22. Saturated Ideals

One of the key concepts in the theory of large cardinals is *saturation* of ideals. In this chapter we investigate σ-saturated, κ-saturated and κ^+-saturated κ-complete ideals on κ.

Let κ be a regular uncountable cardinal. Let I be a κ-complete ideal on κ containing all singletons; thus $X \in I$ whenever $X \subset \kappa$ is such that $|X| < \kappa$. We shall be using the following terminology: X has *measure zero* if $X \in I$, *measure one* if $\kappa - X \in I$, and *positive measure* if $X \notin I$; the phrase *almost all* α means that the set of all contrary α's has measure 0.

Let us consider the Boolean algebra $B = P(\kappa)/I$. Recall that if λ is a cardinal, then B is called λ-*saturated* if every pairwise disjoint family of elements of B has size less than λ; $\text{sat}(B)$ is the least λ such that B is λ-saturated. Let us say that I is λ-*saturated* if B is λ-saturated and let

$$\text{sat}(I) = \text{sat}(B).$$

In other words, I is λ-saturated just in case there exists no collection W of size λ of subsets of κ such that $X \notin I$ for all $X \in W$ and $X \cap Y \in I$ whenever X and Y are distinct members of W. If $\text{sat}(I)$ is finite, then κ is the union of finitely many atoms of I; if $\text{sat}(I)$ is infinite, then it is uncountable and regular, by Theorem 7.15. If $\lambda \le \kappa$, then I is λ-saturated if and only if there is no disjoint collection W of size λ of subsets X of κ such that $X \notin I$ (see Exercise 22.1). Clearly, every I on κ is $(2^\kappa)^+$-saturated. Thus if I is atomless, then $\text{sat}(I)$ is a regular cardinal and

$$\aleph_1 \le \text{sat}(I) \le (2^\kappa)^+.$$

Since I is κ-complete, it follows that $B = P(\kappa)/I$ is a κ-complete Boolean algebra.

Real-Valued Measurable Cardinals

By Ulam's Theorem 10.1, if there exists a nontrivial σ-additive measure then either there exists a measurable cardinal or there exists a real-valued measurable cardinal.

In this section we prove the following theorems:

Theorem 22.1 (Solovay).

(i) *If κ is a real-valued measurable cardinal, then there is a transitive model of set theory in which κ is measurable.*

(ii) *If κ is a measurable cardinal, then there exists a generic extension in which $\kappa = 2^{\aleph_0}$ and κ is real-valued measurable.*

Theorem 22.2 (Prikry). *If $2^{\aleph_0}$ is real-valued measurable, then $2^\lambda = 2^{\aleph_0}$ for all infinite $\lambda < 2^{\aleph_0}$.*

If μ is a κ-additive real-valued measure on κ, then the ideal I_μ of all sets of measure 0 is a σ-saturated κ-complete ideal on κ. We have proved that if an uncountable cardinal κ carries a σ-saturated κ-complete ideal, then κ is weakly inaccessible.

We shall prove Theorem 22.1(i) and Theorem 22.2 for this generalization of real-valued measurability, namely under the assumption that κ is uncountable and carries a σ-saturated κ-complete ideal. Thus let κ be an uncountable cardinal and let I be a σ-saturated κ-complete ideal. Thus let κ be an uncountable cardinal and let I be a σ-saturated κ-complete ideal on κ.

Let us call $A \subset \kappa$ an *atom* if A has positive measure and is not the union of two disjoint sets of positive measure. I is *atomless* if it has no atoms. What we proved in Lemma 10.9(ii) can be formulated as follows: If I is atomless, then $\kappa \leq 2^{\aleph_0}$. It follows that if $2^{\aleph_0} < \kappa$, then every set X of positive measure contains an atom $A \subset X$, and hence there exists an at most countable disjoint collection W of atoms such that $\kappa = \bigcup\{A : A \in W\}$.

We start with the following analog of Theorem 10.20. We recall that a κ-complete ideal on κ is normal if every function $f : S \to \kappa$ regressive on a set $S \subset \kappa$ of positive measure is constant on some $T \subset S$ of positive measure. A real-valued measure μ is *normal* if I_μ is normal.

Lemma 22.3.

(i) *If I is a σ-saturated κ-complete ideal on an uncountable cardinal κ, then there exists a function $f : \kappa \to \kappa$ such that*

$$J = f_*(I) = \{X \subset \kappa : f_{-1}(X) \in I\}$$

is a normal σ-saturated κ-complete ideal on κ.

(ii) *If μ is a κ-additive real-valued measure on κ, then there exists a function $f : \kappa \to \kappa$ such that $\nu = f_*(\mu)$ defined by*

$$\nu(X) = \mu(f_{-1}(X)) \qquad (X \subset \kappa)$$

is a normal κ-additive real-valued measure on κ.

Proof. We shall prove (i) and leave the completely analogous proof of (ii) to the reader. Let us say that a function $g : S \to \kappa$ is *unbounded* on a set S of positive measure if there is no $\gamma < \kappa$ and no $T \subset S$ of positive measure such that $g(\alpha) < \gamma$ for all $\gamma \in T$. Let us consider the family $\mathcal{F}$ of all functions g into κ defined on a set of positive measure and unbounded on its domain. Let us define $g < h$ if $\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and if $g(\alpha) \le h(\alpha)$ everywhere on $\mathrm{dom}(g)$. Let us also define $g \le h$ if $\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and if $g(\alpha) \le h(\alpha)$ everywhere on $\mathrm{dom}(g)$. Let us call $g \in \mathcal{F}$ *minimal* if there is no $h \in \mathcal{F}$ such that $h < g$.

We shall first show that there exists a minimal $g \in \mathcal{F}$. Otherwise, for every $g \in \mathcal{F}$ there is $h \in \mathcal{F}$ such that $h < g$. Thus let $g \in \mathcal{F}$ be arbitrary. Let W be a maximal collection of elements of $\mathcal{F}$ such that $h < g$ for each $h \in W$, and that $\mathrm{dom}(h_1) \cap \mathrm{dom}(h_2) = \emptyset$ whenever h_1 and h_2 are distinct elements of W. Since I is σ-saturated, W is at most countable and by our assumption, the set $\mathrm{dom}(g) - \bigcup\{\mathrm{dom}(h) : h \in W\}$ has measure zero. Thus if we let $f = \bigcup\{h : h \in W\}$, we have $\mathrm{dom}(g) - \mathrm{dom}(f) \in I$, and $f < g$. Since g was arbitrary, we can construct a countable sequence $g_0 > g_1 > \ldots > g_n > \ldots$ such that $\mathrm{dom}(g_n) - \mathrm{dom}(g_{n+1}) \in I$ for each n. It follows that $\bigcap_{n=0}^{\infty} \mathrm{dom}(g_n)$ has positive measure and we get a contradiction since for any $\alpha \in \bigcap_{n=0}^{\infty} \mathrm{dom}(g_n)$ we would have $g_0(\alpha) > g_1(\alpha) > \ldots$.

The same argument shows that for every $h \in \mathcal{F}$ there exists a minimal $g \in \mathcal{F}$ such that $g \le h$. Thus if W is a maximal family of minimal functions $g \in \mathcal{F}$ such that $\mathrm{dom}(g_1) \cap \mathrm{dom}(g_2) = \emptyset$ whenever g_1 and g_2 are distinct elements of W, W is at most countable and $\bigcup\{\mathrm{dom}(g) : g \in W\}$ has measure one. Thus if we let $f = \bigcup\{g : g \in W\}$, then $\mathrm{dom}(f)$ has measure one and f is a *least* unbounded function: On the one hand, if $\gamma < \kappa$, then there is no $S \subset \kappa$ of positive measure such that $f(\alpha) < \gamma$ everywhere on S; on the other hand, if S is a set of positive measure and g is a function on S such that $g(\alpha) < f(\alpha)$ everywhere on S, then g is constant on some $T \subset S$ of positive measure. We can clearly assume that $\mathrm{dom}(f) = \kappa$.

Let $f : \kappa \to \kappa$ be a least unbounded function; we shall show that $J = f_*(I)$ is a normal σ-saturated κ-complete ideal on κ. It is obvious that J is a κ-complete ideal. For every $\gamma \in \kappa$, $f_{-1}(\{\gamma\})$ has measure zero and hence $\{\gamma\} \in J$. If $X \notin J$, then $f_{-1}(X) \notin I$, and if $X \cap Y = \emptyset$, then $f_{-1}(X) \cap f_{-1}(Y) = \emptyset$, and hence J is σ-saturated because I is σ-saturated.

To show that J is normal, let $S \notin I$, and let $g(\alpha) < \alpha$ for all $\alpha \in S$. Then $g(f(\xi)) < f(\xi)$ for all $\xi \in f_{-1}(S)$ and since f is a least unbounded function, $g(f(\xi))$ is constant on some $X \subset f_{-1}(S)$ of positive I-measure. Hence g is constant on $f(X)$ and $f(X) \notin J$. □

Lemma 22.4. *Let I be a normal σ-saturated κ-complete ideal on κ. If S is a set of positive measure and $f : S \to \kappa$ is regressive on S, then f is bounded almost everywhere on S; that is, there exists $\gamma < \kappa$ such that $\{\alpha \in S : f(\alpha) \ge \gamma\} \in I$.*

Proof. For every $X \subset S$ of positive measure there exists $Y \subset X$ of positive measure such that f is constant on Y. Thus let W be a maximal disjoint family of sets $X \subset S$ of positive measure such that f is constant on X. Let

$T = \bigcup\{X : X \in W\}$. The family W is at most countable and hence there is γ such that $f(\alpha) < \gamma$ for all $\alpha \in T$. Clearly, $S - T$ has measure 0. □

Corollary 22.5. *If κ is real-valued measurable (or if κ carries a σ-saturated κ-complete ideal), then κ is a weakly Mahlo cardinal.*

Proof. Let I be a normal σ-saturated κ-complete ideal on κ. Since I is normal, every closed unbounded set has I-measure one (see Lemma 8.11). Because κ is weakly inaccessible, it suffices to show that the set of all regular cardinals $\alpha < \kappa$ has measure one.

