a_u)$, and for every $t \in D$, let $a(t) = \sum\{u \cdot a_u(t) : u \in W\}$. Since the u's are pairwise disjoint, we have $u \cdot a(t) = u \cdot a_u(t)$ for each $u \in W$ and each $t \in D$. In other words, $u \le (a(t) \Rightarrow a_u(t))$ and $u \le (a_u(t) \Rightarrow a(t))$, and so $u \le \|a = a_u\|$. $\square$

Lemma 14.19. *V^B is full. Given a formula $\varphi(x, \ldots)$, there exists some $a \in V^B$ such that* (14.10) *holds, i.e.,*

$$\|\varphi(a, \ldots)\| = \|\exists x\, \varphi(x, \ldots)\|.$$

Proof. In (14.10), $\le$ holds for every a. We wish to find an $a \in V^B$ such that $\ge$ holds. Let $u_0 = \|\exists x\, \varphi(x, \ldots)\|$. Let

$$D = \{u \in B : \text{there is some } a_u \text{ such that } u \le \|\varphi(a_u, \ldots)\|\}.$$

It is clear that D is open and dense below u_0. Let W be a maximal set of pairwise disjoint elements of D; clearly, $\sum\{u : u \in W\} \ge u_0$. By Lemma 14.18 there exists some $a \in V^B$ such that $u \le \|a = a_u\|$ for all $u \in W$. Thus for each $u \in W$ we have $u \le \|\varphi(a, \ldots)\|$, and hence $u_0 \le \|\varphi(a, \ldots)\|$. $\square$

We remark that Lemma 14.19 was the only place in this chapter where we used the Axiom of Choice.

Every set (in V) has a canonical name in the Boolean-valued model V^B:

Definition 14.20 (By $\in$-Induction).

(i) $\check{\emptyset} = \emptyset$;

(ii) for every $x \in V$, let $\check{x} \in V^B$ be the function whose domain is the set $\{\check{y} : y \in x\}$, and for all $y \in x$, $\check{x}(\check{y}) = 1$.

When calculating the Boolean value of a formula, one may find the following observation helpful (cf. Exercise 14.12):

$$\|(\exists y \in x)\,\varphi(y)\| = \sum_{y \in \operatorname{dom} x} (x(y) \cdot \|\varphi(y)\|), \tag{14.21}$$
$$\|(\forall y \in x)\,\varphi(y)\| = \prod_{y \in \operatorname{dom} x} (x(y) \Rightarrow \|\varphi(y)\|).$$

The following lemma is the Boolean-valued version of absoluteness of Δ_0 formulas:

Lemma 14.21. *If $\varphi(x_1, \ldots, x_n)$ is a Δ_0 formula, then*

$$\varphi(x_1, \ldots, x_n) \quad \textit{if and only if} \quad \|\varphi(\check{x}_1, \ldots, \check{x}_n)\| = 1.$$

Proof. By induction on the complexity of φ. □

Corollary 14.22. *If φ is Σ_1, then $\varphi(x, \ldots)$ implies $\|\varphi(\check{x}, \ldots)\| = 1$.* □

The next lemma states that V and V^B "have the same ordinals:"

Lemma 14.23. *For every $x \in V^B$,*

$$\|x \text{ is an ordinal}\| = \sum_{\alpha \in Ord} \|x = \check{\alpha}\|.$$

Proof. Since "x is an ordinal" is Δ_0, we have, by Lemma 14.21,

$$\sum_{\alpha \in Ord} \|x = \check{\alpha}\| \le \|x \text{ is an ordinal}\|.$$

On the other hand, let $\|x \text{ is an ordinal}\| = u$. We first observe that if γ is an ordinal, then

$$\|x \text{ is an ordinal and } x \in \check{\gamma}\| \le \sum_{\alpha \in \gamma} \|x = \check{\alpha}\|.$$

Also, for every α, we have

$$u \le \|x \in \check{\alpha}\| + \|x = \check{\alpha}\| + \|\check{\alpha} \in x\|.$$

However, there is only a set of α's such that $\|\check{\alpha} \in x\| \ne 0$ (because $\|\check{\alpha} \in x\| = \sum_{t \in \operatorname{dom} x}(\|\check{\alpha} = t\| \cdot x(t))$). Hence there is γ such that $u \le \|x \subset \check{\gamma}\|$ and we have $u \le \sum_{\alpha \le \gamma} \|x = \check{\alpha}\|$. □

We show now that V^B is a Boolean-valued model of ZFC.

Theorem 14.24. *Every axiom of* ZFC *is valid in V^B.*

Proof. We show that $\|\sigma\| = 1$ for every axiom of ZFC.

Extensionality. See Lemma 14.17.

Pairing. Given $a, b \in V^B$, let $c = \{a, b\}^B \in V^B$ be such that $\mathrm{dom}(c) = \{a, b\}$ and $c(a) = c(b) = 1$. Then $\|a \in c \wedge b \in c\| = 1$. This, combined with Separation, suffices for the Pairing Axiom. (We could also verify directly that $\|\forall x \in c\,(x = a \vee x = b)\| = 1$.)

