18. Large Cardinals and L

In Chapter 17 we proved that while "smaller" large cardinals (inaccessible, Mahlo, weakly compact) can exist in L, the "bigger" large cardinals (measurable, Ramsey) cannot. In this chapter we isolate and investigate the concept of $0^\sharp$ (*zero-sharp*), a great divide in the landscape of large cardinals.

Silver Indiscernibles

Theorem 18.1 (Silver). *If there exists a Ramsey cardinal, then:*

(i) *If κ and λ are uncountable cardinals and $\kappa < \lambda$, then $(L_\kappa, \in)$ is an elementary submodel of $(L_\lambda, \in)$.*

(ii) *There is a unique closed unbounded class of ordinals I containing all uncountable cardinals such that for every uncountable cardinal κ:*
   - (a) $|I \cap \kappa| = \kappa$,
   - (b) *$I \cap \kappa$ is a set of indiscernibles for $(L_\kappa, \in)$, and*
   - (c) *every $a \in L_\kappa$ is definable in $(L_\kappa, \in)$ from $I \cap \kappa$.*

The elements of the class I are called *Silver indiscernibles.* Before giving the proof of Theorem 18.1 we state some consequences of the existence of Silver indiscernibles.

By the Reflection Principle, if φ is a formula, then there exists an uncountable cardinal κ such that

$$L \vDash \varphi(x_1, \ldots, x_n) \quad \text{if and only if} \quad L_\kappa \vDash \varphi(x_1, \ldots, x_n) \tag{18.1}$$

for all $x_1, \ldots, x_n \in L_\kappa$. By (i), the right hand side holds in and only if $L_\lambda \vDash \varphi(x_1, \ldots, x_n)$ for all cardinals $\lambda \geq \kappa$. In view of this, we can define satisfaction in L for all formulas $\varphi \in \mathit{Form}$: If $\varphi(v_1, \ldots, v_n)$ is a formula of the language $\mathcal{L} = \{\in\}$ and if $\langle a_1, \ldots, a_n \rangle$ is an n-termed sequence in L, we define

$$L \vDash \varphi[a_1, \ldots, a_n] \tag{18.2}$$

as follows: For every uncountable cardinal κ such that $a_1, \ldots, a_n \in L_\kappa$, $L_\kappa \vDash \varphi[a_1, \ldots, a_n]$.

Note that this gives us a truth definition for the constructible universe: $T = \{\#\sigma : L_{\aleph_1} \vDash \sigma\}$. If σ is a sentence, then $\sigma^L \leftrightarrow \#\sigma \in T$. (Note that the set T is constructible but not definable in L: Otherwise, T would be a truth definition in L. Hence the cardinal $\aleph_1$ is not definable in L.)

Moreover, as a consequence of (i) we have $(L_\kappa, \in) \prec (L, \in)$ for every uncountable cardinal κ. As a consequence of (ii) Silver indiscernibles are indiscernibles for L: If $\varphi(v_1, \ldots, v_n)$ is a formula, then

$$L \vDash \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{if and only if} \quad L \vDash \varphi[\beta_1, \ldots, \beta_n] \tag{18.3}$$

whenever $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are increasing sequences in I. Every constructible set is definable from I. If $a \in L$, there exists an increasing sequence $\langle\gamma_1, \ldots, \gamma_n\rangle$ of Silver indiscernibles and a formula φ such that

$$L \vDash a \text{ is the unique } x \text{ such that } \varphi(x, \gamma_1, \ldots, \gamma_n).$$

By (18.3), every formula $\varphi(v_1, \ldots, v_n)$ is either true or false in L for any increasing sequence $\langle\gamma_1, \ldots, \gamma_n\rangle$ of Silver indiscernibles; moreover, the truth value coincides with the truth value of $L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n]$ since $L_{\aleph_\omega} \prec L$ and $\aleph_1, \ldots, \aleph_n$ are Silver indiscernibles. Thus let us define

$$0^\sharp = \{\varphi : L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n]\} \tag{18.4}$$

(*zero-sharp*). Later in this section we shall give another definition of the set $0^\sharp$. We shall show that a set $0^\sharp$ satisfying the definition exists if and only if (i) and (ii) holds, and then $0^\sharp$ is as in (18.4).

Thus the conclusion of Theorem 18.1 is abbreviated as

$$0^\sharp \text{ exists.}$$

In the following corollaries we assume that $0^\sharp$ exists.

Corollary 18.2. *Every constructible set definable in L is countable.*

Proof. If $x \in L$ is definable in L by a formula φ, then the same formula defines x in $L_{\aleph_1}$ and hence $x \in L_{\aleph_1}$. □

In particular, every ordinal number definable in L is countable.

In the following corollary $\aleph_\alpha$ denotes the αth cardinal in V, not $\aleph_\alpha^L$.

Corollary 18.3. *Every uncountable cardinal is inaccessible in L.*

Proof. Since $L \vDash \aleph_1$ is regular, we have

$$L \vDash \aleph_\alpha \text{ is regular}$$

for every $\alpha \geq 1$. Similarly, $L \vDash \aleph_\omega$ is a limit cardinal, and hence

$$L \vDash \aleph_\alpha \text{ is a limit cardinal}$$

for every $\alpha \geq 1$. Thus every uncountable cardinal (and in fact every $\gamma \in I$) is an inaccessible cardinal in L. □

Corollary 18.4. *Every uncountable cardinal is a Mahlo cardinal in L.*

Proof. By Corollary 18.3, every Silver indiscernible is an inaccessible cardinal in L. Since $I \cap \omega_1$ is closed unbounded in ω_1, $\aleph_1$ is a Mahlo cardinal in L. □

Corollary 18.5. *For every $\alpha \geq \omega$, $|V_\alpha \cap L| \leq |\alpha|$. In particular, the set of all constructible reals is countable.*

Proof. The set $V_\alpha \cap L$ is definable in L from α. Thus $V_\alpha \cap L$ is also definable from α in L_κ where κ is the least cardinal $> \alpha$. Hence $V_\alpha \cap L \subset L_\beta$ for some β such that $|\alpha| = |\beta|$. However, $|L_\beta| = |\beta|$. □

Models with Indiscernibles

The proof of Silver's Theorem is based on a theorem of Ehrenfeucht and Mostowski in model theory, stating that every infinite model is elementarily equivalent to a model that has a set of indiscernibles of prescribed order-type. We shall deal only with models $(L_\lambda, \in)$ (and models elementarily equivalent to these); we shall prove below a special case of the Ehrenfeucht-Mostowski Theorem.

We shall use the canonical well-ordering of L to endow the models $(L_\lambda, \in)$ with definable Skolem functions. For each formula $\varphi(u, v_1, \ldots, v_n)$, let h_φ be the n-ary function defined as follows:

$$h_\varphi(v_1, \ldots, v_n) = \begin{cases} \text{the } <_L\text{-least } u \text{ such that } \varphi(u, v_1, \ldots, v_n), \\ \emptyset \qquad \text{otherwise.} \end{cases} \tag{18.5}$$

We call h_φ, $\varphi \in \mathit{Form}$, the *canonical Skolem functions.*

For each limit ordinal λ, $h_\varphi^{L_\lambda}$ is an n-ary function on L_λ, the L_λ-interpretation of h_φ, and is definable in $(L_\lambda, \in)$.

When dealing with models $(L_\lambda, \in)$ we shall freely use terms and formulas involving the h_φ since they as definable functions can be eliminated and the formulas can be replaced by $\in$-formulas. For each limit ordinal λ, the functions $h_\varphi^{L_\lambda}$, $\varphi \in \mathit{Form}$, are Skolem functions for $(L_\lambda, \in)$ and so a set $M \subset L_\lambda$ is an elementary submodel of $(L_\lambda, \in)$ if and only if M is closed under the $h_\varphi^{L_\lambda}$. If $X \subset L_\lambda$, then the closure of X under the $h_\varphi^{L_\lambda}$ is the smallest elementary submodel $M \prec L_\lambda$ such that $X \subset M$, and is the collection of all elements of L_λ definable in L_λ from X.

The fact that the well-ordering $<_\lambda$ of L_λ is definable in L_λ uniformly for all limit ordinals λ (by the same formula) implies the following:

Lemma 18.6. *If α and β are limit ordinals and if $j : L_\alpha \to L_\beta$ is an elementary embedding of $(L_\alpha, \in)$ in $(L_\beta, \in)$, then for each formula φ and all $x_1, \ldots, x_n \in L_\alpha$,*

$$h_\varphi^{L_\beta}(j(x_1), \ldots, j(x_n)) = j(h_\varphi^{L_\alpha}(x_1, \ldots, x_n)). \tag{18.6}$$

Hence j remains elementary with respect to the augmented language $\mathcal{L}^* = \{\in\} \cup \{h_\varphi : \varphi \in Form\}$. □

Let λ be a limit ordinal, and let $\mathfrak{A} = (A, E)$ be a model elementarily equivalent to $(L_\lambda, \in)$. The set $Ord^{\mathfrak{A}}$ of all ordinal numbers of the model $\mathfrak{A}$ is linearly ordered by E; let us use $x < y$ rather than $x \mathrel{E} y$ for $x, y \in Ord^{\mathfrak{A}}$. A set $I \subset Ord^{\mathfrak{A}}$ is a *set of indiscernibles* for $\mathfrak{A}$ if for every formula φ,

$$\text{(18.7)} \qquad \mathfrak{A} \vDash \varphi[x_1, \ldots, x_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[y_1, \ldots, y_n]$$

whenever $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ are elements of I. Let $h_\varphi^{\mathfrak{A}}$ denote the $\mathfrak{A}$-interpretation of the canonical Skolem functions (18.5). Given a set $X \subset A$, let us denote $H^{\mathfrak{A}}(X)$ the closure of X under all $h_\varphi^{\mathfrak{A}}$, $\varphi \in Form$. The set $H^{\mathfrak{A}}(X)$ is the *Skolem hull* of X and is an elementary submodel of $\mathfrak{A}$.

If I is a set of indiscernibles for $\mathfrak{A}$, let $\Sigma(\mathfrak{A}, I)$ be the set of all formulas $\varphi(v_1, \ldots, v_n)$ true in $\mathfrak{A}$ for increasing sequences of elements of I:

$$\text{(18.8)} \quad \varphi(v_1, \ldots, v_n) \in \Sigma(\mathfrak{A}, I) \leftrightarrow \mathfrak{A} \vDash \varphi[x_1, \ldots, x_n] \text{ for some } x_1, \ldots, x_n \in I \text{ such that } x_1 < \ldots < x_n.$$

A set of formulas Σ is called an *E.M.* set (Ehrenfeucht-Mostowski) if there exists a model $\mathfrak{A}$ elementarily equivalent to some L_λ, λ a limit ordinal, and an infinite set I of indiscernibles for $\mathfrak{A}$ such that $\Sigma = \Sigma(\mathfrak{A}, I)$.

Lemma 18.7. *If Σ is an E.M. set and α an infinite ordinal number, then there exists a model $\mathfrak{A}$ and a set of indiscernibles I for $\mathfrak{A}$ such that:*

(i) $\Sigma = \Sigma(\mathfrak{A}, I)$;
(ii) *the order-type of I is α;*
(iii) $\mathfrak{A} = H^{\mathfrak{A}}(I)$.

Moreover, the pair $(\mathfrak{A}, I)$ is unique up to isomorphism.