Let us assume that the set S of all limit ordinals $\alpha < \kappa$ such that $\operatorname{cf}\alpha < \alpha$ has positive measure. Considering the regressive function $\alpha \mapsto \operatorname{cf}\alpha$, we find a set T of positive measure and some $\lambda < \kappa$ such that $\operatorname{cf}\alpha = \lambda$ for all $\alpha \in T$. For each $\alpha \in T$, let $\langle \alpha_\nu : \nu < \lambda \rangle$ be an increasing λ-sequence with limit α.

For each $\nu < \lambda$, the function $\alpha \mapsto \alpha_\nu$ is regressive on T and so, by Lemma 22.4 there is γ_ν such that $\alpha_\nu < \gamma_\nu$ for almost all $\alpha \in T$. Let $\gamma = \sup\{\gamma_\nu : \nu < \lambda\}$. Since $\lambda < \kappa$, we conclude, by κ-completeness of I, that for almost all $\alpha \in T$, $\alpha_\nu < \gamma$ of all $\nu < \lambda$. But this means that for almost all $\alpha \in T$, $\alpha = \lim_\nu \alpha_\nu \le \gamma$. This is a contradiction since T is unbounded. □

Since every closed unbounded set has measure one (if I is a normal σ-saturated κ-complete ideal on κ), every set of positive measure is stationary. It can even be proved that if S has positive measure, then $S \cap \alpha$ is stationary in α for almost all α. Then it follows that κ is the κth weakly Mahlo cardinal, κth cardinal which is a limit of weakly Mahlo cardinals, etc. We shall return to this subject later in this chapter.

We shall now show that every real-valued measurable cardinal is a Rowbottom cardinal; we shall show that the statement of Lemma 17.36 for a measurable cardinal holds under the weaker assumption that κ carries a σ-saturated κ-complete ideal.

Lemma 22.6. *Let I be a normal σ-saturated κ-complete ideal on κ, and let λ be an infinite cardinal less than κ. Let $\mathfrak{A} = (A, \ldots)$ be a model of a language $\mathcal{L}$ such that $|\mathcal{L}| \le \lambda$, and let $A \supset \kappa$. If $P \subset A$ is such that $|P| < \kappa$, then $\mathfrak{A}$ has an elementary submodel $\mathfrak{B} = (B, \ldots)$ such that $B \cap \kappa$ has measure one and $|P \cap B| \le \lambda$. Moreover, if $X \subset A$ has size at most λ, then we can find $\mathfrak{B}$ such that $X \subset B$.*

The proof of Lemma 22.6 uses Skolem functions and arguments similar to those in Theorem 17.27 and Lemma 17.36. The key ingredient is the following lemma:

Lemma 22.7. *Let I be a normal σ-saturated κ-complete ideal on κ, let $\gamma < \kappa$ and let $f : [\kappa]^{<\omega} \to \gamma$ be a partition. Then there exists $H \subset \kappa$ of measure one such that the image of $[H]^{<\omega}$ under f is at most countable.*

Proof. We proceed as in the proof of Theorem 10.22. It suffices to show that for each $n = 1, 2, \ldots$ there is H_n of measure one such that $f([H]^n)$ is at most countable; then we take $H = \bigcap_{n=1}^{\infty} H_n$.

We prove, by induction on n, that for every partition of $[\kappa]^n$ into less than κ pieces there is $H \subset \kappa$ of measure one such that $f([H]^n)$ is at most countable. For $n = 1$, let $f : \kappa \to \gamma$ and $\gamma < \kappa$; let W be a maximal pairwise disjoint family of subsets $X \subset \kappa$ such that X has positive measure and f is constant on X. Let $H = \bigcup\{X : X \in W\}$. Since $|W| \le \aleph_0$, we have $|f(H)| \le \aleph_0$, and since $\gamma < \kappa$ and I is κ-complete, we clearly have $\kappa - H \in I$.

Let us assume that the assertion is true for n and let us prove that it holds also for $n+1$. Let $f : [\kappa]^{n+1} \to \gamma$ where $\gamma < \kappa$. For each $\alpha < \kappa$, we define f_α on $[\kappa - \{\alpha\}]^n$ by $f_\alpha(x) = f(\{\alpha\} \cup x)$. By the induction hypothesis, there exists for each $\alpha < \kappa$ a set X_α of measure one such that $f_\alpha([X_\alpha]^n)$ is at most countable; let A_α be the image of $[X_\alpha]^n$ under f_α. Let X be the diagonal intersection

$$X = \{\alpha < \kappa : \alpha \in \bigcap_{\xi<\alpha} X_\xi\}$$

The set X has measure one since I is normal; also if $\alpha < \alpha_1 < \ldots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\alpha]^n$ and so $f(\{\alpha, \alpha_1, \ldots, \alpha_n\}) = f_\alpha(\{\alpha_1, \ldots, \alpha_n\}) \in A_\alpha$.

For each $\alpha \in X$, let $A_\alpha = \{a_{\alpha,n} : n < \omega\}$. For each n, consider the function $g_n : X \to \gamma$ defined by $g_n(\alpha) = a_{\alpha,n}$. There exists a set $H_n \subset X$ of measure one such that $g_n(H_n)$ is at most countable. Thus let $H = \bigcap_{n=0}^{\infty} H_n$; the set H has measure one, and moreover $\bigcup\{A_\alpha : \alpha \in H\} = \bigcup_{n=0}^{\infty} g_n(H)$ is at most countable. It follows that $f([H]^{n+1})$ is at most countable. □

We can now proceed as in Theorem 19.3 and prove that if I is a normal σ-saturated κ-complete ideal on κ and $V = L[I]$, then GCH holds. In fact, if D denotes the filter dual to I, that is, the filter of all sets of I-measure one, then the proof of Theorem 19.3 goes through in the present context (use $P = \{Y \subset \lambda : Y \le_{L[D]} X\}$).

Now we recall the results of Chapter 18: If κ carries a σ-saturated κ-complete ideal then either $\kappa \le 2^{\aleph_0}$ or κ is measurable. Thus we conclude: If I is a normal σ-saturated κ-complete ideal on κ and $V = L[I]$, then κ is measurable.

Proof of Theorem 22.1(i). Let κ be real-valued measurable. Then there is a normal κ-additive measure μ on κ by Lemma 22.3. Let I be the ideal of sets of measure zero. I is a normal σ-saturated κ-complete ideal on κ. Let $J = I \cap L[I]$. We have $L[J] = L[I]$, and in $L[I]$, J is a normal $\aleph_1$-saturated κ-complete ideal on κ. (If we could assume that $\aleph_1^{L[I]} = \aleph_1$, it would now follow that κ is a measurable cardinal in $L[I]$.)

Since we are not able to show directly that if I is σ-saturated, then $I \cap L[I]$ is σ-saturated in $L[I]$, let us consider a somewhat more general situation.

Let ν be a regular uncountable cardinal less than κ, and let us consider ν-saturated κ-complete ideals on κ.

Lemmas 22.3 and 22.4 hold again; in Lemma 22.7 we have to replace "at most countable" by "of size less than ν." Lemma 22.6 holds for all $\lambda \geq \nu$ and the analog of Theorem 19.3 is: If $V = L[I]$ and I is normal, then $2^{<\nu} = \nu$ and $2^\lambda = \lambda^+$ for all $\lambda \geq \nu$.

Lemma 10.9 can also be generalized, and we get: If I is atomless, then $\kappa \leq 2^{<\nu}$. Hence if $V = L[I]$, every set of positive measure contains a subset that is an atom, and therefore κ is the union of a disjoint family W of atoms such that $|W| < \nu$.

Lemma 22.8. *Let $\nu < \kappa$ be a regular uncountable cardinal, and let I be a normal ν-saturated κ-complete ideal on κ; let F be the dual filter. Then in $L[F]$, $F \cap L[F]$ is a normal measure on κ (and $L[F]$ is the model $L[D]$ of Chapter 19).*

Proof. It is easy to verify that $L[F] = L[I]$, and that in $L[F]$, $I \cap L[I]$ is a normal ν-saturated κ-complete ideal on κ. Thus we may assume that $V = L[F]$; we want to show that F is an ultrafilter.

We know that κ is the union of a disjoint family W of atoms. (What we want to show is that W has only one element.) For $A \in W$, let

$$F_A = \{X \subset \kappa : X \cap A \text{ has positive measure}\}.$$

Since A is an atom, F_A is a filter, and F_A is in fact a normal measure on κ. Hence $F_A \cap L[F_A]$ is the unique normal measure D in $L[F_A]$, and $L[F_A]$ is *the* model $L[D]$.

We shall now show that $F \cap L[D] = D$. Let $X \in L[D]$ be a subset of κ. If $X \in F$, then $X \in F_A$ for all $A \in W$ and hence $X \in F_A \cap L[F_A] = D$. If $X \notin F$, then there is $A \in W$ such that $X \notin F_A$ and hence $X \notin F_A \cap L[F_A] = D$. It follows that $F \cap L[D] = D$ and so $F \cap L[D] \in L[D]$.

Consequently, $L[F] = L[D]$; since we assumed that $V = L[F]$ and because $F \cap L[D] = D$, we have $F = D$. □

Proof of Theorem 22.1(ii). Let κ be a measurable cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. We shall construct a generic extension in which $2^{\aleph_0} = \lambda$ and κ is real-valued measurable.

Let F be a σ-algebra of sets and let μ be a measure on F. Let $I \subset F$ be the ideal of sets of measure 0 and let us consider the Boolean algebra $B = F/I$. That is, the members of B are equivalence classes $[X]$ where $X \in F$ and where $X \equiv Y$ if and only if $\mu(X \triangle Y) = 0$.