Separation. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|Y \subset X\| = 1 \quad \text{and} \quad \|(\forall z \in X)(\varphi(z) \leftrightarrow z \in Y)\| = 1. \tag{14.22}$$

Let $Y \in V^B$ be as follows:

$$\mathrm{dom}(Y) = \mathrm{dom}(X), \qquad Y(t) = X(t) \cdot \|\varphi(t)\|.$$

For every $x \in V^B$ we have $\|x \in Y\| = \|x \in X\| \cdot \|\varphi(x)\|$ and this gives (14.22).

Union. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\forall v \in u)(v \in Y)\| = 1 \tag{14.23}$$

(this is the weak version, cf. (1.8)).

If $X \in V^B$, then letting $Y \in V^B$ as follows verifies (14.23):

$$\mathrm{dom}(Y) = \bigcup\{\mathrm{dom}(u) : u \in \mathrm{dom}(X)\}, \qquad Y(t) = 1 \quad \text{for all } t \in \mathrm{dom}(Y).$$

Power Set. We prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|\forall u\,(u \subset X \to u \in Y)\| = 1; \tag{14.24}$$

(cf. (1.9)). Here we let

$$\mathrm{dom}(Y) = \{u \in V^B : \mathrm{dom}(u) = \mathrm{dom}(X) \text{ and } u(t) \le X(t) \text{ for all } t\},$$
$$Y(u) = 1 \quad \text{for all } u \in \mathrm{dom}(Y).$$

To verify that Y satisfies (14.24) we use the following observation: If $u \in V^B$ is arbitrary, let $u' \in V^B$ be such that $\mathrm{dom}(u') = \mathrm{dom}(X)$ and $u'(t) = X(t) \cdot \|t \in u\|$ for all $t \in \mathrm{dom}(X)$. Then

$$\|u \subset X\| \le \|u = u'\|$$

which makes it possible to include in $\mathrm{dom}(Y)$ only the "representative" u's.

Infinity. See Lemma 14.21 for $\|\check{\omega} \text{ is an inductive set}\| = 1$.

Replacement. It suffices to verify the Collection Principle, cf. (6.5); we prove that for every $X \in V^B$ there is $Y \in V^B$ such that

$$\|(\forall u \in X)(\exists v\,\varphi(u, v) \to (\exists v \in Y)\,\varphi(u, v))\| = 1. \tag{14.25}$$

Here we let

$$\mathrm{dom}(Y) = \bigcup\{S_u : u \in \mathrm{dom}(X)\}, \qquad Y(t) = 1 \quad \text{for all } t \in \mathrm{dom}(Y),$$

where $S_u \subset V^B$ is some set such that

$$\sum_{v \in V^B} \|\varphi(u, v)\| = \sum_{v \in S_u} \|\varphi(u, v)\|.$$

Regularity. We prove that for every $X \in V^B$,

$$\|X \text{ is nonempty} \rightarrow (\exists y \in X)(\forall z \in y)\, z \notin X\| = 1. \tag{14.26}$$

If (14.26) is false, then

$$\|\exists u\,(u \in X) \wedge (\forall y \in X)(\exists z \in y)\, z \in X\| = b \neq 0.$$

Let $y \in V^B$ be of least $\rho(y)$ such that $\|y \in X\| \cdot b \neq 0$. Then $\|y \in X\| \cdot b \leq \|(\exists z \in y)\, z \in X\|$, so there exists a $z \in \operatorname{dom}(y)$ such that $\|z \in X\| \cdot \|y \in X\| \cdot b \neq 0$. Since $\rho(z) < \rho(y)$, this is a contradiction.

Choice. For every S, we have (by Corollary 14.22)

$$\|\check{S} \text{ can be well-ordered}\| = 1.$$

Now, we prove that for every $X \in V^B$ there exist some S and $f \in V^B$ such that

$$\|f \text{ is a function on } \check{S} \text{ and } \operatorname{ran}(f) \supset X\| = 1. \tag{14.27}$$

(This shows that $\|X$ can be well-ordered$\| = 1$.)

We let $S = \operatorname{dom}(X)$ and $f \in V^B$ as follows:

$$\operatorname{dom}(f) = \{(\check{x}, x)^B : x \in S\}, \qquad f(t) = 1 \quad \text{for all } t \in \operatorname{dom}(f)$$

(where $(a,b)^B = \{\{a\}^B, \{a,b\}^B\}^B$). These S and f satisfy (14.27). $\square$

Among elements of V^B, one is of particular significance: the canonical name for a generic ultrafilter on B:

Definition 14.25. The *canonical name* $\dot{G}$ for a generic ultrafilter is the Boolean-valued function defined by

$$\operatorname{dom}(\dot{G}) = \{\check{u} : u \in B\}, \qquad \dot{G}(\check{u}) = u \quad \text{for every } u \in B.$$

See Exercise 14.14.