Proof. We prove uniqueness first. Let $(\mathfrak{A}, I)$ and $(\mathfrak{B}, J)$ be two pairs, each satisfying (i), (ii), (iii). Since both I and J have order-type α, let π be the isomorphism between I and J. We shall extend π to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Since $\mathfrak{A}$ is the Skolem hull of I, there is for each $a \in \mathfrak{A}$ a Skolem term $t(v_1, \ldots, v_n)$ (a combination of the Skolem functions h_φ) such that $a = t^{\mathfrak{A}}[x_1, \ldots, x_n]$ for some $x_1 < \ldots < x_n$ in I; similarly for $\mathfrak{B}$, J. Thus we define

$$\text{(18.9)} \qquad \pi(t^{\mathfrak{A}}[x_1, \ldots, x_n]) = t^{\mathfrak{B}}[\pi(x_1), \ldots, \pi(x_n)]$$

for each Skolem term t and all $x_1, \ldots, x_n \in I$ such that $x_1 < \ldots < x_n$. Since $\Sigma(\mathfrak{A}, I) = \Sigma(\mathfrak{B}, J)$, we have

(18.10)

$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n] = t_2^{\mathfrak{A}}[y_1, \ldots, y_n] \leftrightarrow t_1^{\mathfrak{B}}[\pi x_1, \ldots, \pi x_n] = t_2^{\mathfrak{B}}[\pi y_1, \ldots, \pi y_n],$$
$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n] \mathrel{E^{\mathfrak{A}}} t_2^{\mathfrak{A}}[y_1, \ldots, y_n] \leftrightarrow t_1^{\mathfrak{B}}[\pi x_1, \ldots, \pi x_n] \mathrel{E^{\mathfrak{B}}} t_2^{\mathfrak{B}}[\pi y_1, \ldots, \pi y_n]$$

for any terms t_1, t_2 and indiscernibles x, y: Let $z_1, \ldots, z_{n+m}$ be the enumeration of the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ in increasing order. Then the equality in (18.10) holds (simultaneously in $\mathfrak{A}$ and $\mathfrak{B}$) just in case $\varphi(v_1, \ldots, v_{n+m}) \in \Sigma$ where $\varphi(z_1, \ldots, z_{n+m})$ is the formula that says that $t_1[x_1, \ldots, x_n] = t_2[y_1, \ldots, y_n]$. Hence π is well-defined by (18.9) and is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ extending the order-isomorphism of I and J.

To prove the existence of a model with indiscernibles with properties (i), (ii), and (iii), we use the Compactness Theorem. Since Σ is an E.M. set, there exists $(\mathfrak{A}_0, I_0)$ such that $\Sigma = \Sigma(\mathfrak{A}_0, I_0)$. Let us extend the language $\{\in\}$ by adding α constant symbols c_ξ, $\xi < \alpha$. Let Δ be the following set of sentences:

$$
\begin{array}{lll}
(18.11) & c_\xi \text{ is an ordinal} & (\text{all } \xi < \alpha), \\
 & c_\xi < c_\eta & (\text{all } \xi,\ \eta \text{ such that } \xi < \eta < \alpha), \\
 & \varphi(c_{\xi_1}, \ldots, c_{\xi_n}) & (\text{all } \varphi \in \Sigma \text{ and all } \xi_1 < \ldots < \xi_n < \alpha).
\end{array}
$$

We shall show that every finite subset of Δ has a model. Let $D \subset \Delta$ be finite. There exist $\xi_1 < \ldots < \xi_k$ such that $c_{\xi_1}, \ldots, c_{\xi_k}$ are the only constants mentioned in D. Let $\sigma(c_{\xi_1}, \ldots, c_{\xi_k})$ be the sentence that is the conjunction of all sentences in D.

Since I_0 is infinite, there are $i_1, \ldots, i_k \in I_0$ such that $i_1 < \ldots < i_k$. Let us take the model $\mathfrak{A}_0$ and expand it by interpreting the constant symbols $c_{\xi_1}, \ldots, c_{\xi_k}$ as $i_1, \ldots, i_k$. Since $\Sigma = \Sigma(\mathfrak{A}_0, I_0)$ and $D \subset \Delta$, it is clear that $\mathfrak{A}_0 \vDash \sigma[i_1, \ldots, i_k]$ and hence the expansion $(\mathfrak{A}_0, i_1, \ldots, i_k)$ is a model of σ, hence of D.

By the Compactness Theorem, the set Δ has a model $\mathfrak{M} = (M, E, c_\xi^{\mathfrak{M}})_{\xi<\alpha}$. Let $I = \{c_\xi^{\mathfrak{M}} : \xi < \alpha\}$. I is a set of ordinals of $\mathfrak{M}$ and has order-type α. It is clear that if $\varphi(v_1, \ldots, v_n)$ is an $\in$-formula and $\xi_1 < \ldots < \xi_n$, then $(M, E) \vDash \varphi[c_{\xi_1}^{\mathfrak{M}}, \ldots, c_{\xi_n}^{\mathfrak{M}}]$ if and only if $\varphi \in \Sigma$. Thus I is a set of indiscernibles for (M, E). Now we let A be the Skolem hull of I in (M, E). Since $\mathfrak{A} = (A, E)$ is an elementary submodel of (M, E), it follows that I is a set of indiscernibles for $\mathfrak{A}$, $\Sigma(\mathfrak{A}, I) = \Sigma$, and that $H^{\mathfrak{A}}(I) = H^{(M,E)}(I) = A$. Hence $(\mathfrak{A}, I)$ satisfies (i), (ii), and (iii). □

For each E.M. set Σ and each ordinal α, let us call the (Σ, α)-*model* the unique pair $(\mathfrak{A}, I)$ given by Lemma 18.7. The uniqueness proof of Lemma 18.7 easily extends to give the following:

Lemma 18.8. *Let Σ be an E.M. set, let $\alpha \le \beta$, and let $j : \alpha \to \beta$ be order-preserving. Then j can be extended to an elementary embedding of the (Σ, α)-model into the (Σ, β)-model.*

Proof. Extend j as in (18.9). □

We shall eventually show that the existence of Ramsey cardinal implies the existence of an E.M. set Σ having a certain syntactical property (*remarkability*) and such that every (Σ, α)-model is well-founded. Let us investigate well-foundedness first.

Lemma 18.9. *The following are equivalent, for an E.M. set* Σ*:*

(i) *For every ordinal* α*, the* (Σ, α)*-model is well-founded.*
(ii) *For some ordinal* $\alpha \geq \omega_1$*, the* (Σ, α)*-model is well-founded.*
(iii) *For every ordinal* $\alpha < \omega_1$*, the* (Σ, α)*-model is well-founded.*

Proof. (i) $\to$ (ii) is trivial.

(ii) $\to$ (iii): If $(\mathfrak{A}, I)$ is the (Σ, α)-model and if $\beta \leq \alpha$, let J be the initial segment of the first β elements of I; let $\mathfrak{B} = H^{\mathfrak{A}}(J)$. Clearly, $(\mathfrak{B}, J)$ is the (Σ, β)-model. Since a submodel of a well-founded model is well-founded, it follows that if $\beta \leq \alpha$ and the (Σ, α)-model is well-founded, then the (Σ, β)-model is also well-founded, and thus (ii) implies (iii).

(iii) $\to$ (i): Let us assume that there is a limit ordinal α such that the (Σ, α)-model is not well-founded; let $(\mathfrak{A}, I)$ be the model. There is an infinite sequence a_0, a_1, a_2, ... in $\mathfrak{A}$ such that $a_1 \; E \; a_2$, $a_2 \; E \; a_1$, etc. Each a_n is definable from I; that is, for each n there is a Skolem term t_n such that $a_n = t_n^{\mathfrak{A}}[x_1, \ldots, x_{k_n}]$ for some $x_1, \ldots, x_{k_n} \in I$. Therefore there is a countable subset I_0 of I such that $a_n \in H^{\mathfrak{A}}(I_0)$ for all $n \in \omega$. The order-type of I_0 is a countable ordinal β and $(H^{\mathfrak{A}}(I_0), I_0)$ is the (Σ, β)-model. This model is clearly non-well-founded since it contains all the a_n. Hence for some countable β, the (Σ, β)-model is not well-founded. □

We shall now define remarkability. We consider only (Σ, α)-models where α is an infinite limit ordinal.

Let us say that a (Σ, α)-model $(\mathfrak{A}, I)$ is *unbounded* if the set I is unbounded in the ordinals of $\mathfrak{A}$, that is, if for every $x \in Ord^{\mathfrak{A}}$ there is $y \in I$ such that $x < y$.

Lemma 18.10. *The following are equivalent, for any E.M. set* Σ:

(i) *For all* α, (Σ, α) *is unbounded.*
(ii) *For some* α, (Σ, α) *is unbounded.*
(iii) *For every Skolem term* $t(v_1, \ldots, v_n)$ *the set* Σ *contains the formula*

(18.12) *if* $t(v_1, \ldots, v_n)$ *is an ordinal, then* $t(v_1, \ldots, v_n) < v_{n+1}$.

Proof. (i) $\to$ (ii) is trivial.

(ii) $\to$ (iii): Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where α is a limit ordinal, and assume that I is unbounded in $Ord^{\mathfrak{A}}$. To prove (iii), it suffices to show that for any term t, (18.12) is true in $\mathfrak{A}$ for some increasing sequence $x_1 < \ldots < x_{n+1}$ in I. Let t be a Skolem term. Let us choose $x_1 < \ldots < x_n \in I$ and let $y = t^{\mathfrak{A}}[x_1, \ldots, x_n]$. If $y \notin Ord^{\mathfrak{A}}$, then (18.12) is vacuously true; if $y \in Ord^{\mathfrak{A}}$, then there exists $x_{n+1} \in I$ such that $y < x_{n+1}$, and we have $\mathfrak{A} \models t[x_1, \ldots, x_n] < x_{n+1}$.

(iii) $\to$ (i): Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where α is a limit ordinal, and assume (iii). To prove that I is unbounded in $Ord^{\mathfrak{A}}$, let $y \in Ord^{\mathfrak{A}}$. There exist a Skolem term t and $x_1 < \ldots < x_n \in I$ such that $y = t^{\mathfrak{A}}[x_1, \ldots, x_n]$. Now if x_{n+1} is any element of I greater than x_n, (iii) implies that $y < x_{n+1}$. □

Thus we say that an E.M. set Σ is unbounded if it contains the formulas (18.12) for all Skolem terms t.

Let α be a limit ordinal, $\alpha > \omega$, and let $(\mathfrak{A}, I)$ be the (Σ, α)-model. For each $\xi < \alpha$, let i_ξ denote the ξth element of I. We say that $(\mathfrak{A}, I)$ is *remarkable* if it is unbounded and if every ordinal x of $\mathfrak{A}$ less than i_ω is in $H^{\mathfrak{A}}(\{i_n : n \in \omega\})$.

Lemma 18.11. *The following are equivalent for any unbounded E.M. set* Σ:

(i) *For all* $\alpha > \omega$, *the* (Σ, α)*-model is remarkable.*
(ii) *For some* $\alpha > \omega$, *the* (Σ, α)*-model is remarkable.*
(iii) *For every Skolem term* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$, *the set* Σ *contains the formula*

(18.13) *if* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ *is an ordinal smaller than* y_1, *then* $t(x_1, \ldots, x_m, y_1, \ldots, y_n) = t(x_1, \ldots, x_m, z_1, \ldots, z_n)$.

Moreover, if $(\mathfrak{A}, I)$ *is a remarkable* (Σ, α)*-model and* $\gamma < \alpha$ *is a limit ordinal, then every ordinal* x *of* $\mathfrak{A}$ *less than* i_γ *is in* $H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$.

Proof. (i) $\to$ (ii) is trivial.

(ii) $\to$ (iii): Let $\alpha > \omega$ be a limit ordinal and let $(\mathfrak{A}, I)$ be a remarkable (Σ, α)-model. To prove (iii), it suffices to show that for any t, (18.13) is true in $\mathfrak{A}$ for some increasing sequence $x_1 < \ldots < x_m < y_1 < \ldots < y_n < z_1 < \ldots < z_n$ in I. Let t be a Skolem term. We let $x_1 < \ldots < x_m < y_1 < \ldots < y_n < z_1 < \ldots < z_n \in I$ be such that $x_1, \ldots, x_m$ are the first m members of I and that y_1 is the ωth member of I, $y_1 = i_\omega$. Now if $a = t^{\mathfrak{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is an ordinal of $\mathfrak{A}$ and less than y_1, we have, by remarkability of $(\mathfrak{A}, I)$, $a \in H^{\mathfrak{A}}(\{i_n : n < \omega\})$. Hence there is $k < \omega$, $k \geq m$, and a term s such that

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, y_1, \ldots, y_n] = s[i_0, \ldots, i_k]. \tag{18.14}$$

In other words (18.14) says that $\mathfrak{A}$ satisfies a certain formula $\varphi[i_0, \ldots, i_k, y_1, \ldots, y_n]$. By indiscernibility, $\mathfrak{A}$ also satisfies $\varphi[i_0, \ldots, i_k, z_1, \ldots, z_n]$, i.e.,

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, z_1, \ldots, z_n] = s[i_0, \ldots, i_k].$$

Therefore $t^{\mathfrak{A}}[x_1, \ldots, x_m, y_1, \ldots, y_n] = t^{\mathfrak{A}}[x_1, \ldots, x_m, z_1, \ldots, z_n]$.