Since both F and I are countably complete, it follows that B is countably saturated and $\sum_{n=0}^\infty [X_n] = [\bigcup_{n=0}^\infty X_n]$. Since μ is a measure, I is countably saturated and so B satisfies the countable chain condition. Now a Boolean algebra that is both σ-complete and σ-saturated is complete, and so B is a complete Boolean algebra.

For $[X] \in B$, let us define $m([X]) = \mu(X)$. Clearly, the definition of m does not depend on the particular choice of X, and furthermore, m has the following properties:

(22.1)
(i) m is a real-valued function on B;
(ii) $m(0) = 0$, $m(a) > 0$ if $a \neq 0$, and $m(1) = 1$;
(iii) if $a \leq b$, then $m(a) \leq m(b)$;
(iv) if a_n, $n = 0, 1, \ldots$, are pairwise disjoint, then

$$m\Big(\sum_{n=0}^{\infty} a_n\Big) = \sum_{n=0}^{\infty} m(a_n).$$

A Boolean algebra B with a measure m (satisfying (22.1)) is called a *measure algebra*; a set S with a field of sets $\mathcal{F}$ and a measure μ on $\mathcal{F}$ is called a *measure space*.

We need from measure theory the following basic fact about *products* of measure spaces. Let I be a set (of indices), and for each $i \in I$ let $(S_i, \mathcal{F}_i, \mu_i)$ be a measure space. Let us consider the product $S = \prod_{i\in I} S_i$, and let us consider the following σ-algebra of subsets of S: Let E be a finite subset of I, and for each $i \in E$, let $Z_i \in \mathcal{F}_i$. Let $Z \subset S$ be as follows: If $t \in \prod_{i\in I} S_i$, then

$$t \in Z \quad \text{if and only if} \quad t(i) \in Z_i \text{ for all } i \in E. \tag{22.2}$$

Let $\mathcal{F}$ be the least σ-algebra of subsets of S such that $\mathcal{F}$ contains every $Z \subset S$ of the form (22.2), for any finite $E \subset I$ and any $Z_i \in \mathcal{F}_i$, $i \in E$.

There exists a unique measure μ on $\mathcal{F}$ (the *product measure*) such that for every Z of the form (22.2), $\mu(Z)$ is the product of $\mu_i(Z_i)$, $i \in E$. (In case of the product $S = S_1 \times S_2$, the measure of a "rectangle" $Z_1 \times Z_2$ is equal to $\mu(Z_1) \cdot \mu(Z_2)$.)

We shall use the following simple example of a product measure space (cf. Example 15.31). Let I be an infinite set, and for each $i \in I$ let us consider the space $\{0, 1\}$ of two elements. We give measure $1/2$ to both $\{0\}$ and $\{1\}$:

$$S_i = \{0, 1\}, \qquad \mathcal{F}_i = P(S_i), \tag{22.3}$$
$$\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2, \qquad \mu_i(\emptyset) = 0, \qquad \mu_i(\{0, 1\}) = 1.$$

Let $S = \prod_{i\in I} S_i$, and let μ be the product measure on $\mathcal{F}$, the least σ-algebra of subsets of S containing the sets $\{t \in \{0, 1\}^I : t(i) = 0\}$ for all $i \in I$.

Let M be a transitive model of ZFC (the ground model). In M let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$. Let $(S, \mathcal{F}, \mu)$ be the product measure space $\{0, 1\}^I$ defined above, where $I = \lambda \times \omega$. Let B be the corresponding measure algebra $\mathcal{F}$/the ideal of sets of measure 0.

Let G be an M-generic ultrafilter on B. Since B satisfies the countable chain condition, the generic extension $M[G]$ preserves cardinals. We shall show that in $M[G]$, $2^{\aleph_0} = \lambda$.

On the one hand, an easy computation gives $|\mathcal{F}| = \lambda$ (because $\lambda^{\aleph_0} = \lambda$) and since B satisfies the c.c.c., we get $|B| = \lambda$. Therefore

$$(2^{\aleph_0})^{M[G]} \le (|B|^{\aleph_0})^M = \lambda$$

and we have $(2^{\aleph_0})^{M[G]} \le \lambda$.

On the other hand, we shall exhibit λ distinct subsets of ω in $M[G]$. For each $\alpha < \lambda$ and each $n < \omega$, let $u_{\alpha,n} = [U_{\alpha,n}]$, where $U_{\alpha,n}$ is as follows:

$$U_{\alpha,n} = \{t \in \{0,1\}^{\lambda\times\omega} : t(\alpha,n) = 1\}. \tag{22.4}$$

For $\alpha < \lambda$, let $\dot{x}_\alpha$ be the B-valued subset of ω such that

$$\|n \in \dot{x}_\alpha\| = u_{\alpha,n} \qquad (n < \omega). \tag{22.5}$$

Let x_α be the G-interpretation of $\dot{x}_\alpha$.

We shall show that $x_\alpha \ne x_\beta$ whenever $\alpha \ne \beta$, and in fact that $\|\dot{x}_\alpha = \dot{x}_\beta\| = 0$. Let k be any natural number. Then

$$\|\dot{x}_\alpha \cap k = \dot{x}_\beta \cap k\| = [N_{\alpha,\beta,k}],$$

where

$$N_{\alpha,\beta,k} = \{t : t(\alpha,n) = t(\beta,n) \text{ for all } n < k\}.$$

It is easy to verify that for each k, $\mu(N_{\alpha,\beta,k}) = 1/2^k$. But $\|\dot{x}_\alpha = \dot{x}_\beta\| = \prod_{k=0}^\infty [N_{\alpha,\beta,k}] = [\bigcap_{k=0}^\infty N_{\alpha,\beta,k}] = 0$ since $\mu(\bigcap_{k=0}^\infty N_{\alpha,\beta,k}) = 0$. This completes the proof that $2^{\aleph_0} = \lambda$ in $M[G]$.

Now let us assume that κ is a measurable cardinal in the ground model, and let $\lambda \ge \kappa$ be such that $\lambda^{\aleph_0} = \lambda$. We construct a generic extension $M[G]$ of M, using the measure algebra described above. In $M[G]$, we have $2^{\aleph_0} = \lambda$, and we show that κ is real-valued measurable in $M[G]$. This follows from this general lemma:

Lemma 22.9. *Let κ be a measurable cardinal in the ground model M, let B be (in M) a measure algebra, and let G be an M-generic ultrafilter on B. Then in $M[G]$, there exists a nontrivial κ-additive measure on κ.*

Proof. Let U be a κ-complete nonprincipal ultrafilter on κ. Let B be a complete Boolean algebra and let m be a measure on B. We shall define a B-valued name $\dot{\mu}$ and show that if G is a generic ultrafilter, then the G-interpretation of $\dot{\mu}$ is a nontrivial κ-additive measure on κ.

Let a be a nonzero element of B, and let $\dot{A} \in M^B$ be a B-valued name such that $a \Vdash \dot{A} \subset \kappa$. For each $\alpha < \kappa$, we let

$$f_a(\dot{A}, \alpha) = \frac{m(a \cdot \|\alpha \in \dot{A}\|)}{m(a)}. \tag{22.6}$$

Since U is a κ-complete, there is a unique real number r such that $f_a(\dot{A},\alpha) = r$ for almost all α (mod U). Thus let

(22.7) $\mu_a(A) =$ the unique r such that $f_a(A,\alpha) = r$ almost everywhere (mod U).

Note that if $a \Vdash \dot{A} = \dot{A}'$, then $\mu_a(\dot{A}) = \mu_a(\dot{A}')$. Also, if $a \Vdash \dot{A}_1 \subset \dot{A}_2$, then $\mu_a(\dot{A}_1) \le \mu_a(\dot{A}_2)$. If $X \subset \kappa$ is in M, then $f_a(\check{X},\alpha) = 1$ for all $\alpha \in X$ and $f_a(\check{X},\alpha) = 0$ for all $\alpha \notin X$. Hence $\mu_a(\check{X}) = 1$ if $X \in U$ and $\mu_a(\check{X}) = 0$ if $X \notin U$.

Let $\gamma < \kappa$ and let $\dot{A}_\xi$, $\xi < \gamma$, be such that $a \Vdash \dot{A}_\xi \subset \kappa$ for all $\xi < \gamma$, and that $a \Vdash \dot{A}_\xi \cap \dot{A}_\eta = \emptyset$ whenever $\xi \ne \eta$. Let $\dot{A}$ be such that $a \Vdash \dot{A} = \bigcup_{\xi<\gamma} \dot{A}_\xi$. Then $f_a(\dot{A},\alpha) = \sum_{\xi<\gamma} f_a(\dot{A}_\xi,\alpha)$ for all $\alpha < \kappa$, and hence (because U is κ-complete),

$$\mu_a(\dot{A}) = \sum_{\xi<\gamma} \mu_a(\dot{A}_\xi). \tag{22.8}$$

Let r be a real number, $0 \le r \le 1$, and let $\{a_n\}_{n=0}^\infty$ be a partition of $a \in B$. If $\mu_{a_n}(\dot{A}) < r$ for all n, then for almost all α, $m(a_n \cdot \|\alpha \in \dot{A}\|) < r \cdot m(a_n)$, and it follows that for almost all α, $m(a \cdot \|\alpha \in \dot{A}\|) < r \cdot m(a)$; hence $\mu_a(\dot{A}) < r$.

As a consequence, we obtain:

(22.9) If for every nonzero $b \le a$ there is a nonzero $c \le b$ such that $\mu_c(\dot{A}) < r$, then $\mu_a(\dot{A}) < r$.

(And a similar statement holds when $<$ is replaced by $\le$, $>$ or $\ge$.)