The Forcing Relation

Let M be a transitive model of ZFC (the *ground model*) and let $(P, <) \in M$ be a notion of forcing. We shall now introduce the forcing language by specifying *names*, define the forcing relation $\Vdash$ and prove the fundamental properties of $\Vdash$ (Theorem 14.7). Throughout this section we work inside the ground model.

Let $(P, <)$ be a notion of forcing. By Corollary 14.12 there exists a complete Boolean algebra $B = B(P)$ such that P embeds in B by a mapping $e : P \rightarrow B$ that satisfies (14.6) (and is not one-to-one if P is not separative). We use M^B to denote the B-valued model defined in (14.15) (inside M).

Definition 14.26. $M^P = M^{B(P)}$. The elements of M^P are called *P-names* (or just *names*). P-names are usually denoted by dotted letters. The *forcing language* is the language of set theory with names added as constants. The forcing relation $\Vdash_P$ (or just $\Vdash$) is defined by

$$p \Vdash \varphi(\dot{a}_1, \ldots, \dot{a}_n) \quad \text{if and only if} \quad e(p) \leq \|\varphi(\dot{a}_1, \ldots, \dot{a}_n)\|$$

where φ is a formula of set theory and $\dot{a}_1, \ldots, \dot{a}_n$ are names.

We remark that both names and the forcing relation can be defined directly from P without using the complete Boolean algebra. However, we find the direct definition somewhat less intuitive.

Proof of Theorem 14.7. (i) (a) If $q \leq p$ then $e(q) \leq e(p)$.
 (b) $\|\varphi\| \cdot \|\neg\varphi\| = 0$.
 (c) If $e(p) \cdot \|\varphi\| \neq 0$ then there is a $q \leq p$ such that $e(q) \leq \|\varphi\|$; similarly if $e(p) \cdot \|\neg\varphi\| \neq 0$.
(ii) (a) Left-to-right: Use (i)(a) and (b). Right-to-left: If p does not force $\neg\varphi$ then $e(p) \cdot \|\varphi\| \neq 0$ and proceed as in (i)(c).
 (b) By (14.9)(b) and (c).
 (c) For disjunction, we use $\|\varphi \vee \psi\| = \|\varphi\| + \|\psi\|$ and argue as in (ii)(a). The existential quantifier is similar, using (14.9)(c).
(iii) By Lemma 14.19, M^B is full and so $e(p) \leq \|\varphi(\dot{a})\|$ for some $\dot{a}$. □

Among P-names there are canonical names $\check{x}$ for sets in the ground model. In practice one often abuses the notation by dropping the háček ˇ and confusing $x \in M$ with its name $\check{x}$.

We can also introduce a "name for M;" since $a \in M \leftrightarrow (\exists x \in M)\, a = x$, we define

$$p \Vdash \dot{a} \in \check{M} \quad \text{if and only if} \quad \forall q \leq p\, \exists r \leq q\, \exists x\, (r \Vdash \dot{a} = \check{x}). \tag{14.28}$$

Finally, we consider the canonical name for a generic filter on P. Using Definition 14.25 for $B(P)$ and the relation between generic filters on P and generic ultrafilters on $B(P)$ spelled out in Lemma 14.13, we arrive at the following definition:

$$p \Vdash q \in \dot{G} \quad \text{if and only if} \quad \forall r \leq p\, \exists s \leq r\;\, s \leq q, \tag{14.29}$$

or in terms of the separative quotient mapping h (Lemma 14.11),

$$p \Vdash q \in \dot{G} \quad \text{if and only if} \quad h(p) \preccurlyeq h(q).$$

One final remark: By Theorem 14.24, every axiom of ZFC is forced by every condition. So is every axiom of predicate calculus, and the forcing relation is preserved by the rules of inference. Hence every condition forces every sentence provable in ZFC.

The Forcing Theorem and the Generic Model Theorem

We shall now define the generic extension $M[G]$ and prove Theorems 14.5 and 14.6. We do it first for Boolean-valued models and handle the general case afterward.

Let M be a generic transitive model of ZFC, and let B be a complete Boolean algebra in M. Let G be an M-*generic* ultrafilter on B, i.e., generic over M.

Definition 14.27 (Interpretation by G). For every $x \in M^B$ we define x^G by induction on $\rho(x)$:

(i) $\emptyset^G = \emptyset$,
(ii) $x^G = \{y^G : x(y) \in G\}$.

Using the interpretation by G, we let

$$M[G] = \{x^G : x \in M^B\}. \tag{14.30}$$

Lemma 14.28. *Let G be an M-generic ultrafilter on B. Then for all names $x, y \in M^B$*

(i) $x^G \in y^G$ *if and only if* $\|x \in y\| \in G$,
(ii) $x^G = y^G$ *if and only if* $\|x = y\| \in G$.