(iii) $\to$ (i) and "moreover:" Let $(\mathfrak{A}, I)$ be a (Σ, α)-model, where $\alpha > \omega$ is a limit ordinal, and assume (iii). Let $\gamma \geq \omega$ be a limit ordinal and let $x \in \mathit{Ord}^{\mathfrak{A}}$ be less than i_γ, the γth element of I. We shall show that $x \in H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. Since $\mathfrak{A} = H^{\mathfrak{A}}(I)$, there is a Skolem term t and $x_1 < \ldots < x_m < y_1 \ldots < y_n \in I$ such that $y_1 = i_\gamma$ and $x = t^{\mathfrak{A}}[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Let us choose $w_1, \ldots, w_n$ and $z_1, \ldots, z_n$ in I such that

$$x_1 < \ldots < x_m < w_1 < \ldots < w_n < y_1 < \ldots < y_n < z_1 \ldots < z_n.$$

Now since $x < y_1$, it follows from (18.13) that

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, y_1, \ldots, y_n] = t[x_1, \ldots, x_m, z_1, \ldots, z_n].$$

However, by indiscernibility, this implies that

$$\mathfrak{A} \vDash t[x_1, \ldots, x_m, w_1, \ldots, w_n] = t[x_1, \ldots, x_m, z_1, \ldots, z_n],$$

and hence $x = t^{\mathfrak{A}}[x_1, \ldots, x_m, w_1, \ldots, w_n]$. Therefore $x \in H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. □

Thus we say that an E.M. set Σ is *remarkable* if it is unbounded and contains the formulas (18.13) for all Skolem terms t.

An important consequence of remarkability is the following: Let $(\mathfrak{A}, I)$ be a remarkable (Σ, α)-model and let $\gamma < \alpha$ be a limit ordinal. Let $J = \{i_\xi : \xi < \gamma\}$ and let $\mathfrak{B} = H^{\mathfrak{A}}(J)$. Then $(\mathfrak{B}, J)$ is the (Σ, γ)-model, and the ordinals of $\mathfrak{B}$ form an initial segment of the ordinals of $\mathfrak{A}$.

Another consequence of remarkability is that the indiscernibles form a closed unbounded subset of ordinals. Let $(\mathfrak{A}, I)$ be the (Σ, α)-model. We say that the set I is *closed* in $Ord^{\mathfrak{A}}$ if for every limit $\gamma < \alpha$, i_γ is the least upper bound (in the linearly ordered set $Ord^{\mathfrak{A}}$) of the set $\{i_\xi : \xi < \gamma\}$.

Lemma 18.12. *If* $(\mathfrak{A}, I)$ *is remarkable, then* I *is closed in* $Ord^{\mathfrak{A}}$.

Proof. Let $\gamma < \alpha$ be a limit ordinal. If x is an ordinal of $\mathfrak{A}$ less than i_γ, then by remarkability, x is in the (Σ, γ)-model $\mathfrak{B} = H^{\mathfrak{A}}(\{i_\xi : \xi < \gamma\})$. However, since Σ is unbounded, $\mathfrak{B}$ is an unbounded (Σ, γ)-model and hence $x < i_\xi$ for some $\xi < \gamma$. Hence i_γ is the least upper bound of $\{i_\xi : \xi < \gamma\}$. □

Proof of Silver's Theorem and $0^\sharp$

Let us call an E.M. set Σ *well-founded* if every (Σ, α)-model is well-founded, and let us consider the statement:

(18.15) There exists a well-founded remarkable E.M. set.

We shall prove Theorem 18.1 in two steps: First we shall show that both (i) and (ii) are consequences of the assumption that there exists a well-founded remarkable E.M. set, and then we shall show that if there exists a Ramsey cardinal, then (18.15) holds. (Note that by Lemma 18.9 it suffices to find a well-founded remarkable model with uncountably many indiscernibles.)

Thus let us assume that there exists a well-founded remarkable E.M. set and let Σ be such a set.

For every limit ordinal α, the (Σ, α)-model is a well-founded model elementarily equivalent to some L_γ, and so by (13.13) is (isomorphic to) some L_β.

Lemma 18.13. *If κ is an uncountable cardinal, then the universe of the (Σ, κ)-model is L_κ.*

Proof. The (Σ, κ)-model is (L_β, I) for some β; since $|I| = \kappa$, we clearly have $\beta \geq \kappa$. To prove that $\beta = \kappa$, assume that $\beta > \kappa$. Since I is unbounded in β and has order-type κ, there is a limit ordinal $\gamma < \kappa$ such that $\kappa < i_\gamma$. By remarkability, all ordinals less than i_γ are in the (Σ, γ)-model $\mathfrak{A} = H(\{i_\xi : \xi < \gamma\})$. This is a contradiction since on the one hand we have $\kappa \subset \mathfrak{A}$, and on the other hand $|\mathfrak{A}| = |\gamma| < \kappa$. □

For each uncountable cardinal κ, let I_κ be the unique subset of κ such that (L_κ, I_κ) is the (Σ, κ)-model. By Lemma 18.12, I_κ is closed and unbounded in κ.

Lemma 18.14. *If $\kappa < \lambda$ are uncountable cardinals, then $I_\lambda \cap \kappa = I_\kappa$, and $H^{L_\lambda}(I_\kappa) = L_\kappa$.*

Proof. Let J be the set consisting of the first κ members of I_λ and let $\mathfrak{A} = H^{L_\lambda}(J)$. Then $(\mathfrak{A}, J)$ is a (Σ, κ)-model and the ordinals of $\mathfrak{A}$ are an initial segment of λ, say $Ord^{\mathfrak{A}} = \beta$. Since $(\mathfrak{A}, J)$ is isomorphic to (L_κ, I_κ), it is clear that $\beta = \kappa$ and $J = I_\kappa$. Hence $I_\lambda \cap \kappa = I_\kappa$.

Now since $\mathfrak{A} \prec L_\lambda$, $\mathfrak{A}$ is closed under the definable function $F(\alpha) =$ the αth set in the well-ordering $<_L$, and since $Ord^{\mathfrak{A}} = \kappa$, we have $A = \{F(\alpha) : \alpha < \kappa\} = L_\kappa$. □

Using this lemma, we can now prove both (i) and (ii) of Theorem 18.1 except for the uniqueness of Silver indiscernibles. We let

$$I = \bigcup \{I_\kappa : \kappa \text{ is an uncountable cardinal}\}. \tag{18.16}$$

For each uncountable cardinal κ, $I \cap \kappa = I_\kappa$ is a closed unbounded set of order-type κ, and is a set of indiscernibles for $(L_\kappa, \in)$; moreover, by Lemma 18.7(iii), every $a \in L_\kappa$ is definable in L_κ from I_κ. Let $\kappa < \lambda$ be uncountable cardinals. Since I_λ is closed in L_λ and $I_\lambda \cap \kappa = I_\kappa$, it follows that $\kappa \in I_\lambda$; hence I contains all uncountable cardinals. Also, since $L_\kappa = H^{L_\lambda}(I_\kappa)$, we have $L_\kappa \prec L_\lambda$.

The next two lemmas prove the uniqueness of Silver indiscernibles and of the corresponding E.M. set.

Lemma 18.15. *There is at most one well-founded remarkable E.M. set.*

Proof. Assuming that there is one such Σ, we define the class I in (18.16). Now since $L_{\aleph_\omega}$ is the $(\Sigma, \aleph_\omega)$-model and $\aleph_n \in I$ for each $n \geq 1$, we have

$$\varphi(v_1, \ldots, v_n) \in \Sigma \quad \text{if and only if} \quad L_{\aleph_\omega} \vDash \varphi[\aleph_1, \ldots, \aleph_n] \tag{18.17}$$

which proves that Σ is unique. □

We therefore define $0^\sharp$ (*zero-sharp*):

(18.18) $0^\sharp$ is the unique well-founded remarkable E.M. set if it exists.

The uniqueness of Silver indiscernibles now follows from:

Lemma 18.16. *For every regular uncountable cardinal κ there is at most one closed unbounded set of indiscernibles X for L_κ such that $L_\kappa = H^{L_\kappa}(X)$.*

Proof. Let $\Sigma = \Sigma(L_\kappa, X)$. Since X is closed unbounded, it follows that $X \cap I$ is infinite, and $\Sigma(L_\kappa, X) = \Sigma(L_\kappa, X \cap I) = \Sigma(L_\kappa, I \cap \kappa)$. Hence $\Sigma = 0^\sharp$ and since (L_κ, X) is the (Σ, κ)-model, we have $X = I \cap \kappa$. □

Thus we have proved (i) and (ii) of Theorem 18.1 under the assumption that $0^\sharp$ exists. On the other hand, if (ii) holds, then $0^\sharp$ exists because, e.g., $(L_{\omega_1}, I \cap \omega_1)$ is a remarkable well-founded model with $\aleph_1$ indiscernibles. To complete the proof of Theorem 18.1, it remains to show that if there is a Ramsey cardinal, then $0^\sharp$ exists. That will follow from:

Lemma 18.17. *Let κ be an uncountable cardinal. If there exists a limit ordinal λ such that $(L_\lambda, \in)$ has a set of indiscernibles of order-type κ, then there exist a limit ordinal γ and a set $I \subset \gamma$ of order-type κ such that (L_γ, I) is remarkable.*

It follows that if κ is Ramsey, then by Corollary 17.26 $(L_\kappa, \in)$ has a set of indiscernibles of order-type κ. By Lemma 18.17, there exists a remarkable model (L_γ, I) where I has order-type κ. By Lemma 18.9, $\Sigma(L_\gamma, I)$ is well-founded and remarkable and hence $0^\sharp$ exists.

Proof. Let λ be the least limit ordinal such that $(L_\lambda, \in)$ has a set of indiscernibles $I \subset \lambda$ of order-type κ. We shall show first that there is a set of indiscernibles $I \subset \lambda$ for L_λ, of order-type κ, such that $H^{L_\lambda}(I) = L_\lambda$. Let J be any set of indiscernibles for L_λ, of order-type κ, and let $\mathfrak{A} = H^{L_\lambda}(J)$. Then $\mathfrak{A} \prec L_\lambda$ and hence $\mathfrak{A}$ is isomorphic to some L_β, $\beta \le \lambda$, by the collapsing map π. Now $I = \pi(J)$ is a set of indiscernibles for L_β, and $H^{L_\beta}(I) = L_\beta$. By the minimality of λ, we have $\beta = \lambda$ and hence I is as claimed.

Next we show that any such set I is unbounded in λ. If not, there is a limit ordinal $\alpha < \lambda$ such that $I \subset \alpha$. There is a Skolem term t and $\gamma_1 < \ldots < \gamma_n \in I$ such that $\alpha = t^{L_\lambda}[\gamma_1, \ldots, \gamma_n]$. We claim that the set $J = \{i \in I : i > \gamma_n\}$ is a set of indiscernibles for $(L_\alpha, \in)$. If $\varphi(v_1, \ldots, v_n)$ is a formula, then for any $i_1 < \ldots < i_k \in J$, L_α satisfies $\varphi[i_1, \ldots, i_k]$ if and only if L_λ satisfies the formula

$$L_\alpha \vDash \varphi[i_1, \ldots, i_k]. \tag{18.19}$$

The formula (18.19) is a formula about α, i_1, …, i_k, and since $\alpha = t^{L_\lambda}[\gamma_1, \ldots, \gamma_n]$ there is a formula $\psi(u_1, \ldots, u_n, v_1, \ldots, v_k)$ such that L_λ satisfies (18.19) if and only if

$$L_\lambda \vDash \psi[\gamma_1, \ldots, \gamma_n, i_1, \ldots, i_k]. \tag{18.20}$$

By the indiscernibility of I, the truth of (18.20) is independent of the choice of $i_1 < \ldots < i_k$ in I provided $\gamma_n < i_1$. Hence the truth of (18.19) is independent of the choice of $i_1 < \ldots < i_k$ in J. Hence J is a set of indiscernibles for L_α, and this contradicts the minimality of λ since $\alpha < \lambda$ and the order-type of J is κ.