Now if $b \Vdash \dot{A} \subset \kappa$, we define

$$\mu_b^*(\dot{A}) = \inf\nolimits_{a \le b} \mu_a(\dot{A}). \tag{22.10}$$

Again, if $b \Vdash \dot{A}_1 \subset \dot{A}_2$, then $\mu_b^*(\dot{A}_1) \le \mu_b^*(\dot{A}_2)$, and if $X \in M$, then $\mu_b^*(\check{X}) = 1$ if $X \in U$ and $\mu_b^*(\check{X}) = 0$ if $X \notin U$. However, μ_b^* is not additive and we only have (using (22.8)), for $\gamma < \kappa$:

$$\mu_b^*(\dot{A}) \ge \sum_{\xi<\gamma} \mu_b^*(\dot{A}_\xi) \tag{22.11}$$

under the assumption that $b \Vdash \dot{A}_\xi \cap \dot{A}_\eta = \emptyset$ whenever $\xi \ne \eta$, and that $b \Vdash \dot{A} = \bigcup_{\xi<\gamma} \dot{A}_\xi$.

Note that if $b_1 \le b_2$, then $\mu_{b_1}(\dot{A}) \ge \mu_{b_2}(\dot{A})$.

Now we are ready to define $\dot{\mu}$. Let G be a generic ultrafilter; in $M[G]$, we define $\mu : P(\kappa) \to [0,1]$ as follows:

$$\mu(A) = \sup\nolimits_{b \in G} \mu_b^*(\dot{A}) \tag{22.12}$$

where $\dot{A}$ is a name for A. Let $\dot{\mu}$ be the canonical name for μ (defined in M^B by (22.12) using the canonical $\dot{G}$).

It is clear that μ does not depend on the name $\dot{A}$ for A, that $A_1 \subset A_2$ implies $\mu(A_1) \le \mu(A_2)$ and that if $X \in M$, then $\mu(X) = 1$ if $X \in U$ and $\mu(X) = 0$ if $X \notin U$. It remains to show that μ is κ-additive.

Let r be a real number (in M) such that $0 \le r \le 1$. We claim that

$$\mu^*_b(\dot{A}) \ge r \quad \text{if and only if} \quad b \Vdash \dot{\mu}(\dot{A}) \ge \check{r}. \tag{22.13}$$

If $\mu^*_b(\dot{A}) \ge r$, then for every generic G such that $b \in G$, $\mu(A) \ge r$, and hence $b \Vdash \dot{\mu}(\dot{A}) \ge \check{r}$. Thus assume that $b \Vdash \dot{\mu}(\dot{A}) \ge \check{r}$. Then

$$b \Vdash \forall q < \check{r}\ \exists d \in \dot{G}\ \mu^*_d(\dot{A}) \ge q,$$

that is,

$$\forall q < r\ \forall c \le b\ \exists d \le c\ \mu^*_d(\dot{A}) \ge q. \tag{22.14}$$

Let $q < r$; we claim that $\mu^*_b(\dot{A}) \ge q$. If $a \le b$, then $\forall c \le a\ \exists d \le c$ such that $\mu^*_d(\dot{A}) \ge q$ and hence (by a variant of (22.9)), $\mu_a(\dot{A}) \ge q$. Thus $\mu^*_b(\dot{A}) \ge q$. Since this holds for any $q < r$, we have $\mu^*_b(\dot{A}) \ge r$.

Next we show that μ is finitely additive. Let $\dot{A}$, $\dot{A}_1$, and $\dot{A}_2$ be such that every condition forces that $\dot{A}$ is the disjoint union of $\dot{A}_1$ and $\dot{A}_2$. If r_1 and r_2 are real numbers and if $b \Vdash (\dot{\mu}(\dot{A}_1) \ge \check{r}_1$ and $\mu(\dot{A}_2) \ge \check{r}_2)$, then by (22.13) and (22.11), $b \Vdash \dot{\mu}(\dot{A}) \ge \check{r}_1 + \check{r}_2$; hence $\mu(A) \ge \mu(A_1) + \mu(A_2)$. Conversely, let us assume that $\mu(A) > \mu(A_1) + \mu(A_2)$. There are reals $r_1, r_2 \in M$ and $b \in G$ such that

$$b \Vdash \dot{\mu}(\dot{A}_1) < \check{r}_1,\ \dot{\mu}(\dot{A}_2) < \check{r}_2, \text{ and } \dot{\mu}(\dot{A}) \ge \check{r}_1 + \check{r}_2.$$

Since $b \Vdash \dot{\mu}(\dot{A}_1) < \check{r}_1$, there is for each $c \le b$ some $d \le c$ such that $\mu_d(\dot{A}_1) < r_1$; hence by (22.9), $\mu_b(\dot{A}_1) < r_1$. Similarly, $\mu_b(\dot{A}_2) < r_2$, and so $\mu^*_b(\dot{A}) \le \mu_b(\dot{A}) < r_1 + r_2$. This is a contradiction.

Now when we know that μ is finitely additive, it suffices to show that $\mu(\bigcup_{\xi<\gamma} A_\xi) \le \sum_{\xi<\gamma} \mu(A_\xi)$ for any family $\{A_\xi : \xi < \gamma\}$ of fewer than κ subsets of κ. Thus let $\gamma < \kappa$ and let $\dot{A}_\xi$, $\xi < \gamma$, and $\dot{A}$ be such that $\|\dot{A} = \bigcup_{\xi<\gamma} \dot{A}_\xi\| = 1$, and let us assume that $\mu(A) > \sum_{\xi<\gamma} \mu(A_\xi)$. Then there exist $r \in M$ and $b \in G$ such that

$$b \Vdash \sum_{\xi<\gamma} \dot{\mu}(\dot{A}_\xi) < \check{r} \text{ and } \dot{\mu}(\dot{A}) > \check{r}.$$

Let $E \subset \gamma$ be an arbitrary finite set, let $A_E = \bigcup_{\xi\in E} A_\xi$. Since $\|\dot{\mu}(\dot{A}_E) \le \sum_{\xi\in E} \dot{\mu}(\dot{A}_\xi)\| = 1$, we have $b \Vdash \dot{\mu}(\dot{A}_E) < \check{r}$. By (22.9), we get $\mu_b(\dot{A}_E) < r$.

Since $\mu_b(\dot{A}_E) < r$ for all finite $E \subset \gamma$, it follows from (22.8) that $\mu_b(\dot{A}) \le r$. Hence $\mu^*_b(\dot{A}) \le r$, a contradiction.

This completes the proof that in $M[G]$ μ is a nontrivial κ-additive measure on κ. ∎

Example 22.10 (A model in which $2^{\aleph_0}$ carries a σ-saturated ideal). Let κ be a measurable cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. We shall construct a generic extension that satisfies $2^{\aleph_0} = \lambda$ and such that there is a σ-saturated κ-complete ideal on κ.

Let P be the notion of forcing that adjoins λ Cohen reals; i.e., a condition is a finite 0–1 function p with $\mathrm{dom}(p) \subset \lambda$. If G is a generic filter on P, then $V[G] \vDash 2^{\aleph_0} = \lambda$, and all cardinals are preserved in $V[G]$ because P satisfies the countable chain condition. That κ carries in $V[G]$ a σ-saturated ideal follows from this lemma:

Lemma 22.11. *Let κ be a measurable cardinal and let I be a nonprincipal κ-complete prime ideal on κ. Let P be a notion of forcing that satisfies the countable chain condition. Then in $V[G]$, the ideal J generated by I is a σ-saturated κ-complete ideal on κ.*

Proof. Let J be the ideal in $V[G]$ defined as follows:

$$X \in J \quad \text{if and only if} \quad X \subset Y \text{ for some } Y \in I.$$

First we show that J is κ-complete. Let $\mathcal{X} = \{X_\xi : \xi < \gamma\}$ be a family of fewer than κ elements of J; let $\dot{\mathcal{X}}$ be a name for $\mathcal{X}$ and let $p_0 \in G$ be such that $p_0 \Vdash \forall \xi < \gamma\ \dot{X}_\xi \in \dot{J}$.

For each $\xi < \gamma$ and each $p \leq p_0$, there exist $q \leq p$ and some $Y \in I$ such that $q \Vdash \dot{X}_\xi \subset \check{Y}$. Let W_ξ be a maximal antichain of $q \leq p_0$ for which there is $Y_{\xi,q}$ such that $q \Vdash \dot{X}_\xi \subset \check{Y}_{\xi,q}$. Since P satisfies the countable chain condition, each W_ξ is countable, and hence $Y = \bigcup\{Y_{\xi,q} : \xi < \gamma \text{ and } q \in W_\xi\}$ belongs to I. Now it is easy to verify that $p_0 \Vdash \bigcup_{\xi<\gamma} \dot{X}_\xi \subset \check{Y}$ and hence $\bigcup \mathcal{X} \in J$.

To prove that J is countably saturated, let us assume that $\mathcal{X} = \{X_\xi : \xi < \omega_1\}$ is a family of pairwise disjoint sets of positive J-measure. Let $\dot{\mathcal{X}}$ be a name for $\mathcal{X}$ and let $p \in G$ be such that $p \Vdash \dot{X}_\xi \notin \dot{J}$, for each $\xi < \omega_1$, and $p \Vdash \dot{X}_\xi \cap \dot{X}_\eta = \emptyset$ for all $\xi \neq \eta$.

For each $\xi < \omega_1$, let $Y_\xi = \{\alpha < \kappa : \text{some } q \leq p \text{ forces } \alpha \in \dot{X}_\xi\}$. Clearly $p \Vdash \dot{X}_\xi \subset \check{Y}_\xi$, and so $Y_\xi \notin I$. By the κ-completeness of I, we have $Y = \bigcap_{\xi<\omega_1} Y_\xi \notin I$. Thus $Y \neq \emptyset$, and let α be some element of Y. For each $\xi < \omega_1$, let $q_\xi \leq p$ be such that $q_\xi \Vdash \alpha \in \dot{X}_\xi$. Since P satisfies the countable chain condition, there are ξ, η such that q_ξ and q_η are compatible. Let q be stronger than both q_ξ and q_η; then $q \Vdash \alpha \in \dot{X}_\xi \cap \dot{X}_\eta$, a contradiction. □

Proof of Theorem 22.2. We shall prove that if $2^{\aleph_0}$ carries a σ-saturated $2^{\aleph_0}$-complete ideal then $2^\lambda = 2^{\aleph_0}$ for all $\lambda < 2^{\aleph_0}$. Let λ be a regular cardinal; two functions f, g on λ are *almost disjoint* if there is $\gamma < \lambda$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \gamma$.