Proof. We prove (i) and (ii) simultaneously, by induction on pairs $(\rho(x), \rho(y))$.

$$\begin{aligned}
\text{(i)}\quad \|x \in y\| \in G &\leftrightarrow \exists t \in \operatorname{dom}(y)\,(y(t) \in G \text{ and } \|x = t\| \in G)\\
&\leftrightarrow \exists t\,(y(t) \in G \text{ and } x^G = t^G)\\
&\leftrightarrow x^G \in \{t^G : y(t) \in G\}\\
&\leftrightarrow x^G \in y^G.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad \|x \subset y\| \in G &\leftrightarrow \textstyle\prod_{t \in \operatorname{dom} x}(x(t) \Rightarrow \|t \in y\|) \in G\\
&\leftrightarrow \forall t \in \operatorname{dom}(x)\,(x(t) \in G \text{ implies } \|t \in y\| \in G)\\
&\leftrightarrow \forall t\,(x(t) \in G \text{ implies } t^G \in y^G)\\
&\leftrightarrow \{t^G : x(t) \in G\} \subset y^G\\
&\leftrightarrow x^G \subset y^G.
\end{aligned}$$

□

$M[G]$ is a transitive class. The following is the Forcing Theorem for Boolean-valued models.

Theorem 14.29. *If G is an M-generic ultrafilter on B, then for all $x_1, \ldots, x_n \in M^B$,*

$$M[G] \vDash \varphi(x_1^G, \ldots, x_n^G) \quad \textit{if and only if} \quad \|\varphi(x_1, \ldots, x_n)\| \in G. \tag{14.31}$$

Proof. Lemma 14.28 proves (14.31) for atomic formulas. The rest of the proof is by induction on the complexity of φ.

(a) φ is $\neg\psi$, $\psi\wedge\chi$, $\psi\vee\chi$, etc. Assuming (14.31) for ψ and χ, the induction step works because G is an ultrafilter. For instance,

$$\begin{aligned} M[G] \vDash \psi \wedge \chi &\leftrightarrow M[G] \vDash \psi \text{ and } M[G] \vDash \chi \\ &\leftrightarrow \|\psi\| \in G \text{ and } \|\chi\| \in G \\ &\leftrightarrow \|\psi\| \cdot \|\chi\| \in G \\ &\leftrightarrow \|\psi \wedge \chi\| \in G. \end{aligned}$$

Similarly for $\neg$, $\vee$, etc.

(b) φ is $\exists x\, \psi(x,\ldots)$ or $\forall x\, \psi(x,\ldots)$. We assume (14.31) for ψ and use the genericity of G:

$$\begin{aligned} M[G] \vDash \exists x\, \psi(x,\ldots) &\leftrightarrow (\exists x \in M[G])\, M[G] \vDash \psi(x,\ldots) \\ &\leftrightarrow (\exists x \in M^B)\, M[G] \vDash \psi(x^G,\ldots) \\ &\leftrightarrow (\exists x \in M^B)\, \|\psi(x,\ldots)\| \in G \\ &\leftrightarrow \sum_{x\in M^B} \|\psi(x,\ldots)\| \in G \\ &\leftrightarrow \|\exists x\, \psi(x,\ldots)\| \in G. \end{aligned}$$

The penultimate equivalence holds because if we let $A = \{\|\psi(x,\ldots)\| : x \in M^B\}$, then $A \subset B$ and $A \in M$, and since G is generic we have

$$(\exists a \in A)\, a \in G \quad \text{if and only if} \quad \sum A \in G.$$

Similarly for $\forall x\, \psi(x,\ldots)$. ☐

Corollary 14.30. *$M[G]$ is a model of* ZFC.

Proof. By Theorem 14.24, every axiom σ of ZFC is valid in M^B, therefore $\|\sigma\| = 1 \in G$ and hence σ is true in $M[G]$. ☐

The following completes the proof of both Theorems 14.5 and 14.6 when forcing with a complete Boolean algebra:

Lemma 14.31.

(i) *$M \subset M[G]$, and both models have the same ordinals.*
(ii) *$G \in M[G]$ and if $N \supset M$ is a transitive model of* ZFC *such that $G \in N$, then $N \supset M[G]$.*

Proof. (i) For every $x \in M$, the G-interpretation of the canonical name $\check{x}$ is $\check{x}^G = x$ (proved by $\in$-induction). Hence $M \subset M[G]$. To show that every ordinal in $M[G]$ is in M (that $M[G]$ is not "longer" than M), we use Lemma 14.23.

(ii) Let $\dot{G}$ be the canonical name of a generic ultrafilter (Definition 14.25). Its interpretation is $\dot{G}^G = G$ and so $G \in M[G]$. If $N \supset M$ is a transitive model containing G, then the construction of $M[G]$ can be carried out inside N, and thus $M[G] \subset N$. □

We shall now prove Theorems 14.5 and 14.6:

Let $(P, <)$ be a notion of forcing in the ground model M, and let $G \subset P$ be generic over M. Let $B = B(P)$, and let $M^P = M^B$ be the class of the P-names. First we define *G-interpretation* of P-names: For every $x \in M^P$,

$$\text{(14.32)} \qquad \begin{aligned} &\text{(i)}\ \ \emptyset^G = \emptyset, \\ &\text{(ii)}\ \ x^G = \{y^G : (\exists p \in G)\, e(p) \le x(y)\}. \end{aligned}$$

Then we let

$$M[G] = \{x^G : x \in M^P\}.$$

Now let H be the ultrafilter on B generated by $e(G)$: $H = \{u \in B : \exists p \in G\, e(p) \le u\}$. H is M-generic, and it is easily seen that $x^G = x^H$ for all $x \in M^B$. Thus $M[G] = M[H]$.