Finally, let I be a set of indiscernibles for L_λ of order-type κ such that $H^{L_\lambda}(I) = L_\lambda$, and that i_ω, the ωth element of I, is least possible. We will show that (L_λ, I) is remarkable.

Let us assume that (L_λ, I) is not remarkable. Then there is a Skolem term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ such that the following holds in L_λ for any $x_1 < \ldots < x_m < y_1 < \ldots < y_n < z_1 < \ldots < z_n$:

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) < y_1 \tag{18.21}$$

and

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) \neq t(x_1, \ldots, x_m, z_1, \ldots, z_n). \tag{18.22}$$

Let $x_1, \ldots, x_m$ be the first m elements of I. We now consider the following increasing n-termed sequences in I: Let u_0 be the sequence of first n indiscernibles after x_m, let u_1 be the first n indiscernibles after u_0, etc.; for each $\alpha < \kappa$, let

$$\gamma_\alpha = t(x_1, \ldots, x_m, u_\alpha).$$

By indiscernibility, applied to the formula (18.22), we have $\gamma_\alpha \neq \gamma_\beta$ whenever $\alpha \neq \beta$. In fact, in (18.22) we have either $<$ or $>$ (in place of $\neq$); but $>$ is impossible since that would mean that $\gamma_\alpha > \gamma_\beta$ whenever $\alpha < \beta$. Thus $\langle \gamma_\alpha : \alpha < \kappa \rangle$ is an increasing sequence of ordinals.

We claim that $J = \{\gamma_\alpha : \alpha < \kappa\}$ is a set of indiscernibles for L_λ. This is so because for any formula φ, the truth value of $\varphi(\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_k})$ in L_λ does not depend on the choice of $\gamma_{\alpha_1} < \ldots < \gamma_{\alpha_k}$ in J because by the definition of the u_α, the truth value of $\varphi(t(x_1 \ldots, x_m, u_{\alpha_1}), \ldots, t(x_1 \ldots, x_m, u_{\alpha_k}))$ does not depend on the choice of $\alpha_1 < \ldots < \alpha_k$.

Hence $\{\gamma_\alpha : \alpha < \kappa\}$ is a set of indiscernibles for L_λ. Since i_ω is the first member of u_ω, it follows by (18.21) that $\gamma_\omega < i_\omega$. Now if $A = H(J)$ and π is the transitive collapse of A, then, as we proved in the first paragraph, $\pi(A) = L_\lambda$, and $K = \pi(J)$ is a set of indiscernibles for L_λ of order-type κ such that $H^{L_\lambda}(K) = L_\lambda$. However, $\pi(\gamma_\omega) \leq \gamma_\omega < i_\omega$, and so the ωth member of K is smaller than i_ω, contrary to our assumption. Hence (L_λ, I) is remarkable.

□

This completes the proof of Theorem 18.1. Lemma 18.17 also gives the following equivalence:

Corollary 18.18. $0^\sharp$ *exists if and only if for some limit ordinal* λ, *the model* $(L_\lambda, \in)$ *has an uncountable set of indiscernibles.* □

The set $0^\sharp$ is, strictly speaking, a set of formulas. But as formulas can be coded by natural numbers, we can regard $0^\sharp$ as a subset of ω. This convention has become standard. Moreover, in Chapter 25 we show that $0^\sharp$ is a Π^1_2 singleton, and so $\{0^\sharp\}$ is a Δ^1_3 set. At this point we outline the proof of absoluteness of $0^\sharp$:

Lemma 18.19. *The property "Σ is a well-founded remarkable E.M. set" is absolute for every inner model of* ZF. *Hence $M \vDash 0^\sharp$ exists if and only if $0^\sharp \in M$ in which case $(0^\sharp)^M = 0^\sharp$.*

Proof. We first replace the property "Σ is an E.M. set" by a syntactical condition.

Let $\hat{\mathcal{L}}$ be the language $\{\in, c_1, c_2, \ldots, c_n, \ldots\}$ where c_n, $n < \omega$, are constant symbols. For every $\in$-formula $\varphi(v_1, \ldots, v_n)$ let $\hat{\varphi}$ be the sentence $\varphi(c_1, \ldots, c_n)$ of $\hat{\mathcal{L}}$. For each set of formulas Σ, let $\hat{\Sigma}$ be the set containing (i) all $\hat{\varphi}$ for $\varphi \in \Sigma$, (ii) the sentence "c_1 is an ordinal and $c_1 < c_2$," and (iii) the sentence "$\varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n})$" for every $\varphi \in \Sigma$ and any $i_1 < \ldots < i_n$, $j_1 < \ldots < j_n$, (iv) all axioms of $\text{ZFC} + V = L$. Let us consider the condition

(18.23) $\hat{\Sigma}$ is consistent.

Clearly, if Σ is an E.M. set, then $\hat{\Sigma}$ is consistent, for we simply interpret the constants c_n, $n < \omega$, as some Silver indiscernibles. Conversely, if $\hat{\Sigma}$ is consistent, then $\hat{\Sigma}$ has a model and that model provides us with a (Σ, ω)-model (with indiscernibles c_n, $n < \omega$) and the proof of Lemma 18.7 goes through. Therefore (18.23) holds if and only if Σ is an E.M. set.

As remarkability can also be expressed as a syntactical property, it follows that "Σ is a remarkable E.M. set" can be written as a Δ_0 property (with parameters V_ω and *Form*). As such it is absolute for transitive models.

If Σ is a remarkable E.M. set, then for every limit ordinal α there is a unique (up to isomorphism) (Σ, α)-model and we can find one $((A, E), I)$ such that $I = \alpha$ and that $<^{\mathfrak{A}}$ (i.e., E) agrees with $<$ on α. If $((A, E), \alpha)$ is such, we say that "$((A, E), \alpha)$ is a (Σ, α)-model." This last property is a Δ_1 property of Σ, (A, E), α, V_ω and *Form*. Then $\Sigma = 0^\sharp$ if and only if

(18.24) $\forall \alpha\ \forall (A, E)$ (if $((A, E), \alpha)$ is a (Σ, α)-model, then (A, E) is well-founded).

As well-foundedness is absolute for transitive models of ZF, it follows that (18.24) is absolute for inner models of ZF (which contain all ordinals), and therefore "$\Sigma = 0^\sharp$" is absolute. □

Elementary Embeddings of L

In Chapter 17 we proved that a well-founded ultrapower of the universe induces an elementary embedding $j_U : V \to \text{Ult}$, and conversely, if $j : V \to M$

is a nontrivial elementary embedding, then (17.2) defines a normal measure on the least ordinal moved by j.

Let j be a nontrivial elementary embedding of the universe, and let M be a transitive model of ZFC, containing all ordinals. Let $N = j(M) = \bigcup_{\alpha \in Ord} j(M \cap V_\alpha)$. Then N is a transitive model of ZF and $j : M \to N$ is elementary:

(18.25) $$M \vDash \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad N \vDash \varphi(j(a_1), \ldots, j(a_n)).$$

((18.25) is proved by induction on the complexity of φ). In particular, if $M = L$, then $j(V) \vDash$ (N is the constructible universe), and so $N = L$, and $j \restriction L$ is an elementary embedding of L in L. Note that by Scott's Theorem, the function $j \restriction L$ is not a class in L; thus if there exists an elementary embedding of L (into L), then $V \neq L$.

If $0^\sharp$ exists, then there are nontrivial elementary embeddings of L. In fact, let j be any order-preserving function from the class I of all Silver indiscernibles into itself. Then j can be extended to an elementary embedding of L; we simply let

(18.26) $$j(t^L[\gamma_1, \ldots, \gamma_n]) = t^L[j(\gamma_1), \ldots, j(\gamma_n)]$$

for every Skolem term t and any Silver indiscernibles $\gamma_1 < \ldots < \gamma_n$. We shall prove that the converse is true, that if there is a nontrivial elementary embedding of L, then $0^\sharp$ exists:

Theorem 18.20 (Kunen). *The following are equivalent*:

(i) $0^\sharp$ *exists.*
(ii) *There is a nontrivial elementary embedding* $j : L \to L$.

Toward the proof of Kunen's Theorem, let us investigate elementary embeddings $j : M \to N$ where M is a transitive model of ZFC.

Definition 18.21. Let M be a transitive model of ZFC, and let κ be a cardinal in M. An *M-ultrafilter* on κ is a collection $D \subset P^M(\kappa)$ that is an ultrafilter on the algebra of sets $P^M(\kappa)$. Explicitly,

(18.27)
(i) $\kappa \in D$ and $\emptyset \notin D$;
(ii) if $X \in D$ and $Y \in D$, then $X \cap Y \in D$;
(iii) if $X \in D$ and $Y \in M$ is such that $X \subset Y$, then $Y \in D$;
(iv) for every $X \subset \kappa$ such that $X \in M$, either X or $\kappa - X$ is in D.

D is *κ-complete* if whenever $\alpha < \kappa$ and $\{X_\xi : \xi < \alpha\} \in M$ is such that $X_\xi \in D$ for all $\xi < \alpha$, then $\bigcap_{\xi<\alpha} X_\alpha \in D$; D is *normal* if whenever $f \in M$ is a regressive function on $X \in D$, then f is constant on some $Y \in D$.

If $j : M \to N$ is an elementary embedding, then the least ordinal moved by j is called the *critical point* of j.

Lemma 18.22. *If $j : M \to N$ is an elementary embedding and κ is the critical point of j then κ is a regular uncountable cardinal in M, and $D = \{X \in P^M(\kappa) : \kappa \in j(X)\}$ is a nonprincipal normal κ-complete M-ultrafilter on κ.*

Proof. Exactly as the proof of Lemma 17.2. Note that κ-completeness of D implies that κ is regular in M. □

If D is an M-ultrafilter on κ, one can construct the ultrapower of M by D as follows: Consider, in M, the class of all functions f with domain κ. Using D, define an equivalence relation $=^*$ and the relation $\in^*$ as usual;

$$f =^* g \leftrightarrow \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D,$$
$$f \in^* g \leftrightarrow \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in D.$$

Then define equivalence classes mod $=^*$, and the model $\text{Ult} = \text{Ult}_D(M)$. An analog of Theorem 12.3 is easily verified:

$$\text{Ult} \vDash \varphi([f_1], \ldots, [f_n]) \text{ if and only if } \{\alpha < \gamma : M \vDash \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in D.$$

If for each $a \in M$, c_a denotes the constant function with value a, then

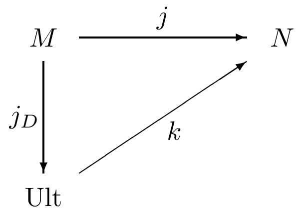
$$j_D(a) = [c_a]$$

defines an elementary embedding of M in Ult.

The ultrapower of M by an M-ultrafilter D is not necessarily well-founded, even if D is countably complete.

If $j : M \to N$ is an elementary embedding with M and N being transitive models, and if D is the M-ultrafilter $\{X : \kappa \in j(X)\}$, then, as in Lemma 17.4, we have the commutative diagram

(18.28)



and it follows that $\text{Ult}_D(M)$ is well-founded. (If $[f_0] \ni^* [f_1] \ni^* \ldots$ were a descending sequence in Ult, then $k([f_0]) \ni k([f_1]) \ni \ldots$ would be a descending sequence in N.)

We proceed with the proof of Kunen's Theorem.

Let $j : L \to L$ be an elementary embedding. We shall first replace j by a more manageable embedding. We let D be the L-ultrafilter $\{X \in P^L(\gamma) : \gamma \in j(X)\}$ where γ is the critical point of j. The ultrapower $\text{Ult}_D(L)$ is well-founded and so we identify Ult with its transitive collapse L; let j_D be the canonical embedding, $j_D : L \to L$. The critical point of j_D is γ because D is γ-complete.

Lemma 18.23. *If κ is a limit cardinal such that* $\operatorname{cf}\kappa > \gamma$, *then* $j_D(\kappa) = \kappa$.