Lemma 22.12. *Let κ carry a σ-saturated κ-complete ideal, and let $\lambda < \kappa$ be a regular uncountable cardinal. If $\mathcal{F}$ is a family of almost disjoint functions $f : \lambda \to \kappa$, then $|\mathcal{F}| \leq \kappa$.*

Proof. If $|\mathcal{F}| > \kappa$, then because every $f : \lambda \to \kappa$ is bounded by some $\beta < \kappa$, there exist some $\mathcal{G} \subset \mathcal{F}$ and some $\beta < \kappa$ such that $|\mathcal{G}| = \kappa$ and every $f \in \mathcal{G}$ is bounded by β.

Let $F : [\mathcal{G}]^2 \to \lambda$ be the following partition: $F(\{f, g\}) =$ some γ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \gamma$. By Lemma 22.7, there exists $\mathcal{H} \subset \mathcal{G}$ of size κ such that the image A of $[\mathcal{H}]^2$ under F is at most countable. Let $\alpha > \sup(A)$. Then $f(\alpha) \neq g(\alpha)$ whenever $f, g \in \mathcal{H}$, which is a contradiction since $|\mathcal{H}| = \kappa$ and $f(\alpha) < \beta$ for all $f \in \mathcal{H}$. □

Now we are ready to prove the theorem. Let $\kappa = 2^{\aleph_0}$. We prove $2^\lambda = \kappa$ by induction on $\lambda < \kappa$. If λ is a singular cardinal and $2^\nu = \kappa$ for all $\nu < \lambda$, then $2^\lambda = \kappa$ by Corollary 5.17. Thus let $\lambda < \kappa$ be regular and let us assume that $2^\nu = \kappa$ for all $\nu < \lambda$.

For each $X \subset \lambda$, let $f_X = \langle X \cap \alpha : \alpha < \lambda \rangle$. If $X \neq Y$, then f_X and f_Y are almost disjoint. For each $\alpha < \lambda$, the set $\{f_X(\alpha) : X \subset \lambda\}$ has size κ, and hence $\{f_X : X \subset \lambda\}$ yields a family $\mathcal{F}$ of 2^λ almost disjoint functions from λ into κ. By Lemma 22.12 we get $|\mathcal{F}| \leq \kappa$ and therefore $2^\lambda = \kappa$. □

Generic Ultrapowers

We shall now introduce a powerful method for dealing with ideals on regular uncountable cardinals. The method is a generalization of the method of ultrapowers from Chapter 10.

Let κ be a regular uncountable cardinal and let I be an ideal on κ. Let us view the universe as a ground model, let us denote this ground model M and let us consider the generic extension of M given by the completion of the Boolean algebra $P(\kappa)/I$. In other words, consider the notion of forcing $(P, \subset)$, where P is the collection of all subsets of κ of positive measure:

(22.15) (i) $X \in P$ if and only if $X \notin I$;
(ii) X is stronger than Y if and only if $X \subset Y$

Let G be a generic filter on P.

Lemma 22.13.

(i) *G is an M-ultrafilter on κ extending the filter dual to I.*
(ii) *If I is κ-complete in M, then G is a κ-complete M-ultrafilter.*
(iii) *If I is normal, then G is normal.*

Proof. (i) If $X \subset \kappa$ has measure one, then $\{Y \in P : Y \subset X\}$ is dense in P and hence $X \in G$. That G is an M-ultrafilter is obvious.

(ii) If $\{X_\alpha : \alpha < \gamma\}$, $\gamma < \kappa$, is (in M) a partition of κ, then by the κ-completeness of I, the set $\{Y \in P : Y \subset \text{some } X_\alpha\}$ is dense in P and hence $X_\alpha \in G$ for some α.

(iii) If $X \in G$ and if $f \in M$ is a regressive function on X, then $\{Y \subset X : f \text{ is constant on } Y\}$ is dense below X, and hence f is constant on some $Y \in G$. □

From now on assume that I is a κ-complete ideal on κ containing all singletons. Then G is a nonprincipal κ-complete M-ultrafilter on κ. Note that if I is atomless, then $G \notin M$ (if I is prime, then G is the dual of I and so $G \in M$).

Let us consider (in $M[G]$) the ultrapower $\text{Ult}_G(M)$; let us call this ultrapower a *generic ultrapower*. The generic ultrapower is a model of ZFC, but is not necessarily well-founded. We have the analog of Łoś's Theorem, in this form

(22.16) $\quad \text{Ult}_G(M) \vDash \varphi([f_1], \ldots, [f_n])$

$$\text{if and only if} \quad \{\alpha : M \vDash \varphi(f_1(\alpha), \ldots, f_n(\alpha))\} \in G$$

whenever $f_1, \ldots, f_n \in M$ are functions defined on a set $X \in G$. In particular, we have an elementary embedding, the *canonical embedding* $j_G : M \to \text{Ult}_G(M)$, defined by

$$j_G(x) = [c_x]$$

where c_x is the constant function on κ with value x, and $[c_x]$ is its equivalence class in the ultrapower.

Let us denote the generic ultrapower by N and $j_G = j$. The ordinal numbers of the model N form a linearly ordered class, not necessarily well-ordered, but we shall show that (because I is κ-complete), Ord^N has an initial segment of order-type κ. If $x \in Ord^N$, let us call the *order-type* of x the order-type of the set $\{y \in Ord^N : y <^N x\}$. If the order-type of x is an ordinal number, we take the liberty of identifying x with this ordinal.

Lemma 22.14.

(i) *For every $\gamma < \kappa$, $j(\gamma) = \gamma$; hence Ord^N has an initial segment of order-type κ.*

(ii) *$j(\kappa) \neq \kappa$.*

(iii) *If I is normal, then there exists $x \in Ord^N$ such that $x = \kappa$; in fact $[d] = \kappa$ where d is the diagonal function $d(\alpha) = \alpha$.*

Proof. G is κ-complete, nonprincipal, and if I is normal then G is normal. □

Let us mention again the fact that we already mentioned and that is fairly easy to verify: If P is the notion of forcing (22.15) then $B(P) = B(P(\kappa)/I)$; the mapping $X \mapsto [X]$ gives the natural correspondence.

To illustrate the method of generic ultrapowers we present two examples. The first is (a modification of) Silver's proof of Theorem 8.12; the other is a theorem of Jech and Prikry.

Example 22.15 (Proof of Silver's Theorem 8.12). Let us consider this typical special case: Let κ be a singular cardinal of cofinality $\aleph_1$, and assume that $2^\lambda = \lambda^+$ for all $\lambda < \kappa$. We shall show that $2^\kappa = \kappa^+$, using a generic ultrapower.

Let I be the ideal of nonstationary subsets of ω_1, let P be the corresponding notion of forcing (i.e., forcing conditions are stationary sets) and let G be a generic filter on P. Note that since $|P| = 2^{\aleph_1} < \kappa$ (in M), all cardinals $\geq \kappa$ remain cardinals in $M[G]$.

Let us work in $M[G]$. G is a normal σ-complete M-ultrafilter on ω_1^M. Let $N = \text{Ult}_G(M)$ be the generic ultrapower and let $j : M \to N$ be the canonical elementary embedding. N is not necessarily well-founded.

Let $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ be (in M) an increasing continuous sequence of cardinals converging to κ. Let e be the cardinal number in N represented by the function $e(\alpha) = \kappa_\alpha$. Let e^+ denote the successor cardinal of e in N.

For each $X \subset \kappa$ in M let f_X be the function on ω_1^M defined by $f_X(\alpha) = X \cap \kappa_\alpha$. Clearly, each f_X represents in N a subset of e. Moreover, if $X \neq Y$, then f_X and f_Y are almost disjoint and hence represent distinct subsets of e. It follows that $|P^M(\kappa)| \leq |P^N(e)|$, where $P^N(e)$ denotes the collection of all subsets of e in N.

Now $N \vDash 2^e = e^+$ (because $M \vDash 2^{\kappa_\alpha} = \kappa_\alpha^+$ for all α), which means that in the model N there is a one-to-one correspondence between the power set of e and e^+. It follows that there is a one-to-one correspondence between $P^N(e)$ and the set $\text{ext}(e^+) = \{x \in \mathit{Ord}^N : x <^N e^+\}$. Thus we have so far $|P^M(\kappa)| \leq |\,\text{ext}(e^+)|$.

Next we observe that $e = \sup\{j(\kappa_\gamma) : \gamma < \omega_1^M\}$. This is because if f represents an ordinal less than e, then there is a set of limit ordinals $X \in G$ such that $f(\alpha) < \kappa_\alpha$ for all $\alpha \in X$; thus $f(\alpha) < \kappa_{\gamma(\alpha)}$ for some $\gamma(\alpha) < \alpha$, and by normality of G, there is γ such that $[f] <^N \kappa_\gamma$. Now for each $\gamma < \omega_1^M$, $|\,\text{ext}(j(\kappa_\gamma))| \leq |(\kappa_\gamma^{\aleph_1})^M| < \kappa$, and therefore $|\,\text{ext}(e)| \leq \kappa$.