The Forcing Theorem now follows from the definition of $\Vdash$ and Theorem 14.29. As for the Generic Model Theorem 14.5, (a), (c), (d), and the first part of (b) are immediate consequences of Lemma 14.31; it only remains to verify that $G \in M[G]$. For that, we can either observe that $G = \{p \in P : e(p) \in H\}$ is in $M[H]$, or invoke (14.29) and verify that $\dot{G}^G = G$.

Consistency Proofs

Forcing is used mainly (but not exclusively) in consistency proofs. In practice, a consistency result is usually presented as follows: Suppose that A is some sentence (in the language of set theory) and we wish to prove that A is consistent with ZFC, or more generally, that A is consistent with some extension T of ZFC. This is accomplished by assuming that T holds (in V, the universe) and by exhibiting a forcing notion P such that the generic extension $V[G]$ satisfies A.

One way to make this argument legitimate is to assume that there exists a countable transitive model M of T. Using a forcing notion $P \in M$, there exists a P-generic filter G over M, and $M[G]$ is a transitive model that satisfies A. Hence A is consistent relative to T.

The assumption of a countable transitive model is unnecessary, as statements about generic extensions can be considered merely as an informal reformulation of statements about the forcing relation. In particular, "$V[G]$ satisfies A" is to be understood to mean "every $p \in P$ forces A." Then (assuming that T is consistent), the negation $\neg$A is not provable: If it were then every condition would force $\neg$A (or, the Boolean value $\|\neg\text{A}\|$ would be 1). Note that for consistency of A, it is enough to show that *some* $p \in P$ forces A; in the language of generic extensions, one finds a $p \in P$ such that when G is generic and $p \in G$, then $V[G] \vDash$ A.

In some cases, forcing results are stated as *independence* results: A sentence A is *independent* of the axioms T. This usually means that both A and ¬A are consistent with T.

Independence of the Continuum Hypothesis

We now present Cohen's proof of independence of CH.

Theorem 14.32. *There is a generic extension $V[G]$ that satisfies $2^{\aleph_0} > \aleph_1$.*

Proof. We describe the notion of forcing that produces a generic extension with the desired property. Let P be the set of all functions p such that

(14.33) (i) $\operatorname{dom}(p)$ is a finite subset of $\omega_2 \times \omega$,
(ii) $\operatorname{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

If G is a generic set of conditions, we let $f = \bigcup G$. We claim that

(14.34) (i) f is a function;
(ii) $\operatorname{dom}(f) = \omega_2 \times \omega$.

(Of course, ω_2 means ω_2 in the ground model.)

Part (i) of (14.34) holds because G is a filter. For part (ii), the sets $D_{\alpha,n} = \{p \in P : (\alpha, n) \in \operatorname{dom}(p)\}$ are dense in P, hence G meets each of them, and so $(\alpha, n) \in \operatorname{dom}(f)$ for all $(\alpha, n) \in \omega_2 \times \omega$.

Now, for each $\alpha < \omega_2$, let $f_\alpha : \omega \to \{0, 1\}$ be the function defined as follows:

$$f_\alpha(n) = f(\alpha, n).$$

If $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$; this is because the set

$$D = \{p \in P : p(\alpha, n) \neq p(\beta, n) \text{ for some } n\}$$

is dense in P and hence $G \cap D \neq \emptyset$. Thus in $V[G]$ we have a one-to-one mapping $\alpha \mapsto f_\alpha$ of ω_2 into $\{0, 1\}^\omega$.

Each f_α is the characteristic function of a set $a_\alpha \subset \omega$. As in Example 14.2, we call these sets *Cohen generic reals*. Thus P adjoins $\aleph_2$ Cohen generic reals to the ground model.

The proof of Theorem 14.32 is almost complete, except for one detail: We don't know that the ordinal ω_2^V is the cardinal $\aleph_2$ of $V[G]$. We shall complete the proof by showing that $V[G]$ has the same cardinals as the ground model (*P preserves cardinals*).

Definition 14.33. A forcing notion P satisfies the *countable chain condition* (c.c.c.) if every antichain in P is at most countable.

The following theorem is one of the basic tools of forcing:

Theorem 14.34. *If P satisfies the countable chain condition then V and $V[G]$ have the same cardinals and cofinalities.*

In other words $\mathrm{cf}^V \alpha = \mathrm{cf}^{V[G]} \alpha$ for all limit ordinals α; the statement on cardinals follows.

Proof. It suffices to show that if κ is a regular cardinal then κ remains regular in $V[G]$. Thus let $\lambda < \kappa$; we show that every function $f \in V[G]$ from λ into κ is bounded.