Proof. Every constructible function $f : \gamma \to \kappa$ is bounded by some $\alpha < \kappa$ and hence $[f] < [c_\alpha]$ (where c_α is the constant function with value α). Thus $j_D(\kappa) = \lim_{\alpha\to\kappa} j_D(\alpha)$. Now if $\alpha < \kappa$, then $|j_D(\alpha)| \le |(\alpha^\gamma)^L|$, hence $j_D(\alpha) < \kappa$. It follows that $j_D(\kappa) = \kappa$. □

Let us drop the subscript D and simply assume that $j : L \to L$ is an elementary embedding, that γ is its critical point and that $j(\kappa) = \kappa$ for every limit cardinal κ such that $\operatorname{cf}\kappa > \gamma$.

Let U_0 be the class of all limit cardinals κ with $\operatorname{cf}\kappa > \gamma$; by transfinite induction we define a sequence of classes $U_0 \supset U_1 \supset \ldots \supset U_\alpha \supset \ldots$ as follows:

(18.29)
$$U_{\alpha+1} = \{\kappa \in U_\alpha : |U_\alpha \cap \kappa| = \kappa\},$$
$$U_\lambda = \bigcap_{\alpha<\lambda} U_\alpha \qquad (\lambda \text{ limit}).$$

(That is, $U_{\alpha+1}$ consists of fixed points of the increasing enumeration of U_α.) Each U_α is nonempty, and in fact a proper class. To see this, verify, by induction on α, that each U_α is a proper class and is δ-*closed*, for each δ with $\operatorname{cf}\delta > \gamma$; that is, whenever $\langle \kappa_\xi : \xi < \delta\rangle$ is an increasing sequence in U_α, then $\lim_{\xi\to\delta} \kappa_\xi \in U_\alpha$. Hence each U_α is nonempty, and we choose a cardinal $\kappa \in U_{\omega_1}$.

Thus κ is such that $\operatorname{cf}\kappa > \gamma$ and κ is the κth element of each U_α, $\alpha < \omega_1$. We shall find a set of $\aleph_1$ indiscernibles for $(L_\kappa, \in)$.

Since $j : L \to L$ is an elementary and $j(\kappa) = \kappa$, it is clear that the mapping $i = j{\restriction}L_\kappa$ is an elementary embedding of $(L_\kappa, \in)$ into $(L_\kappa, \in)$. We shall use i and the sets $U_\alpha \cap \kappa$, $\alpha < \omega_1$, to produce indiscernibles γ_α, $\alpha < \omega_1$, for L_κ. Let $X_\alpha = U_\alpha \cap \kappa$ for each $\alpha < \omega_1$, and recall that γ is the critical point of i.

For each $\alpha < \omega_1$, we let

(18.30)
$$M_\alpha = H^{L_\alpha}(\gamma \cup X_\alpha).$$

M_α is an elementary submodel of L_κ.

If π_α is the transitive collapse of M_α, then because $|X_\alpha| = \kappa$, we have $\pi_\alpha(M_\alpha) = L_\kappa$. Thus if we denote $i_\alpha = \pi_\alpha^{-1}$, then i_α is an elementary embedding of L_κ in L_κ. Let $\gamma_\alpha = i_\alpha(\gamma)$.

Lemma 18.24.

(i) *The ordinal γ_α is the least ordinal greater than γ in M_α.*
(ii) *If $\alpha < \beta$ and $x \in M_\beta$, then $i_\alpha(x) = x$. In particular, $i_\alpha(\gamma_\beta) = \gamma_\beta$.*
(iii) *If $\alpha < \beta$, then $\gamma_\alpha < \gamma_\beta$.*

Proof. (i) Since $\gamma \subset M_\alpha$, $i_\alpha(\gamma)$ is the least ordinal in M_α greater than or equal to γ; thus it suffices to show that $\gamma \notin M_\alpha$. If $x \in M_\alpha$, then $x = t[\eta_1, \dots, \eta_n]$ where t is a Skolem term and the η's are either smaller than γ or elements of X_α. For all such η, $i(\eta) = \eta$ and hence $i(x) = i(t(\eta_1, \dots, \eta_n)) = t(i(\eta_1), \dots, i(\eta_n)) = x$. However, $i(\gamma) \neq \gamma$ and so $\gamma \notin M_\alpha$.

(ii) Each $x \in M_\beta$ is of the form $t[\eta_1, \dots, \eta_n]$ where the η's are either $< \gamma$ or in X_β. If $\eta < \gamma$, then clearly $i_\alpha(\eta) = \eta$. If $\eta \in X_\beta$, then because $\alpha < \beta$, we have $|X_\alpha \cap \eta| = \eta$ and hence $\pi_\alpha(\eta) = \eta$; in other words, $i_\alpha(\eta) = \eta$. Therefore $i_\alpha(x) = x$.

(iii) If $\alpha < \beta$, then $M_\alpha \supset M_\beta$ and hence $\gamma_\alpha \le \gamma_\beta$. To see that $\gamma_\alpha \neq \gamma_\beta$, note that because $\gamma_\alpha > \gamma$, we have $i_\alpha(\gamma_\alpha) > i_\alpha(\gamma) = \gamma_\alpha$, while $i_\alpha(\gamma_\beta) = \gamma_\beta$. □

Lemma 18.25. *If $\alpha < \beta$, then there is an elementary embedding $i_{\alpha,\beta} : L_\kappa \to L_\kappa$ such that for every ξ that is either smaller than α or greater than β we have $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$, and $i_{\alpha,\beta}(\gamma_\alpha) = \gamma_\beta$.*

Proof. Let $M_{\alpha,\beta} = H^{L_\kappa}(\gamma_\alpha \cup X_\beta)$, and let $i_{\alpha,\beta} = \pi^{-1}_{\alpha,\beta}$ where $\pi_{\alpha,\beta}$ is the transitive collapse of $M_{\alpha,\beta}$. The mapping $i_{\alpha,\beta}$ is an elementary embedding of L_κ in L_κ.

If $\eta < \gamma_\alpha$, then clearly $i_{\alpha,\beta}(\eta) = \eta$; in particular $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ if $\xi < \alpha$. If $x \in M_{\beta+1}$, then $x = t(\eta_1, \dots, \eta_n)$ where the η's are either smaller than γ or elements of $X_{\beta+1}$. If $\eta \in X_{\beta+1}$, then $|X_\beta \cap \eta| = \eta$ and therefore $i_{\alpha,\beta}(\eta) = \eta$. Hence $i_{\alpha,\beta}(x) = x$ for every $x \in M_{\beta+1}$, and in particular $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ if $\xi > \beta$.

Now we shall show that $i_{\alpha,\beta}(\gamma_\alpha) = \gamma_\beta$. Since $M_{\alpha,\beta} \supset M_\beta$, we have $\gamma_\beta \in M_{\alpha,\beta}$; and since $\gamma_\alpha \subset M_{\alpha,\beta}$, $i_{\alpha,\beta}(\gamma_\alpha)$ is the least ordinal in $M_{\alpha,\beta}$ greater than or equal to γ_α; hence we have $\gamma_\alpha \le i_{\alpha,\beta}(\gamma_\alpha) \le \gamma_\beta$.

Thus it suffices to show that there is no ordinal $\delta \in M_{\alpha,\beta}$ such that $\gamma_\alpha \le \delta < \gamma_\beta$. Otherwise there is some $\delta = t(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_k)$ such that the ξ's are $< \gamma_\alpha$ and the η's are in X_β (and t is a Skolem term) and that $\gamma_\alpha \le \delta < \gamma_\beta$. Thus we have:

$$(L_\kappa, \in) \vDash \exists \xi < \gamma_\alpha \text{ such that } \gamma_\alpha \le t(\xi, \eta) < \gamma_\beta. \tag{18.31}$$

The formula in (18.31) is a formula φ about γ_α, η, and γ_β. At this point, we apply the elementary embedding $i_\alpha : L_\kappa \to L_\kappa$ backward. That is, γ_α, the η's and γ_β are all in the range of i_α: $\gamma_\alpha = i_\alpha(\gamma)$, $\eta = i_\alpha(\eta)$, and $\gamma_\beta = i_\alpha(\gamma_\beta)$; and since $L_\kappa \vDash \varphi[i_\alpha(\gamma), i_\alpha(\eta), i_\alpha(\gamma_\beta)]$, we conclude that $L_\kappa \vDash \varphi[\gamma, \eta, \gamma_\beta]$, namely

$$(L_\kappa, \in) \vDash \exists \xi < \gamma \text{ such that } \gamma \le t(\xi, \eta) < \gamma_\beta.$$

Thus pick some ξ's less than γ such that $\gamma \le t(\xi, \eta) < \gamma_\beta$. Since $\xi \in \gamma$ and $\eta \in X_\beta$, we have $t(\xi, \eta) \in M_\beta$, which means that $t(\xi, \eta)$ is an ordinal in M_β between γ and γ_β, and that contradicts Lemma 18.24(i). □

The proof of Kunen's Theorem will be complete when we show:

Lemma 18.26. *The set $\{\gamma_\alpha : \alpha < \omega_1\}$ is a set of indiscernibles for $(L_\kappa, \in)$.*

Proof. Let φ be a formula and let $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ be two sequences of countable ordinals. We wish to show that

$$(18.32) \qquad L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \quad \text{if and only if} \quad L_\kappa \vDash \varphi[\gamma_{\beta_1}, \ldots, \gamma_{\beta_n}].$$

Let us pick $\delta_1 < \ldots < \delta_n$ such that $\alpha_n < \delta_1$ and $\beta_n < \delta_1$. First we apply the elementary embedding i_{α_n,δ_n} and get

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\alpha_n}] \quad \text{if and only if} \quad L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_{n-1}}, \gamma_{\delta_n}]$$

because $i_{\alpha_n,\delta_n}(\gamma_{\alpha_n}) = \gamma_{\delta_n}$, and preserves the other γ's. The we apply $i_{\alpha_{n-1},\delta_{n-1}}$ with a similar effect, and by a successive application of $i_{\alpha_{n-2},\delta_{n-2}}$, …, i_{α_1,δ_1} we get

$$L_\kappa \vDash \varphi[\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_n}] \quad \text{if and only if} \quad L_\kappa \vDash \varphi[\gamma_{\delta_1}, \ldots, \gamma_{\delta_n}].$$

Then we do the same for the β's and δ's as we did for the α's and δ's, and (18.32) follows. □

This completes the proof of Theorem 18.20.

The following result is related to Kunen's Theorem:

Theorem 18.27. *Let $j : L_\alpha \to L_\beta$ be an elementary embedding and let γ be the critical point of j. If $\gamma < |\alpha|$, then $0^\sharp$ exists.*

Proof. Let γ be the critical point of j. Since $\gamma < |\alpha|$, every $X \subset \gamma$ is in L_α, and so $D = \{X \subset \gamma : \gamma \in j(X)\}$ is an L-ultrafilter.

Let us consider the ultrapower $\mathrm{Ult}_D(L)$. If the ultrapower is well-founded, then we are done because then the canonical embedding $j_D : L \to \mathrm{Ult}_D(L)$ is a nontrivial elementary embedding of L in L. Thus we shall prove that $\mathrm{Ult}_D(L)$ is well-founded.

Let us assume that f_0, f_1, …, f_n, … is a counterexample to well-foundedness of the ultrapower. Each f_n is a constructible function on γ and $\{\xi < \gamma : f_{n+1}(\xi) \in f_n(\xi)\} \in D$ for all $n < \omega$. Let θ be a limit ordinal such that $f_n \in L_\theta$ for all n and let M be an elementary submodel of $(L_\theta, \in)$ such that $|M| = |\gamma|$, $\gamma \subset M$, and $f_n \in M$ for all n. Let π be the transitive collapse of M, $\pi(M) = L_\eta$, and let $g_n = \pi(f_n)$, for all n.