If $x <^N e^+$, then there is in N a one-to-one mapping of x into e, and therefore, $|\,\text{ext}(x)| \leq |\,\text{ext}(e)| \leq \kappa$. Thus $\text{ext}(e^+)$ is a linearly ordered set whose each initial segment has size at most κ. Therefore $|\,\text{ext}(e^+)| \leq \kappa^+$, and we have

$$|P^M(\kappa)| \leq \kappa^+.$$

We have argued so far in $M[G]$; in other words, we have proved that $|P^M(\kappa)|^{M[G]} \leq (\kappa^+)^{M[G]}$. But since all cardinals $\geq \kappa$ in M remain cardinals in $M[G]$, it is necessary that $|P^M(\kappa)|^M \leq (\kappa^+)^M$; in other words we have proved that $2^\kappa = \kappa^+$ (in M). ◻

Theorem 22.16. *Let I be a σ-complete ideal on ω_1. If $2^{\aleph_0} < \aleph_{\omega_1}$ then $2^{\aleph_1} \leq 2^{\aleph_0} \cdot \text{sat}(I)$.*

Corollary 22.17. *If there exists an $\aleph_2$-saturated ideal on ω_1, then*

(i) $2^{\aleph_0} = \aleph_1$ *implies* $2^{\aleph_1} = \aleph_2$;

(ii) $\aleph_1 < 2^{\aleph_0} < \aleph_{\omega_1}$ *implies* $2^{\aleph_0} = 2^{\aleph_1}$. □

Proof of Theorem 22.16. Let $2^{\aleph_0} = \aleph_\gamma < \aleph_{\omega_1}$ and let I be a σ-complete λ-saturated ideal on ω_1. We shall show that $2^{\aleph_1} \le \aleph_\gamma \cdot \lambda$.

Let P be the notion of forcing corresponding to I, and let G be generic on P. Since $\operatorname{sat}(P) = \operatorname{sat}(I) \le \lambda$, all cardinals $\ge \lambda$ in M are cardinals in $M[G]$.

Let us work in $M[G]$, and let $N = \operatorname{Ult}_G(M)$ and $j = j_G : M \to N$. For each $X \subset \omega_1$ in M let f_X be the function on ω_1^M defined by $f_X(\alpha) = X \cap \alpha$. Each f_X represents in N a subset of the countable ordinal d represented by the function $d(\alpha) = \alpha$; moreover, if $X = Y$, then f_X and f_Y are almost disjoint and hence $[f_X] \ne [f_Y]$. It follows that $|P^M(\omega_1^M)| \le |P^N(d)|$. Let e be the cardinal number in N such that $N \vDash 2^{\aleph_0} = e$ (we recall that $\omega^N = \omega$). Since $N \vDash |P(d)| = e$, we have $|P^N(d)| = |\operatorname{ext}(e)|$ and so

$$|P^M(\omega_1^M)| \le |\operatorname{ext}(e)|. \tag{22.17}$$

Next we shall compute the size of $\operatorname{ext}(e)$. Since $M \vDash 2^{\aleph_0} = \aleph_\gamma$ and $j : M \to N$ is elementary, we have $e = j(\omega_\gamma)$. We shall now prove by induction on $\gamma < \omega_1^M$ that

$$|\operatorname{ext}(j(\omega_\gamma^M))| \le \lambda \cdot |\omega_\gamma^M|. \tag{22.18}$$

Let us denote $j(\omega_\gamma^M) = e_\gamma$ for all $\gamma < \omega_1^M$. By Lemma 22.14, the ordinals of N have an initial segment of order-type ω_1^M; thus the *infinite cardinals* of N also have an initial segment of order-type ω_1^M, namely $\{e_\gamma : \gamma < \omega_1^M\}$.

If $\gamma = 0$, then $e_\gamma = \omega$ and (22.18) is true. If γ is a limit ordinal, then $e_\gamma = \sup\{e_\delta : \delta < \gamma\}$ and (22.18) is again true provided it is true for all $\delta < \gamma$. If $\gamma = 1$, then $\operatorname{ext}(e_1)$ is a linearly ordered set whose each initial segment is countable, and hence $|\operatorname{ext}(e_1)| \le \aleph_1$. Since λ is a cardinal (now we are in $M[G]$), we have $\aleph_1 \le \lambda$, and (22.18) holds.

Let us assume that (22.18) holds for γ and let us show that it also holds for $\gamma + 1$. Every function $f : \omega_1 \to \omega_{\gamma+1}$ in M is bounded by some constant function, and therefore $j(\omega_{\gamma+1}) = \sup\{j(\xi) : \xi < \omega_{\gamma+1}\}$. Hence the linearly ordered set $\operatorname{ext}(e_{\gamma+1})$ has a cofinal set of order-type $\omega_{\gamma+1}^M$ and each its initial segment has size $\le \lambda \cdot |\omega_\gamma^M|$ (because if $\xi < \omega_{\gamma+1}$, then $|\operatorname{ext}(j(\xi))| \le |\operatorname{ext}(e_\gamma)| \le \lambda \cdot |\omega_\gamma^M|$). It follows that $|\operatorname{ext}(e_{\gamma+1})| \le \lambda \cdot |\omega_{\gamma+1}^M|$.

Now we put (22.17) and (22.18) together and get

$$|P^M(\omega_1^M)| \le \lambda \cdot \aleph_\gamma^M.$$

This we proved in $M[G]$; but since all cardinals $\ge \lambda$ in M remains cardinals in $M[G]$, the same must be true in M. Hence (in M)

$$2^{\aleph_1} \le \lambda \cdot \aleph_\gamma.$$ □

Precipitous Ideals

In an early application of generic ultrapowers, Solovay proved that if I is κ^+-saturated then the generic ultrapower is well-founded (see Lemma 22.22 in the next section). It has been recognized that this property of ideals is important enough to single out and study such ideals.

Definition 22.18. A κ-complete ideal on κ is *precipitous* if the generic ultrapower $\text{Ult}_G(M)$ is well-founded.

We give below several necessary and sufficient (combinatorial) conditions on I to be precipitous.

Let I be a κ-complete ideal on κ containing all singletons. Let S be a set of positive measure. An *I-partition* of S is a maximal family W of subsets of S of positive measure such that $X \cap Y \in I$ for any distinct $X, Y \in W$. An I-partition W_1 of S is a *refinement* of an I-partition W_2 of S, $W_1 \le W_2$, if every $X \in W_1$ is a subset of some $Y \in W_2$. A *functional* on S is a collection F of functions such that $W_F = \{\text{dom}(f) : f \in F\}$ is an I-partition of S and $\text{dom}(f) \ne \text{dom}(g)$ whenever $f \ne g \in F$.

We define $F < G$ for two functionals on S to mean that:

(i) each $f \in F \cup G$ is a function into the ordinals;
(ii) $W_F \le W_G$; and
(iii) if $f \in F$ and $g \in G$ are such that $\text{dom}(f) \subset \text{dom}(g)$, then $f(\alpha) < g(\alpha)$ for all $\alpha \in \text{dom}(f)$.

The reason we define functionals is that they represent functions in the Boolean-valued model M^B (and so are canonical representatives for elements of $\text{Ult}_G(M)$): Let $\dot{f} \in M^B$ be such that

$$S \Vdash \dot{f} \text{ is a function with } \text{dom}(\dot{f}) \in \dot{G} \text{ and } \dot{f} \in M. \tag{22.19}$$

Then there is an I-partition W of S, and for each $X \in W$ a function f_X on X such that for all $X \in W$, $X \Vdash \dot{f}\restriction\check{X} = \check{f}_X$. Thus the functional $\{f_X : X \in W\}$ represents the Boolean-valued $\dot{f}$ on S.

Conversely, if F is functional on S, then there is some $\dot{f} \in M^B$ such that (22.19) holds; and for each $f \in F$, if $X = \text{dom}(f)$, then $X \Vdash \dot{f}\restriction\check{X} = \check{f}$.

Note also that if $F < G$ are functionals on S and $\dot{f}$, $\dot{g}$ are corresponding Boolean-valued names, then

$$S \Vdash \dot{f}, \dot{g} \in M \text{ and } \text{dom}(\dot{f}) \subset \text{dom}\,\dot{g} \text{ and } \dot{f}(\alpha) < \dot{g}(\alpha) \text{ for all } \alpha \in \text{dom}(\dot{f}). \tag{22.20}$$

Conversely, if $\dot{f}$ and $\dot{g}$ satisfy (22.20), then there are functionals F and G that represent $\dot{f}$ and $\dot{g}$, and $F < G$.

Lemma 22.19. *The following are equivalent*:

(i) *I is precipitous.*
(ii) *Whenever S is a set of positive measure and $\{W_n : n < \omega\}$ are I-partitions of S such that $W_0 \geq W_1 \geq \ldots \geq W_n \geq \ldots$, then there exists a sequence of sets $X_0 \supset X_1 \supset \ldots \supset X_n \supset \ldots$ such that $X_n \in W_n$ for each n, and $\bigcap_{n=0}^{\infty} X_n$ is nonempty.*
(iii) *For no set S of positive measure is there a sequence of functionals on S such that $F_0 > F_1 > \ldots > F_n > \ldots$.*

Proof. In view of the preceding discussion on functionals, (ii) is equivalent to (i): If $F_0 > F_1 > \ldots$ are functionals on S, and $\dot{f}_0, \dot{f}_1, \ldots$, the corresponding elements of M^B, then S forces that $[\dot{f}_0], [\dot{f}_1], \ldots$ is a descending sequence of ordinals in the generic ultrapower. Conversely, if S forces that $\mathrm{Ult}_G(M)$ has a descending sequence of ordinals, we construct $F_0, F_1, \ldots$ on S such that $F_0 > F_1 > \ldots$.

The implication (ii) $\rightarrow$ (iii) is easy. If $F_0 > F_1 > \ldots$ are functionals on S, then the partitions $W_{F_0}, W_{F_1}, \ldots$ constitute a counterexample: If $X_0 \supset X_1 \supset \ldots$ are elements of $W_{F_0}, W_{F_1}, \ldots$, let $f_0 \in F_0$ be the function with domain X_0, $f_1 \in F_1$ with domain X_1, etc.; now if $\bigcap_{n=0}^{\infty} X_n$ were nonempty, we would get $f_0(\alpha) > f_1(\alpha) > \ldots$ for $\alpha \in \bigcap_{n=0}^{\infty} X_n$.

To show (iii) $\rightarrow$ (ii), let $W_0 \geq W_1 \geq \ldots$ be partitions of some $S \notin I$ that fail (ii). We shall construct functionals on S such that $F_0 > F_1 > \ldots$.