Let $\dot{f}$ be a name, let $p \in P$ and assume

$$p \Vdash \dot{f} \text{ is a function from } \check{\lambda} \text{ to } \check{\kappa}. \tag{14.35}$$

For every $\alpha < \lambda$ consider the set

$$A_\alpha = \{\beta < \kappa : \exists q < p \; q \Vdash \dot{f}(\alpha) = \beta\}.$$

We claim that every A_α is at most countable: If $W = \{q_\beta : \beta \in A_\alpha\}$ is a set of witnesses to $\beta \in A_\alpha$ then W is an antichain, and therefore countable by c.c.c. Hence $|A_\alpha| \leq \aleph_0$.

Now, because κ is regular, the set $\bigcup_{\alpha<\kappa} A_\alpha$ is bounded, by some $\gamma < \kappa$. It follows that for each $\alpha < \lambda$, p forces $\dot{f}(\alpha) < \gamma$.

Thus for every $\dot{f} \in V^P$ and every $p \in P$, if (14.35) then $p \Vdash \dot{f}$ is bounded below κ. It follows that in $V[G]$, every function $f : \lambda \to \kappa$ is bounded. $\square$

Now we complete the proof of Theorem 14.32 by showing that the forcing notion that we employed satisfies c.c.c. That follows from the following consequence of Theorem 9.18 on Δ-systems. $\square$

Lemma 14.35. *Let P be a set of finite functions, with values in a given countable set C. Let $p < q$ be defined as $p \supset q$, and assume that for all $p, q \in P$, if $p \cup q$ is a function then $p \cup q \in P$ (or more generally, $\exists r \in P \, (r \supset p \cup q)$). Then P satisfies the countable chain condition.*

Proof. Let F be an uncountable subset of P, and let W be the set $\{\mathrm{dom}(p) : p \in F\}$. As C is countable, the set W must be uncountable. By Theorem 9.18 there exists an uncountable Δ-system $Z \subset W$; let $S = X \cap Y$ for any $X \neq Y$ in Z. Let G be the set of all $p \in F$ such that $\mathrm{dom}(p) \in Z$; again because C is countable there are uncountably many $p \in G$ with the same $p{\restriction}S$. Now if p and q are two such functions, i.e., $\mathrm{dom}(p) \cap \mathrm{dom}(q) = S$ and $p{\restriction}S = q{\restriction}S$, then p and q are compatible functions and therefore compatible conditions. Hence F is not an antichain. $\square$

Independence of the Axiom of Choice

If the ground model M satisfies the Axiom of Choice, then so does the generic extension. However, we can still use the method of forcing to construct a model in which AC fails; namely, we find a suitable submodel of the generic model, a model N such that $M \subset N \subset M[G]$.

Theorem 14.36 (Cohen). *There is a model of* ZF *in which the real numbers cannot be well-ordered. Thus the Axiom of Choice is independent of the axioms of* ZF*.*

Before we construct a model without Choice, we shall prove an easy but useful lemma on automorphisms of Boolean-valued models. Let B be a complete Boolean algebra and let π be an automorphism of B. We define, by induction on $\rho(x)$ an automorphism of the Boolean-valued universe V^B, and denote it also π:

(14.36) (i) $\pi(\emptyset) = \emptyset$;
(ii) $\operatorname{dom}(\pi x) = \pi(\operatorname{dom}(x))$, and $(\pi x)(\pi y) = \pi(x(y))$ for all $\pi(y) \in \operatorname{dom}(\pi x)$.

Clearly, π is a one-to-one function of V^B onto itself, and $\pi(\check{x}) = \check{x}$ for every x.

Lemma 14.37. *Let $\varphi(x_1, \ldots, x_n)$ be a formula. If π is an automorphism of B, then for all $x_1, \ldots, x_n \in V^B$,*

$$\|\varphi(\pi x_1, \ldots, \pi x_n)\| = \pi(\|\varphi(x_1, \ldots, x_n)\|). \tag{14.37}$$

Proof. (a) If φ is an atomic formula, (14.37) is proved by induction (as in the definition of $\|x \in y\|$, $\|x = y\|$). For instance,

$$\begin{aligned}
\|\pi x \in \pi y\| &= \sum_{t \in \operatorname{dom}(\pi y)} (\|\pi x = t\| \cdot (\pi y)(t)) \\
&= \sum_{z \in \operatorname{dom}(y)} (\|\pi x = \pi z\| \cdot (\pi y)(\pi z)) \\
&= \pi\Big(\sum_{z \in \operatorname{dom}(y)} (\|x = z\| \cdot y(z))\Big) = \pi(\|x \in y\|).
\end{aligned}$$

(b) In general, the proof is by induction on the complexity of φ. □

In practice, (14.37) is used as follows: Let $(P, <)$ be a separative partially ordered set. If π is an automorphism of P, then π extends to an automorphism of the complete Boolean algebra $B(P)$, by $\pi(u) = \sum\{\pi(p) : p \leq u\}$. Then (14.37) takes this form: For all P-names $\dot{x}_1, \ldots, \dot{x}_n$,

$$p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n) \quad \text{if and only if} \quad \pi p \Vdash \varphi(\pi\dot{x}_1, \ldots, \pi\dot{x}_n). \tag{14.38}$$

For the proof of Theorem 14.36, let us assume that the ground model M satisfies $V = L$. We first extend M by adding countably many Cohen generic reals: Let P be the set of all functions p such that

(14.39) (i) $\mathrm{dom}(p)$ is a finite subset of $\omega \times \omega$,
(ii) $\mathrm{ran}(p) \subset \{0, 1\}$,

and let $p < q$ if and only if $p \supset q$.