Since $\pi(\xi) = \xi$ for all $\xi < \gamma$, we see that for each $\xi < \gamma$ and all n, $g_{n+1}(\xi) \in g_n(\xi)$ if and only if $f_{n+1}(\xi) \in f_n(\xi)$, and hence g_0, g_1, …, g_n, … is also a counterexample to well-foundedness of the ultrapower. However, since each g_n is in L and $|\eta| = |\gamma| < |\alpha|$, we have $g_n \in L_\alpha$ for all n. Thus $j(g_n)$ is defined for all n, and we have,

$$\{\xi < \gamma : g_{n+1}(\xi) \in g_n(\xi)\} \in D \quad \text{if and only if} \quad (j(g_{n+1}))(\gamma) \in (j(g_n))(\gamma).$$

Now we reached a contradiction because $(j(g_0))(\gamma) \ni (j(g_1))(\gamma) \ni \ldots$ would be a descending sequence. □

Corollary 18.28. *If there is a Jónsson cardinal, then $0^\sharp$ exists.*

Proof. Let κ be a Jónsson cardinal and let us consider the model $(L_\kappa, \in)$. Let A be an elementary submodel, of size κ, such that $A \neq L_\kappa$. Let π be the transitive collapse of A; clearly, $\pi(A) = L_\kappa$. Thus $j = \pi^{-1}$ is a nontrivial elementary embedding of L_κ in L_κ. Since κ is a cardinal, $0^\sharp$ exists by Theorem 18.27. □

Chang's Conjecture is the statement that every model of type $(\aleph_2, \aleph_1)$ has an elementary submodel of type $(\aleph_1, \aleph_0)$.

Corollary 18.29. *Chang's Conjecture implies that $0^\sharp$ exists.*

Proof. Consider the model $(L_{\omega_2}, \omega_1, \in)$, and let $\mathfrak{A} = (A, \omega_1 \cap A, \in)$ be its elementary submodel such that $|A| = \aleph_1$ and $|\omega_1 \cap A| = \aleph_0$. Let π be the transitive collapse of A; we have $\pi(A) = L_\alpha$ for some α such that $\omega_1 \leq \alpha < \omega_2$. Also, $\pi(\omega_1 \cap A)$ is a countable ordinal, and hence $\pi(\omega_1) < \omega_1$. Then $j = \pi^{-1}$ is an elementary embedding of L_α in L_{ω_2}, and its critical point is a countable ordinal. Hence $0^\sharp$ exists. □

All results about $0^\sharp$ and Silver indiscernibles for L proved in the present section can be relativized to obtain similar results for the models $L[x]$, where $x \subset \omega$.

In particular, if there exists a Ramsey cardinal there is for every $x \subset \omega$ a unique class I_x containing all uncountable cardinals such that for each uncountable cardinal κ, $I_x \cap \kappa$ is a set of indiscernibles for the model $(L_\kappa[x], \in, x)$ and all elements of $L_\kappa[x]$ are definable in the model from $I_x \cap \kappa$. Here x is considered a one-place predicate. Also, for every regular uncountable cardinal κ, $I_x \cap \kappa$ is closed unbounded in κ.

The proof of the relativization of Silver's Theorem uses models with indiscernibles $(\mathfrak{A}, I)$ where $\mathfrak{A}$ is elementarily equivalent to some $(L_\lambda[x], \in, x)$ where $\lambda > \omega$ is a limit ordinal. If κ is a Ramsey cardinal, then $(L_\kappa[x], \in, x)$ has a set of indiscernibles of size κ, and the theorem follows.

We define $x^\sharp$ as the unique set $\Sigma = \Sigma((L_\lambda[x], \in, x), I)$ that is well-founded and remarkable. If $x^\sharp$ exists, then we have

$$x^\sharp = \{\varphi : (L_{\aleph_\omega}[x], \in, x) \vDash \varphi[\aleph_1, \ldots, \aleph_n]\}.$$

Here φ is a formula of the language $\{\in, P\}$ where P is a one-place predicate (interpreted as x). Note that x is definable in the model $(L_\lambda[x], \in, x)$ (by the formula $P(v)$).

The real $x^\sharp$ is absolute for all transitive models M of ZF containing all ordinals such that $x^\sharp \in M$.

Also, "$x^\sharp$ exists" is equivalent to the existence of a nontrivial elementary embedding $j : L[x] \to L[x]$.

Jensen's Covering Theorem

The theorem presented in this section shows that in the absence of $0^\sharp$ the universe does not differ drastically from the constructible model. In particular, the cofinality function is closely related to the cofinality function in L, and every singular cardinal is a singular cardinal in L. Moreover, the Singular Cardinal Hypothesis holds and cardinal exponentiation is determined by the continuum function on regular cardinals.

Theorem 18.30 (Jensen's Covering Theorem). *If $0^\sharp$ does not exist, then for every uncountable set X of ordinals there exists a constructible set $Y \supset X$ such that $|Y| = |X|$.*

The Covering Theorem expresses the closeness between V and L: Every uncountable set of ordinals can be *covered* by a constructible set of the same cardinality. In other words, every set X of ordinals can be covered by some $Y \in L$ such that $|Y| \le |X| \cdot \aleph_1$. (This is best possible: In Chapter 28 we give an example of a forcing extension of L in which there is a countable set of ordinals that cannot be covered by a countable (in V) constructible set.)

The converse of the Covering Theorem is also true: If $0^\sharp$ exists then every uncountable cardinal is regular in L, and in particular, since $\aleph_\omega$ is regular in L, the countable set $\{\aleph_n : n < \omega\}$ cannot be covered by a constructible set of cardinality less than $\aleph_\omega$. This shows:

$$0^\sharp \text{ exists if and only if } \aleph_\omega \text{ is regular in } L.$$

Corollary 18.31. *If $0^\sharp$ does not exist then for every $\lambda \ge \aleph_2$, if λ is a regular cardinal in L then $\operatorname{cf} \lambda = |\lambda|$. Consequently, every singular cardinal is a singular cardinal in L.*

The assumption $\lambda \ge \aleph_2$ is necessary: The forcing mentioned above yields a model where $\lambda = \aleph_2^L$ is such that $|\lambda| = \aleph_1$ and $\operatorname{cf} \lambda = \omega$.

Proof. Let λ be a limit ordinal such that $\lambda \ge \omega_2$ and that λ is a regular cardinal in L. Let X be an unbounded subset of λ such that $|X| = \operatorname{cf} \lambda$. By the Covering Theorem, there exists a constructible set Y such that $X \subset Y \subset \lambda$ and that $|Y| = |X| \cdot \aleph_1$. Since Y is unbounded in λ and λ is a regular cardinal in L, we have $|Y| = |\lambda|$. This gives $|\lambda| = \aleph_1 \cdot \operatorname{cf} \lambda$ and since $\lambda \ge \aleph_2$, we have $|\lambda| = \operatorname{cf} \lambda$. ◻

Corollary 18.32. *If $0^\sharp$ does not exist then for every singular cardinal κ, $(\kappa^+)^L = \kappa^+$.*

Proof. Let κ be a singular cardinal and let λ be the successor cardinal of κ in L; we want to show that $\lambda = \kappa^+$. If not, then $|\lambda| = \kappa$, and since κ is singular, we have $\operatorname{cf} \lambda < \kappa$. However, this means that $\operatorname{cf} \lambda < |\lambda|$ which contradicts Corollary 18.31. ◻

Corollary 18.33. *If $0^\sharp$ does not exist then the Singular Cardinal Hypothesis holds.*

Proof. Let κ be such that $2^{\operatorname{cf}\kappa} < \kappa$, and let $A = [\kappa]^{\operatorname{cf}\kappa}$ be the set of all subsets of κ of size $\operatorname{cf}\kappa$. We shall show that $|A| \le \kappa^+$. By the Covering Theorem, for every $X \in A$ there exists a constructible $Y \subset \kappa$ such that $X \subset Y$ and $|Y| = \lambda$ where $\lambda = \aleph_1 \cdot \operatorname{cf}\kappa$. Thus

$$A \subset \bigcup\{[Y]^{\operatorname{cf}\kappa} : Y \in C\} \tag{18.33}$$

where $C = \{Y \subset \kappa : |Y| = \lambda \text{ and } Y \in L\}$. If $Y \in C$, then $|[Y]^{\operatorname{cf}\kappa}| = \lambda^{\operatorname{cf}\kappa} = (\aleph_1 \cdot \operatorname{cf}\kappa)^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa} < \kappa$. Since $|C| \le |P^L(\kappa)| = |(\kappa^+)^L| \le \kappa^+$ it follows from (18.33) that $|A| \le \kappa^+$. □

Corollary 18.34. *If $0^\sharp$ does not exist then if κ is a singular cardinal and if there exists a nonconstructible subset of κ, then some $\alpha < \kappa$ has a nonconstructible subset.*

Proof. Let κ be a singular cardinal and assume that each $\alpha < \kappa$ has only constructible subsets; we shall show that every subset of κ is constructible. It suffices to show that each subset of κ of size $\operatorname{cf}\kappa$ is constructible: If $A \subset \kappa$, let $\{\alpha_\nu : \nu < \operatorname{cf}\kappa\}$ be such that $\lim_\nu \alpha_\nu = \kappa$; then $\mathcal{A} = \{A \cap \alpha_\nu : \nu < \operatorname{cf}\kappa\}$ is a subset of L_κ of size $\le \operatorname{cf}\kappa$ and hence constructible. It follows that A is constructible.

Let $X \subset \kappa$ be such that $|X| \le \operatorname{cf}(\kappa)$. By the Covering Theorem, there exists a constructible set of ordinals $Y \supset X$ such that $|Y| < \kappa$. Let π be the isomorphism between Y and its order-type α; the function π is constructible and one-to-one. Since $|\alpha| = |Y| < \kappa$, we have $\alpha < \kappa$.

Let $Z = \pi(X)$. Then $Z \subset \alpha$ is constructible by the assumption, and hence $X = \pi^{-1}(Z)$ is also constructible. □

The rest of this chapter is devoted to the proof of the Covering Theorem. Jensen's proof of the Covering Theorem used a detailed analysis of construction of sets in L, the *fine structure theory*, see [1972]. The proof appeared in Devlin and Jensen [1975]. Subsequently, Silver and Magidor gave proofs that did not use the fine structure. The outline below is based on Magidor [1990] (and on Kanamori's presentation in [∞]).

Let us assume that there exists an uncountable set X of ordinals that cannot be covered by a constructible set of the same size. The goal is to produce a nontrivial elementary embedding from L into L. In fact, by Theorem 18.27 it suffices to find some $j : L_\alpha \to L_\beta$ with critical point below $|\alpha|$.

Let τ be the least ordinal such that there exists a set $X \subset \tau$ that cannot be covered, and let $X \subset \tau$ be such a set with $|X|$ least possible. Let $\nu = |X|$.

Lemma 18.35.

(i) *τ is a cardinal in L.*
(ii) *If $Y \in L$ covers X then $|Y|^L \geq \tau$.*
(iii) *ν is a regular cardinal, $\nu < \tau$, and $\nu = \aleph_1 \cdot \operatorname{cf} \tau$.*

Proof. (i) and (ii) follow from the minimality of τ.

(iii) $|X| < \tau$, because otherwise, $Y = \tau$ would cover X. Clearly, $|X| \geq \aleph_1 \cdot \operatorname{cf} \tau$; thus assume that $\nu > \aleph_1 \cdot \operatorname{cf} \tau$. Let $\tau = \lim_{\xi \to \operatorname{cf} \tau} \tau_\xi$. For each ξ, let $Y_\xi \in L$ cover $X \cap \tau_\xi$. Let $\{E_\alpha : \alpha < \tau\}$ be a constructible enumeration of all bounded constructible subsets of τ, and let $Z = \{\alpha < \tau : E_\alpha = Y_\xi$ for some $\xi\}$. By the minimality of ν, Z can be covered by some $W \in L$ of size $\aleph_1 \cdot \operatorname{cf} \tau$. Then the set $Y = \bigcup_{\alpha \in Z} E_\alpha$ covers X, a contradiction. □

Now let M be an elementary submodel of $(L_\tau, \in)$ such that $X \subset M$ and $|M| = \nu$. Let L_η be the transitive collapse of M, and let $j = \pi^{-1}$ where π is the collapsing isomorphism. Hence $j : L_\eta \to L_\tau$ is an elementary embedding. As X is cofinal in τ, and $|\eta| = \nu < \tau$, j is nontrivial.