Without loss of generality, let us assume that if $X \in W_{n+1}$, $Y \in W_n$, and $X \subset Y$, then $X \neq Y$. Let $T = \bigcup_{n=0}^{\infty} W_n$; note that the partially ordered set $(T, \subset)$ is an upside-down tree (of height ω).

For each $z \in S$, let us consider the set $T_z = \{X \in T : z \in X\}$. Since every descending sequence $X_0 \supset X_1 \supset \ldots$ in T has empty intersection, it follows that for every z, T_z has no infinite descending sequence $X_0 \supset X_1 \supset \ldots$; hence the relation $\subset$ on T_z is well-founded. Thus there is, for each z, an ordinal function ρ_z on T (the rank function) such that $\rho_z(X) < \rho_z(Y)$ when $X \subset Y$. It is clear that if $X \in W_{n+1}$, $Y \in W_n$, and $z \in X \subset Y$, then $\rho_z(X) < \rho_z(Y)$.

Thus we define, for each $X \in T$, a function f_X on X as follows:

$$f_X(z) = \rho_z(X) \qquad (\text{all } z \in X).$$

Now it is clear that if we let $F_n = \{f_X : X \in W_n\}$ for each n, then $F_0, F_1, \ldots$ are functionals on S and $F_0 > F_1 > \ldots$. $\square$

Lemma 22.20. *Let κ be a regular uncountable cardinal. The ideal $I = \{X \subset \kappa : |X| < \kappa\}$ is not precipitous.*

Proof. Let $I = \{X \subset \kappa : |X| < \kappa\}$. A set $X \subset \kappa$ has positive measure just in case $|X| = \kappa$. For each such X, let f_X be the unique order-preserving function from X onto κ.

For each set X of positive measure there exists a set $Y \subset X$ of positive measure such that $f_Y(\alpha) < f_X(\alpha)$ for all $\alpha \in Y$; namely if we let $Y =$

$\{\alpha \in X : f_X(\alpha) \text{ is a successor ordinal}\}$, then $f_X(\alpha) = f_Y(\alpha) + 1$ for all $\alpha \in Y$. Thus for each $X \notin I$ there is an I-partition W_X of X such that for all $Y \in W_X$, $f_Y(\alpha) < f_X(\alpha)$ on Y.

Now we construct I-partitions $W_0 \geq W_1 \geq \ldots$ as follows: We let $W_0 = \{\kappa\}$, and for each n, we let $W_{n+1} = \bigcup\{W_X : X \in W_n\}$. For each n, we let F_n be the functional $F_n = \{f_X : X \in W_n\}$. It is clear that $F_0 > F_1 > \ldots > F_n > \ldots$, and therefore I is not precipitous. □

An alternate characterization of precipitousness is in terms of infinite games. $\mathcal{G}_I$ is the infinite game played by two players, Empty and Nonempty, who alternately choose sets S_n of positive I-measure such that $S_{n+1} \subset S_n$. Empty plays first and wins if $\bigcap_{n=0}^{\infty} S_n = \emptyset$.

Lemma 22.21. *I is precipitous if and only if Empty has no winning strategy in the game $\mathcal{G}_I$.*

Proof. If I is not precipitous then there is a set S of positive measure and a sequence of functionals on S such that $F_0 > F_1 > \ldots > F_n > \ldots$. Empty chooses $S_0 = S$ for his first move. When Nonempty plays S_{2n-1}, Empty finds some $f \in F_n$ such that the set $X = \operatorname{dom}(f) \cap S_{2n-1}$ has positive measure and chooses $S_{2n} = X$ for his move. It follows that $\bigcap_{n=0}^{\infty} S_n$ is empty, and hence Empty wins.

Now suppose that I is precipitous and σ is a strategy for Empty; we will show that σ is not a winning strategy. Let S_0 be Empty's first move by σ. Then S_0 forces that in $M[G]$ there is an infinite sequence $\langle S_n : n \in \omega\rangle$ of moves in which Empty follows σ and each $S_n \in G$. If $j : M \to \operatorname{Ult}_G(M)$ is the canonical embedding then $\langle j(S_n) : n \in \omega\rangle$ is an infinite sequence of moves (of $j(\mathcal{G}_I)$) in which Empty follows $j(\sigma)$ and $\kappa \in \bigcap_{n=0}^{\infty} j(S_n)$. Since $\operatorname{Ult}_G(M)$ is well-founded, there exists (by absoluteness) such a sequence in $\operatorname{Ult}_G(M)$, and since j is elementary, there exists a sequence $\langle S_n : n \in \omega\rangle$ in M in which Empty follows σ but for some $\alpha < \kappa$, $\alpha \in \bigcap_{n=0}^{\infty} S_n$. Thus σ is not a winning strategy. □

Saturated Ideals

Results from Chapter 10 and those proved earlier in this chapter establish the following facts about the existence of a σ-saturated κ-complete ideal on κ: If κ carries a σ-saturated κ-complete ideal then either κ is measurable, or $\kappa \leq 2^{\aleph_0}$ and κ is weakly inaccessible (Lemma 10.9 and 10.14). If a σ-saturated ideal exists then there exists a normal one (Lemma 22.3), and its consistency strength is that of a measurable cardinal (Lemma 22.8). These results generalize easily to ν-saturated ideals for $\nu < \kappa$; the analog of Lemma 10.9 (with the same proof) is that either κ is measurable, or $\kappa \leq 2^{<\nu}$.

In this section we investigate κ-saturated and κ^+-saturated ideals. We shall employ the technique of generic ultrapowers; this is particularly useful because the generic ultrapower is well-founded:

Lemma 22.22. *Let κ be a regular uncountable cardinal. Every κ^+-saturated κ-complete ideal on κ is precipitous.*

Proof. Let I be a κ^+-saturated κ-complete ideal on κ. Let S be a set of positive measure and let $W_0 \geq W_1 \geq \ldots$ be I-partitions of S. We shall find $X_0 \supset X_1 \supset \ldots$ in $W_0, W_1, \ldots$ such that $\bigcap_{n=0}^\infty X_n$ is nonempty.

We shall first modify each W_n to obtain a new I-partition W'_n that is almost like W_n but is disjoint. We proceed by induction on n. Since $|W_0| \leq \kappa$, let $W_0 = \{X_\alpha : \alpha < \theta\}$ where $\theta \leq \kappa$, and for each $\alpha < \theta$, let $X'_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$; then we let $W'_0 = \{X' : X \in W\}$. Since I is κ-complete, we have $X - X' \in I$ for all $X \in W_0$ and thus W'_0 is an I-partition of S; moreover, W'_0 is disjoint, and is a partition of $S_0 = \bigcup W'_0$ and $S - S_0 \in I$. Having constructed W'_n, we enumerate $W_{n+1} = \{X_\alpha : \alpha < \theta\}$ where $\theta \leq \kappa$, and for each $\alpha < \theta$, let $X'_\alpha = (X_\alpha - \bigcup_{\beta<\alpha} X_\beta) \cap Z$ where Z is the unique $Z \in W'_n$ that is almost all of the unique $Y \in W_n$ such that $X_\alpha \subset Y$. We let $W'_{n+1} = \{X' : X \in W_{n+1}\}$; W'_n is a partition of $S_{n+1} = \bigcup W'_{n+1}$, $S - S_{n+1} \in I$, and $X - X' \in I$ for all $X \in W_{n+1}$.

Since each S_n is almost all of S, the set $\bigcap_{n=0}^\infty S_n$ is nonempty; let z be an element of this intersection. For each n there is a unique $Y_n \in W'_n$ such that $z \in Y_n$; let X_n be the unique $X_n \in W_n$ such that $Y_n \subset X_n$. It is clear that $X_0 \supset X_1 \supset \ldots \supset X_n \supset \ldots$, and $\bigcap_{n=0}^\infty X_n \neq \emptyset$. □

In the next section we shall generalize Lemma 22.8: We shall prove that if κ carries a precipitous ideal then it is measurable in an inner model.

Let us first consider κ-saturated ideals. First we observe that the proof of Lemmas 22.3(i) and 22.4 works as well when I is only κ-saturated, and so we have:

Lemma 22.23.

(i) *If there exists a κ-saturated κ-complete ideal on an uncountable cardinal κ, then there exists a normal κ-saturated κ-complete ideal on κ.*

(ii) *Let I be a normal κ-saturated κ-complete ideal on κ. If $S \notin I$ and if $f : S \to \kappa$ is regressive on S, then there is $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all $\alpha \in S$.* □

Also, the proof of Lemma 10.14 works also for κ-saturated ideals, and so we have:

Lemma 22.24. *If κ carries a κ-saturated κ-complete ideal then κ is weakly inaccessible.* □

It is consistent (relative to the existence of a measurable cardinal) that an inaccessible cardinal κ carries an ideal I such that $\operatorname{sat}(I) = \kappa$. (Such κ cannot be weakly compact, see Exercise 22.13.)

We shall now prove the main result on κ-saturated ideals, using generic ultrapowers. First we need a lemma on preservation of stationary sets by forcing:

Lemma 22.25. *Let κ be a regular uncountable cardinal. Let $V[G]$ be a generic extension of V by a κ-c.c. notion of forcing. Then every closed unbounded $C \subset \kappa$ in $V[G]$ has a closed unbounded subset $D \in V$. Consequently, if $S \in V$ is stationary in V, then S remains stationary in $V[G]$.*

Proof. Let $\dot{C}$ be a name such that every condition forces that $\dot{C}$ is a closed unbounded subset of κ. Let $D = \{\alpha : \|\alpha \in \dot{C}\| = 1\}$. Clearly, D is a subset of C and is closed; we have to prove only that D is unbounded.