Let G be a generic set of conditions. For each $i \in \omega$, let

$$a_i = \{n \in \omega : (\exists p \in G)\, p(i, n) = 1\}$$

and let $A = \{a_i : i \in \omega\}$. Let $\dot{a}_i$, $i \in \omega$, and $\dot{A}$ be the canonical names for a_i and A:

(14.40) $\mathrm{dom}(\dot{a}_i) = \{\check{n} : n \in \omega\}$, and $\dot{a}_i(\check{n}) = \sum\{p \in P : p(i, n) = 1\}$,

(14.41) $\mathrm{dom}(\dot{A}) = \{\dot{a}_i : i \in \omega\}$, and $\dot{A}(\dot{a}_i) = 1$.

Lemma 14.38. *If $i \neq j$, then every p forces $\dot{a}_i \neq \dot{a}_j$.*

Proof. For every p there exists a $q \supset p$ such that for some $n \in \omega$, $q(i, n) = 1$ and $q(j, n) = 0$. □

In the model $M[G]$, let N be the class of all sets hereditarily ordinal-definable over A, $N = \mathit{HOD}(A)$. As we have seen in Chapter 13, N is a transitive model of ZF. Since the elements of A are sets of integers, it is clear that $A \in N$. We shall show that A cannot be well-ordered in the model N. For that, it suffices to show that there is no one-to-one function $f \in N$ from A into the ordinals.

Lemma 14.39. *In $M[G]$, there is no one-to-one function $f : A \to \mathit{Ord}$, ordinal-definable over A.*

Proof. Assume that $f : A \to \mathit{Ord}$ is one-to-one and ordinal-definable over A. Then there is a finite sequence $s = \langle x_0, \ldots, x_k \rangle$ in A such that f is ordinal-definable from s and A. Since f is one-to-one, it is easy to see that every $a \in A$ is ordinal definable from s and A. In particular, pick some $a \in A$ that is not among the x_i, $i \leq k$.

Since $a \in \mathit{OD}[s, A]$, there is a formula φ such that

(14.42) $M[G] \vDash a$ is the unique set such that $\varphi(a, \alpha_1, \ldots, \alpha_n, s, A)$

for some ordinals $\alpha_1, \ldots, \alpha_n$. We shall show that (14.42) is impossible.

Let $\dot{a}$ be a name for a, let $\dot{x}_0, \ldots, \dot{x}_k$ be names for $x_0, \ldots, x_k$ and let $\dot{s}$ be a name for the sequence $\langle x_0, \ldots, x_k \rangle$. We shall show the following:

(14.43) For every p_0 that forces $\varphi(\dot{a}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$ there exist $\dot{b}$ and $q \leq p_0$ such that q forces $\dot{a} \neq \dot{b}$ and $\varphi(\dot{b}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \dot{s}, \dot{A})$.

Let $p_0 \Vdash \varphi(\dot{a}, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A})$. There exist i, $i_0, \ldots, i_k$ and $p_1 \leq p_0$ such that p_1 forces $\dot{a} = \dot{a}_i$, $\dot{x}_0 = \dot{a}_{i_0}, \ldots, \dot{x}_k = \dot{a}_{i_k}$. Let $j \in \omega$ be such that $j \neq i$, and that for all m, $(j, m) \notin \mathrm{dom}(p_1)$.

Now let π be the permutation of ω that interchanges i and j, and $\pi x = x$ otherwise. This permutation induces an automorphism of P: For every $p \in P$,

$$\begin{aligned} \text{(14.44)} \qquad \operatorname{dom}(\pi p) &= \{(\pi x, m) : (x, m) \in \operatorname{dom}(p)\}, \\ (\pi p)(\pi x, m) &= p(x, m). \end{aligned}$$

In turn, π induces an automorphism of B, and of M^B. It is easy to see (cf. (14.40) and (14.41)) that $\pi(\dot{a}_i) = \dot{a}_j$, $\pi(\dot{a}_j) = \dot{a}_i$, $\pi(\dot{a}_x) = \dot{a}_x$ for all $x \neq i, j$, $\pi(\dot{A}) = \dot{A}$ and $\pi(\dot{s}) = \dot{s}$. Since $(j, m) \notin \operatorname{dom}(p_1)$ for all m, it follows that $(i, m) \notin \operatorname{dom}(\pi p_1)$ for all m, and thus p_1 and πp_1 are compatible. Let $q = p_1 \cup \pi p_1$.