The goal is to extend $j : L_\eta \to L_\tau$ to an elementary embedding $J : L_\delta \to L_\varepsilon$ where $|\delta|$ is greater than the critical point of j. This can be achieved by finding $M \prec L_\tau$ that satisfies certain closure conditions. These closure conditions guarantee that if L_η is the transitive collapse of M then η is a cardinal in L, and furthermore, that for any $\delta > \tau$, j extends to an elementary embedding J with domain L_δ.

The precise nature of the closure conditions will be spelled out in (18.41). For the remainder of this chapter, we use the phrase "M is sufficiently closed" to indicate that M satisfies (18.41).

We defer the issue of η being a cardinal in L, as its proof requires a finer analysis of the constructible hierarchy. We start with the proof of extendibility of j.

Lemma 18.36. *Let M be sufficiently closed, $X \subset M \prec L_\tau$ such that $|X| = \nu = |M|$, let $\pi : M \simeq L_\eta$ be the transitive collapse, let $j = \pi^{-1}$, and assume that η is a cardinal in L. Then for every limit ordinal $\delta \geq \eta$ there exists an elementary embedding $J : L_\delta \to L_\varepsilon$ such that $J{\restriction}L_\eta = j$.*

Proof. Let $\delta \geq \eta$ be a limit ordinal. We consider the following directed system of models: Let D be the set of all pairs $i = (\alpha, p)$ where $\alpha < \eta$ and p is a finite subset of L_δ, ordered by $(\alpha, p) \leq (\beta, q)$ if and only if $\alpha \leq \beta$ and $p \subset q$. $(D, <)$ is a directed set. Let $i = (\alpha, p)$, and let $M_i = H^\delta(\alpha \cup p)$ be the Skolem hull of $\alpha \cup p$ in $(L_\delta, \in)$. Let L_{η_i} be the transitive collapse of M_i and let $e_i : L_{\eta_i} \to L_\delta$ be the inverse of the collapsing map $\pi_i : M \simeq L_{\eta_i}$. For $i \leq k$, let $e_{i,k} = \pi_k \circ e_i$.

Let us consider the directed system of models

(18.34) $$\{L_{\eta_i}, e_{i,k} : i, k \in D\}.$$

Clearly, every $x \in L_\delta$ is in some M_i, and so L_δ is the direct limit of $\{L_{\eta_i}, e_{i,k}\}_{i,k\in D}$. For every $i \in D$, $|M_i|^L < \eta$, and since η is a cardinal in L, we have $\eta_i < \eta$. We claim that for all $i, k \in D$, $e_{i,k} \in L_\eta$. This is because $L_{\eta_i} = H^{\eta_i}(\alpha \cup \pi_i(p))$, $L_{\eta_k} = H^{\eta_k}(\beta \cup \pi_k(q))$, and for every Skolem term t, $e_{i,k}(t^{L_{\eta_i}}(\xi, x)) = t^{L_{\eta_k}}(\xi, e_{i,k}(x))$, so $e_{i,k}$ is definable in L_η from η_i, η_k, $\pi_i(p)$, and $\pi_k(q)$.

Now we consider the directed system

$$\{j(L_{\eta_i}), j(e_{i,k}) : i, k \in D\}. \tag{18.35}$$

The closure properties (18.41) of M guarantee that the direct limit of the system (18.35) is well-founded. Let N be the direct limit, and for each $i \in D$, $\tilde{e}_i : L_{j(\eta_i)} \to N$ be an elementary embedding such that $\tilde{e}_i = \tilde{e}_k \circ j(\tilde{e}_{i,k})$ whenever $i < k$. As N is well-founded, we may assume that N is transitive, and then (by (13.13)), $N = L_\varepsilon$ for some limit ordinal ε.

We can extend $j : L_\eta \to L_\tau$ to $J : L_\delta \to L_\varepsilon$ as follows:

$$J(x) = \tilde{e}_i(j(e_i^{-1}(x))) \tag{18.36}$$

where i is some (any) $i \in D$ such that $x \in M_i$.

It remains to show that $J(x) = j(x)$ for all $x \in L_\eta$. So let $x \in L_\eta$, and let $\alpha < \eta$ be such that $x \in L_\alpha$. Let $i = (\alpha, \{x\})$. Since $L_\alpha \subset M_i = H^\delta(\alpha \cup \{x\})$, it follows that $e_i \restriction L_\alpha$ is the identity, as is $e_{k,l} \restriction L_\alpha$ whenever $i \le k \le l$. Thus $j(e_{k,l}) \restriction j(L_\alpha)$ is the identity, for all $l \ge k \ge i$, and therefore $\tilde{e}_i \restriction j(L_\alpha)$ is the identity. Hence $e_i(x) = x$ and $\tilde{e}_i(jx) = jx$, and therefore $J(x) = j(x)$. □

The crucial step in the proof of the Covering Theorem is the following.

Lemma 18.37. *Let M be sufficiently closed, $X \subset M \prec L_\tau$, such that $|X| = \nu = |M|$, and let L_η be the transitive collapse of M. Then η is a cardinal in L.*

The proof is by contradiction. Assuming that η is not a cardinal in L, we shall produce a constructible set of size ν that covers X. It is in this proof that we need a finer analysis of constructibility. We start by refining Gödel's Condensation Lemma:

Lemma 18.38. *For every infinite ordinal ρ, if $M \prec_{\Sigma_1} (L_\rho, \in)$ then the transitive collapse of M is L_γ for some γ. Moreover, there is a Π_2 sentence σ such that for every transitive set M, $(M, \in) \vDash \sigma$ if and only if $M = L_\rho$ for some infinite ordinal ρ.* □

We omit the proof of Lemma 18.38. It can be found in Magidor [1990] or in Kanamori [∞]. A related fact is the following lemma that is not difficult to deduce from Lemma 18.38:

Lemma 18.39. *Let $\{(L_{\eta_i}, \in), e_{i,k} : i, k \in D\}$ be a directed system of models, $e_{i,k}$ being Σ_0-elementary embeddings. If the direct limit of this system is well-founded, then it is isomorphic to some L_γ.* □

We also need the concept of Σ_n *Skolem terms* and Σ_n *Skolem hull*:

Definition 18.40. Let $n \geq 1$.

(i) A Σ_n *Skolem term* is a composition of canonical Skolem functions (18.5) for Σ_n formulas.
(ii) If $Z \subset L_\rho$, the Σ_n *Skolem hull* of Z is the set $H_n^\rho(Z) = \{t^{L_\rho}[z_1, \ldots, z_k] : t$ is a Σ_n Skolem term and $z_1, \ldots, z_n \in Z\}$.

While a Σ_n Skolem function is not necessarily a Σ_n function, we have the following:

Lemma 18.41.

(i) *$H_n^\rho(Z)$ is a Σ_n-elementary submodel of L_ρ.*
(ii) *If $j : L_\alpha \to L_\beta$ is Σ_n-elementary, then for every Σ_n Skolem term t and all $x_1, \ldots, x_k \in L_\alpha$, $j(t^{L_\alpha}[x_1, \ldots, x_k]) = t^{L_\beta}[j(x_1), \ldots, j(x_k)]$.* □

Proof of Lemma 18.37. Let us assume that η is not a cardinal in L. Then there exists a constructible function that maps some $\alpha < \eta$ onto η. Consequently, there exists an ordinal $\rho \geq \eta$, such that for some $\alpha < \eta$ and some finite set $p \subset L_\rho$,

$$H^\rho(\alpha \cup p) \supset \eta. \tag{18.37}$$

We say that η is not a cardinal at ρ. Let ρ be the least ordinal such that η is not a cardinal at ρ.

There are three possible cases.

Case I. There exists some $n > 1$ such that $H_n^\rho(\alpha \cup p) \supset \eta$ for some $\alpha < \eta$ and some finite $p \subset L_\rho$, but $H_{n-1}^\rho(\beta \cup q) \not\supset \eta$, for all $\beta < \eta$ and all finite $q \subset L_\rho$.

Case II. $H_1^\rho(\alpha \cup p) \supset \eta$ for some $\alpha < \eta$ and some finite $p \subset L_\rho$.

Case III. $H_n^\rho(\alpha \cup p) \not\supset \eta$, for all $\alpha < \eta$ and all finite $p \subset L_\rho$.

We start with Case I.

Case I. We consider the following directed system of models. Let D be the set of all pairs $i = (\alpha, p)$ where $i < \eta$ and $p \subset L_\rho$ is finite, ordered by $(\alpha, p) \leq (\beta, q)$ if and only if $\alpha \leq \beta$ and $p \subset q$. For each $i \in D$, let $M_i = H_{n-1}^\rho(\alpha \cup p)$. Let L_{η_i} be the transitive collapse of M_i and let $e_i : L_{\eta_i} \to L_\rho$ be the inverse of the collapsing map. For $i \leq k$, let $e_{i,k} = e_k^{-1} \circ e_i$. Clearly, L_ρ is the direct limit of the directed system

$$\{L_{\eta_i}, e_{i,k} : i, k \in D\}, \tag{18.38}$$

with $e_{i,k}$ being Σ_{n-1}-elementary embeddings.

For each $i \in D$, $\eta_i < \eta$ because otherwise $\eta \subset H_{n-1}^\rho(\alpha \cup e^{-1}(p))$, contradicting the assumption about n. Also, $e_{i,k} \in L_\eta$ for all $i, k \in D$, because

$e_{i,k}$ is defined in L_η by its action on Σ_{n-1} Skolem terms: $e_{i,k}(t^{L_{\eta_i}}(\xi, x)) = t^{L_{\eta_k}}(\xi, e_{i,k}(x))$.

Now we consider the directed system

$$\{j(L_{\eta_i}), j(e_{i,k}) : i, k \in D\}. \tag{18.39}$$

The closure properties (18.41) of M guarantee that the direct limit of (18.39) is well-founded, and by Lemma 18.39, it is equal to L_γ for some γ. Let $\tilde{e}_i$ be the embedding of $j(L_{\eta_i})$ into L_γ; $\tilde{e}_i$ is Σ_{n-1}-elementary. We extend $j : L_\eta \to L_\tau$ to $J : L_\rho \to L_\gamma$ as follows:

$$J(x) = \tilde{e}_i(j(e_i^{-1}(x))) \tag{18.40}$$

where $i \in D$ is such that $x \in M_i$. As in the proof of Lemma 18.36, J extends j, and it is easily verified that J is Σ_{n-1}-elementary.

The key observation is that J is even Σ_n-elementary. To prove that, it is enough to show that for every Σ_{n-1} formula φ, if $L_\gamma \vDash \exists x\, \varphi(x, J(y))$ then $L_\rho \vDash \exists x\, \varphi(x, y)$. Thus let $y \in L_\rho$ and $x \in L_\gamma$ be such that $L_\gamma \vDash \varphi(x, J(y))$. Let $i \in D$ be such that $x \in \operatorname{ran}(\tilde{e}_i)$ and $y \in \operatorname{ran}(e_i)$. If $u \in L_{j(\eta_i)}$ and $v \in L_{\eta_i}$ are such that $x = \tilde{e}_i(u)$ $y = e_i(v)$ then $J(y) = \tilde{e}_i(j(v))$, and $L_\gamma \vDash \varphi(\tilde{e}_i(u), \tilde{e}_i(j(v)))$. Since $\tilde{e}_i$ is Σ_{n-1}-elementary, we have $L_{j(\eta_i)} \vDash \varphi(u, j(v))$. The statement $L_{j(\eta_i)} \vDash \exists z\, \varphi(z, j(v))$ is Σ_0 (with parameters $j(L_{\eta_i})$ and $j(v)$) and true in L_τ; hence in L_η, $L_{\eta_i} \vDash \exists z\, \varphi(z, v)$. Let $z \in L_{\eta_i}$ be such that $L_{\eta_i} \vDash \varphi(z, v)$; since e_i is Σ_{n-1}-elementary, we get $L_\rho \vDash \varphi(e_i(z), e_i(v))$, and so $L_\rho \vDash \exists x\, \varphi(x, y)$.

Now we reach a contradiction. Let $\alpha < \eta$ and a finite $p \subset L_\rho$ be such that $\eta \subset H_n^\rho(\alpha \cup p)$. First we have

$$X \subset M \cap \tau = j``\eta = J``\eta,$$

and since J is Σ_n-elementary, Lemma 18.41 gives

$$J``\eta \subset J``H_n^\rho(\alpha \cup p) = H^\gamma(J``\alpha \cup J``p).$$

By the minimality of τ, the set $J``\alpha \subset j(\alpha) < \tau$ can be covered by a constructible set $Y$ of size $|Y| \le \nu$. Hence $X$ can be covered by the constructible set $H_n^\gamma(Y \cup J``p)$, which has cardinality $\le \nu$, contrary to Lemma 18.35.

This completes the proof of Case I.

Case II. We use the fact that in this case, ρ must be a limit ordinal. This is an immediate consequence of this:

Lemma 18.42. *If γ is infinite, $\alpha < \gamma$ and $p \subset L_{\gamma+1}$ is finite, then there exists a finite set $q \subset L_\gamma$ such that*

$$H_1^{\gamma+1}(\alpha \cup q) \cap L_\gamma \subset H^\gamma(\alpha \cup p).$$

Proof. This is quite routine when $p = \emptyset$. When p is nonempty, the idea is to replace members of p by the parameters used in their definitions over L_γ. We omit the proof. □

Continuing Case II, we consider the directed system of models (18.38) with $\eta_i < \eta$ where all embeddings are Σ_0-elementary embeddings. The index set D is the set of all $i = (\alpha, p, \xi)$ where $\alpha < \eta$, $p \subset L_\rho$ is finite and $\xi < \rho$ such that $p \in L_\xi$. Each model L_{η_i} is the transitive collapse of $H^\xi(\alpha \cup p)$.

The closure properties (18.41) of M guarantee that the direct limit of the system (18.39) is well-founded, say L_γ. We extend j to $J : L_\rho \to L_\gamma$ as before, and as in Case I prove that J is not just Σ_0-elementary, but Σ_1-elementary. As in Case I, we reach a contradiction by covering X by a constructible set of size $\leq \nu$.

Case III. In this case, we consider the directed system (18.38) indexed by triples $i = (\alpha, p, n)$ where α and p are as before and $n \geq 1$; $(\alpha, p, n) \leq (\beta, q, m)$ means $\alpha \leq \beta$, $p \subset q$ and $n \leq m$. For each $i = (\alpha, p, n)$, $M_i = H_n^\rho(\alpha \cup p)$; by the assumption on ρ, the transitive collapse of M_i is some L_{η_i} with $\eta_i < \eta$, and if for each $k \geq i$, $e_{i,k}$ is Σ_n-elementary (and $e_{i,k} \in L_\eta$).

Again, by (18.41) the direct limit of (18.39) is some L_γ, and for each $i = (\alpha, p, n)$, $\tilde{e}_i$ is Σ_n-elementary. Extending j to $J : L_\rho \to L_\gamma$ as before, we get J elementary, and reach a contradiction in much the same way as before. □

It remains to find a model $M \supset X$ with the right closure conditions. This is provided by the following technical lemma:

Lemma 18.43. *There exists a model $M \prec L_\tau$ such that $X \subset M$, $|M| = \nu = |X|$, and if j^{-1} is the transitive collapse of M onto L_η, then*

(18.41) *for every directed system $\{L_{\eta_i}, e_{i,k} : i, k \in D\}$ with $L_{\eta_i}, e_{i,k} \in L_\eta$, with limit L_ρ for some $\rho \geq \eta$, and D as in the proof of Lemmas 18.36 and 18.37, the direct limit of $\{j(L_{\eta_i}), j(e_{i,k}) : i, k \in D\}$ is well-founded.* □

The construction of M proceeds in ν steps. At each step $\xi < \nu$ let $(\eta(\xi), \rho(\xi))$ be the least (η, ρ) such that for some increasing $\{i_n\}_{n=0}^\infty \subset D$, there are ordinals $\beta_n \in L_{\eta_{i_n}}$ such that $\beta_{n+1} < e_{i_n, i_{n+1}}(\beta_n)$ for $n = 0$, 1, 2, ... We add the ordinals β_n to M at this stage ξ. Using the fact that ν is a regular uncountable cardinal, one can verify that the resulting model M satisfies (18.41). As the proof is rather long and tedious, we omit it and refer the reader to either Magidor [1990] or Chapter 32 in Kanamori's book.

Exercises

18.1. If there exists a cardinal κ such that $\kappa \to (\omega_1)^{<\omega}$ then $0^\sharp$ exists.

18.2. Let M be a transitive model of ZFC, let B be a complete Boolean algebra in M and let G be an M-generic ultrafilter on B. If $M \vDash 0^\sharp$ does not exist, then $M[G] \vDash 0^\sharp$ does not exist.

[All cardinals $\geq |B^+|$ remain cardinals in $M[G]$. Let $\gamma_1 < \gamma_2 < \ldots < \gamma_n < \ldots < \gamma_\omega$ be an increasing sequence of cardinals in M such that $\gamma_1 \geq |B^+|$. If $0^\sharp$ exists in $M[G]$, then $0^\sharp = \{\varphi : L_{\gamma_\omega} \vDash \varphi[\gamma_1, \ldots, \gamma_n]\}$ and hence $0^\sharp \in M$.]

18.3. Assume that $0^\sharp$ exists. If $A \subset \omega_1$ is such that $A \cap \alpha \in L$ for every $\alpha < \omega_1$, then $A \in L$.

[For every $\alpha \in I \cap \omega_1$ there is t_α such that $A \cap \alpha = t_\alpha(\gamma_1^\alpha, \ldots, \gamma_{n(\alpha)}^\alpha, \alpha, \delta_1^\alpha, \ldots, \delta_{k(\alpha)}^\alpha)$. Clearly $A \cap \alpha = t_\alpha(\gamma_1^\alpha, \ldots, \gamma_{n(\alpha)}^\alpha, \alpha, \aleph_2, \ldots, \aleph_{k(\alpha)+1})$. Since there are only countably many Skolem terms, and by Fodor's Theorem, there is a stationary subset X of $I \cap \omega_1$ and t, γ_1, …, γ_n such that for all $\alpha \in X$, $A \cap \alpha = t(\gamma_1, \ldots, \gamma_n, \alpha, \aleph_2, \ldots, \aleph_{k+1})$. Show that $A = t(\gamma_1, \ldots, \gamma_n, \aleph_1, \aleph_2, \ldots, \aleph_{k+1})$.]

18.4. Let κ be an uncountable regular cardinal. If $0^\sharp$ exists, then for every constructible set $X \subset \kappa$, either X or $\kappa - X$ contains a closed unbounded subset.

[Let $X = t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ where $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_m$ are Silver indiscernibles such that $\alpha_n < \kappa \leq \beta_1$. Show that either X or $\kappa - X$ contains all Silver indiscernibles γ such that $\alpha_n < \gamma < \kappa$: The truth value of $\gamma \in t(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ is the same for all such γ.]

18.5. Let us assume that for some uncountable regular cardinal κ, every constructible $X \subset \kappa$ either contains or is disjoint from a closed unbounded set. Then $0^\sharp$ exists.

[Let D be the collection of all constructible subsets of κ containing a closed unbounded subset. D is an L-ultrafilter and every intersection of less than κ elements of D is nonempty; hence the ultrapower $\text{Ult}_D(L)$ is well-founded and gives an elementary embedding of L in L.]

18.6. If κ is weakly compact and if $|(\kappa^+)^L| = \kappa$, then $0^\sharp$ exists.

[Let B be the least nontrivial κ-complete algebra of subsets of κ closed under inverses of constructible functions $f : \kappa \to \kappa$; we have $|B| = \kappa$. Let U be a κ-complete ultrafilter on B containing all final segments $\{\alpha : \aleph_0 \leq \alpha < \kappa\}$. $U \cap L$ is a nonprincipal L-ultrafilter, and $\text{Ult}_{U\cap L}(L)$ is well-founded. Thus there is a nontrivial elementary embedding of L in L.]

18.7. Let i_n $(n \leq \omega)$ be the nth Silver indiscernible, and let $j : I \to I$ be order-preserving such that $j(i_n) = i_n$ for $n < \omega$ and $j(i_\omega) > i_\omega$. Then j extends to an elementary embedding $j : L \to L$ with i_ω its critical point.

18.8. Every Silver indiscernible is ineffable (hence weakly compact) in L.

[Show that i_ω is ineffable in L, by Lemma 17.32.]

18.9. If $0^\sharp$ exists then $L \vDash \exists\kappa\ \kappa \to (\omega)^{<\omega}$.

[Let $\kappa = i_\omega$. If $f : [\kappa]^{<\omega} \to \{0, 1\}$ is in L, there is some $n < \omega$ such that the set $\{i_k : k \leq n < \omega\}$ is homogeneous for f.]

18.10. If $0^\sharp$ exists then the Erdős cardinal η_ω in L is smaller than the least Silver indiscernible.

[$(\eta_\omega)^L$ is definable in L.]

18.11. If $j : L \to L$ is elementary, then the critical point of j is a Silver indiscernible.

[Let κ be the critical point, let $D = \{X : \kappa \in j(X)\}$, and let $j_D : L \to \mathrm{Ult}_D(L) = L$ be the canonical embedding. κ is the critical point of j_D, and $j_D(\lambda) = \lambda$ for all regular $\lambda \geq \kappa^+$. If $\kappa \notin I$ then $\kappa = t(\alpha_1, \ldots, \alpha_k, \lambda_1, \ldots, \lambda_n)$ where $\alpha_i < \kappa < \lambda_j$ and $j_D(\alpha_i) = \alpha_i$, $j_D(\lambda_j) = \lambda_j$. Hence $j_D(\kappa) = \kappa$, a contradiction.]

18.12. If both ω_1 and ω_2 are singular, then $0^\sharp$ exists.

[Let $\kappa = \omega_1$ and let λ be the successor cardinal of κ in L. Since $\mathrm{cf}\,\kappa = \mathrm{cf}\,\lambda = \omega$, there are sets $X \subset \kappa$ and $Y \subset \lambda$, both of order-type ω such that $\sup X = \kappa$ and $\sup Y = \lambda$. Let $M = L[X, Y]$; M is a model of ZFC and in M, κ is a singular cardinal, and λ is not a cardinal. Hence $0^\sharp$ exists in M.]

18.13. For every $x \subset \omega$, either $0^\sharp \in L[x]$ or $x^\sharp \in L[0^\sharp, x]$.

[If $0^\sharp \notin L[x]$, then the Covering Theorem for L holds in $L[x]$ but fails in $L[0^\sharp, x]$, and hence the Covering Theorem for $L[x]$ fails in $L[0^\sharp, x]$. Therefore $x^\sharp \in L[0^\sharp, x]$.]

Historical Notes

Theorem 18.1 was discovered by Gaifman (assuming the existence of a measurable cardinal). Gaifman's results were announced in [1964] and the proof was published in [1974], Gaifman's proof used iterated ultrapowers (see also Gaifman [1967]). Silver in his thesis (1966, published in [1971b]) developed the present method of proof, using infinitary combinatorics, and proved the theorem under the weaker assumption of existence of κ with the property $\kappa \to (\aleph_1)^{<\omega}$. Gaifman proved that if there is a measurable cardinal, then there exists $A \subset \omega$ such that the conclusion of Theorem 18.1 holds in $L[A]$. Solovay formulated $0^\sharp$ and proved that it is a Δ^1_3 set of integers; Silver deduced the existence of $0^\sharp$ under weaker assumptions.

Construction of models with indiscernibles was introduced by Ehrenfeucht and Mostowski in [1956].

The equivalence of the existence of $0^\sharp$ with the existence of a nontrivial elementary embedding of L (Theorem 18.20) is due to Kunen; the present proof is due to Silver. Kunen also derived $0^\sharp$ from the existence of Jónsson cardinals and from Chang's Conjecture.

Theorem 18.30 (and its corollaries) is due to Jensen. A proof of the theorem appeared in Devlin and Jensen [1975]. Jensen's proof makes use of his fine structure theory, see Jensen [1972]. The present proof is due to Magidor [1990]. Lemma 18.38 appears in Magidor [1990] and in Kanamori's book [∞]; Magidor attributes the proof to Boolos [1970].

Exercise 18.3: Solovay.

Exercise 18.6: Kunen.

Exercise 18.12: Magidor.