Now, on the one hand we have

$$p_1 \Vdash \varphi(\dot{a}_i, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A}),$$

and on the other hand, since $\pi\check{\alpha} = \check{\alpha}$, $\pi\dot{s} = \dot{s}$ and $\pi\dot{A} = \dot{A}$, we have

$$\pi p_1 \Vdash \varphi(\dot{a}_j, \alpha_1, \ldots, \alpha_n, \dot{s}, \dot{A}).$$

Hence

$$q \Vdash \varphi(\dot{a}_i, \ldots) \text{ and } \varphi(\dot{a}_j, \ldots)$$

and by Lemma 14.38, $q \Vdash \dot{a}_i \neq \dot{a}_j$. Thus we have proved (14.43), which contradicts (14.42). □

Exercises

14.1. Show that in the definition of generic set one can replace (14.1)(iii) by the following weaker property: If $p, q \in G$, then p and q are compatible.

[To prove (14.1)(iii), show that $D = \{r \in P :$ either r is incompatible with p, or r is incompatible with q, or $r \leq p$ and $r \leq q\}$ is dense.]

14.2. A filter G on P is generic over M if and only if for every $p \in G$, if $D \in M$ is dense below p then $G \cap D \neq \emptyset$.

14.3. A filter G on P is generic over M if and only if $G \cap D \neq \emptyset$ whenever $D \in M$ is open and dense in P.

14.4. A filter G on P is generic over M if and only if $G \cap D \neq \emptyset$ whenever $D \in M$ is predense in P.

14.5. A filter G on P is generic over M if and only if $G \cap D \neq \emptyset$ whenever $D \in M$ is a maximal antichain in P.

14.6. Let $(P, <)$ be a notion of forcing in M with the following property: For every $p \in P$ there exist q and r such that $q \leq p$, $r \leq p$ and such that q and r are incompatible. Show that if $G \subset P$ is generic over M, then $G \notin M$.

[If F is a filter on P, then $\{p \in P : p \notin F\}$ is dense in P.]

14.7. If $\{q : q \Vdash \varphi\}$ is dense below p then $p \Vdash \varphi$.

14.8. Assume that for every $p \in P$ there exists a $G \subset P$ generic over M such that $p \in G$ (e.g., if M is countable). Show that $p \Vdash \sigma$ if and only if $M[G] \vDash \sigma$ for all generic G such that $p \in G$.

14.9. The separative quotient is unique up to isomorphism.

[If $(Q, <)$ is separative, then $\leq$ can be defined in terms of compatibility: $x \leq y$ if and only if every z compatible with x is compatible with y.]

14.10. If B is a complete Boolean algebra in the ground model M, then $G \subset B$ is a generic ultrafilter over M if and only if G is a generic filter on B^+ over M.

14.11. An ultrafilter G on B is generic over M if and only if for every partition W of B such that $W \in M$, there exists a unique $u \in G \cap W$.

14.12. (i) $\|(\exists y \in x)\,\varphi(y)\| = \sum_{y \in \operatorname{dom} x}(x(y) \cdot \|\varphi(y)\|)$.
(ii) $\|(\forall y \in x)\,\varphi(y)\| = \prod_{y \in \operatorname{dom} x}(x(y) \Rightarrow \|\varphi(y)\|)$.

14.13. (i) If $x = y$ then $\|\check{x} = \check{y}\| = 1$ and if $x \neq y$ then $\|\check{x} = \check{y}\| = 0$.
(ii) If $x \in y$ then $\|\check{x} \in \check{y}\| = 1$ and if $x \notin y$ then $\|\check{x} \in \check{y}\| = 0$.

14.14. Let $\dot{G}$ be the canonical name for a generic ultrafilter on B. Show that

(i) $\|\dot{G}$ is an ultrafilter on $B\| = 1$.
(ii) For every $X \subset B$, $\|$if $\check{X} \subset \dot{G}$ then $\prod X \in \dot{G}\| = 1$.

14.15. If G is an M-generic ultrafilter on B, let M^B/G be defined by (14.11) and (14.12). Prove that M^B/G is isomorphic to $M[G]$.

14.16. If G is an M-generic ultrafilter on B and π an automorphism of B (in M), then $H = \pi(G)$ is M-generic and $M[H] = M[G]$.

Historical Notes

The method of forcing was invented by Paul Cohen who used it to prove the independence of the Continuum Hypothesis and the Axiom of Choice (see [1963, 1964] and the book [1966]). The Boolean-valued version of Cohen's method has been formulated by Scott, Solovay, and Vopěnka. Following an observation of Solovay that the forcing relation can be viewed as assigning Boolean-values to formulas, Scott formulated his version of Boolean-valued models in [1967]. Vopěnka developed a theory of Cohen's method of forcing, using open sets in a topological space as forcing conditions (in [1964, 1965a, 1965b, 1965c, 1966, 1967a] and Vopěnka-Hájek [1967]), eventually arriving at the Boolean-valued version of forcing more or less identical to Scott-Solovay's version (Vopěnka [1967b]).