Let $\alpha_0 < \kappa$; we wish to find $\alpha > \alpha_0$ such that every condition forces $\alpha \in \dot{C}$. For every p, there is $q \le p$ and some $\beta > \alpha_0$ such that $q \Vdash \beta \in \dot{C}$. Thus there is a maximal incompatible set W of conditions, and for each $q \in W$ an ordinal $\beta = \beta_q$ such that $q \Vdash \beta \in \dot{C}$. Since $|W| < \kappa$, we let $\alpha_1 = \sup\{\beta_q : q \in W\}$; we have $\alpha_1 < \kappa$ and

$$p \Vdash (\exists \beta \in \dot{C})\ \alpha_0 < \beta \le \alpha_1$$

for all conditions p. Similarly, we find $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ such that for every n and every condition p,

$$p \Vdash (\exists \beta \in \dot{C})\ \alpha_n < \beta \le \alpha_{n+1}.$$

If we let $\alpha = \lim_n \alpha_n$, it is clear that $\|\alpha \in \dot{C}\| = 1$. □

Theorem 22.26 (Solovay). *Let κ be a regular uncountable cardinal and assume that κ carries a κ-saturated ideal.*

(i) *κ is weakly Mahlo*;
(ii) *$\{\alpha < \kappa : \alpha$ is weakly Mahlo$\}$ is stationary*;
(iii) *if $X \subset \kappa$ has measure one in a normal κ-saturated ideal, then $X \cap M(X)$ has measure one, where*

$$M(X) = \{\alpha < \kappa : \operatorname{cf} \alpha > \omega \textit{ and } X \cap \alpha \textit{ is stationary in } \alpha\}.$$

Proof. If there exists a κ-saturated ideal on κ, then κ is weakly inaccessible by Lemma 22.24, and there exists a normal κ-saturated ideal on κ (by Lemma 22.23). Let I be a normal κ-saturated ideal on κ. We first prove:

Lemma 22.27. *If $S \subset \kappa$ is stationary, then for I-almost all $\alpha < \kappa$, $S \cap \alpha$ is stationary in α.*

Proof. If not, then there is a set X of positive measure such that $S \cap \alpha$ is not stationary in α (or $\operatorname{cf} \alpha = \omega$) for all $\alpha \in X$. Let G be a generic ultrafilter on κ (corresponding to I) such that $X \in G$. Let $N = \operatorname{Ult}_G(M)$. N is a transitive model. Since I is normal, κ is represented in N by the function $d(\alpha) = \alpha$. Since $S = j(S) \cap \kappa$, we have $N \vDash S$ is not stationary.

However, the notion of forcing is κ-saturated and hence κ is a regular cardinal in $M[G]$, and by Lemma 22.25, $M[G] \vDash S$ is stationary. Now $N \subset M[G]$ and so $N \vDash S$ is stationary. A contradiction. □

Since I is normal, every set of positive measure is stationary. Thus (iii) follows since if X has measure one then $M(X)$ has measure one by Lemma 22.27, and so does $X \cap M(X)$.

To prove (i), it suffices to show that almost all $\alpha < \kappa$ are regular cardinals. Otherwise, let X be a set of positive measure such that all $\alpha \in X$ are singular. Let $G \ni X$ be generic and let $N = \mathrm{Ult}_G(M)$. Then $N \vDash \kappa$ is singular, contrary to the fact that κ is regular in $M[G]$ and $N \subset M[G]$.

Now (ii) follows by an application of (iii): Let $X = \{\alpha < \kappa : \alpha \text{ is regular}\}$, then $X \cap M(X) = \{\alpha < \kappa : \alpha \text{ is weakly Mahlo}\}$. □

As a corollary of Theorem 22.26 we have Solovay's original proof of Theorem 8.10:

Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. We claim that S is the disjoint union of κ stationary subsets.

Otherwise, the ideal $I = \{X \subset \kappa : X \cap S \text{ is nonstationary}\}$ is a normal κ-saturated ideal. By Lemma 22.27, $S - M(S)$ has measure zero and hence is nonstationary, which contradicts Lemma 8.9. □

Now let us consider κ^+-saturated ideals.

Lemma 22.28. *Let I be a κ^+-saturated κ-complete ideal on κ.*

(i) *There exists a least unbounded function, i.e., a function $f : \kappa \to \kappa$ such that for any $\gamma < \kappa$ there is no S of positive measure such that $f(\alpha) < \gamma$ on S (unbounded) and that for any $g : \kappa \to \kappa$, if $g(\alpha) < f(\alpha)$ on a set of positive measure then g is constant on a set of positive measure.*
(ii) *There exists a normal κ^+-saturated κ-complete ideal on κ.*

Proof. By Lemma 22.22 I is precipitous. Since I is κ^+-saturated, the Boolean-valued names for functions on κ in the ground model can be represented not by functionals but by ordinary functions: Let F be a functional (on κ). Let $W = \{\mathrm{dom}(f) : f \in F\}$; since I is κ^+-saturated, W can be replaced by a *disjoint* W' such that for each $X \in W$ there is $X' \in W'$ such that $X' \subset X$ and $X - X' \in I$. If we replace each $f \in F$ by its restriction to the corresponding $X' \in W'$, we get a functional F' whose elements have disjoint domains. Then $f = \bigcup F'$ is a function, and if $\dot{f} \in V^B$ is the name corresponding to F, then $\|\dot{f} = \check{f}\| = 1$.

Let $\dot{f} \in V^B$ be such that $\|\dot{f}$ represents $\check{\kappa}$ in the generic ultrapower$\| = 1$ and let $f : \kappa \to \kappa$ be such that $\|\dot{f} = \check{f}\| = 1$. Then f is the least unbounded function.

If f is the least unbounded function then $f_*(I) = \{X \subset \kappa : f_{-1}(X) \in I\}$ is a normal κ-complete ideal and is κ^+-saturated. □

Unlike κ-saturation, the existence of a κ^+-saturated ideal on κ does not imply that κ be a limit cardinal. However, the consistency strength of a κ^+-saturated ideal on a successor cardinal κ is considerably stronger than measurability (while the existence of an ideal I on an inaccessible κ such that sat$(I) = \kappa^+$ is equiconsistent with measurability). It is consistent, relative to a Woodin cardinal, that the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated; we shall study this problem in Part III. We shall also return to the subject of saturation of the nonstationary ideal in general in Chapter 23.

Saturated ideals have influence on cardinal arithmetic, similar to measurable cardinals:

Lemma 22.29. *Let κ be a regular uncountable cardinal and let I be a κ^+-saturated ideal on κ. If $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.*

Proof. Let M be the ground model. Let P be the notion of forcing corresponding to I, let G be generic on P, and let $N = \mathrm{Ult}_G(M)$. Since I is κ^+-saturated, N is well-founded and hence we identify it with a transitive model $N \subset M[G]$. Let $j : M \to N$ be the canonical embedding. We have $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and $j(\kappa) > \kappa$. If $X \subset \kappa$ and $X \in M$, then $X \in N$ because $X = j(X) \cap \kappa$. Thus $P^M(\kappa) \subset P^N(\kappa)$.

We assume that $M \vDash (2^\lambda = \lambda^+$ for all $\lambda < \kappa)$ and hence $N \vDash (2^\lambda = \lambda^+$ for all $\lambda < j(\kappa))$, and in particular, $N \vDash |P(\kappa)| = \kappa^+$, where α^+ denotes the least cardinal greater than α. Now $(\kappa^+)^N \le (\kappa^+)^{M[G]}$; and because sat$(P) = \kappa^+$ (in M), $(\kappa^+)^M$ is a cardinal in $M[G]$ and we have also $(\kappa^+)^{M[G]} = (\kappa^+)^M$. Thus we have, in $M[G]$,

$$|P^M(\kappa)| \le (\kappa^+)^M$$

and since all cardinals above κ^+ in M are preserved, the last formula is also true in M, and we have $2^\kappa = \kappa^+$. □

Lemma 22.30. *Let I be an $\aleph_2$-saturated ideal on ω_1. Then*

(i) *If $2^{\aleph_0} = \aleph_1$, then $2^{\aleph_1} = \aleph_2$.*
(ii) *If $\aleph_1 < 2^{\aleph_0} < \aleph^{\omega_1}$, then $2^{\aleph_1} = 2^{\aleph_0}$.*
(iii) *If $2^{\aleph_0} = \aleph_{\omega_1}$, then $2^{\aleph_1} \le \aleph_{\omega_2}$.*
(iv) *If $\aleph_{\omega_1}$ is strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.*
(v) *Let $\Phi(\alpha)$ denote the αth member of the class $\{\kappa : \aleph_\kappa = \kappa\}$. If $\Phi(\omega_1)$ is strong limit, then $2^{\Phi(\omega_1)} < \Phi(\omega_2)$.*

Proof. (i) and (ii) are as in Corollary 22.17.

Let G be a generic ultrafilter on ω_1, let $N = \mathrm{Ult}_G(M)$ and let $j : M \to N$. N is a transitive model, $N \subset M[G]$.

Let us denote $\kappa = \omega_1^M$. We have $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and $j(\kappa) > \kappa$. Thus κ is a countable ordinal in N. Moreover, every $f : \kappa \to Ord$ in M belongs to N, and so every $\gamma < \omega_2^M$ is countable in N. Since sat$(I) = \aleph_2$, ω_2^M is a cardinal in $M[G]$, hence in N, and so

$$j(\omega_1^M) = \omega_2^M.$$

We shall now prove (iii), (iv), and (v). To prove (iii), let us assume that $M \vDash 2^{\aleph_0} = \aleph_{\omega_1}$. Since $N \vDash |\omega_1^M| = \aleph_0$, and j is elementary, we have $M[G] \vDash$

$|P^M(\kappa)| \le (2^{\aleph_0})^N$ and $N \vDash 2^{\aleph_0} = \aleph_{j(\kappa)}$. Now $\aleph^N_{j(\kappa)} \le \aleph^{M[G]}_{j(\kappa)}$; and since $j(\kappa) = \omega_2^M$, $\aleph^{M[G]}_{j(\kappa)}$ is the ω_2^Mth cardinal in $M[G]$. However, all cardinals $\ge \aleph_2^M$ are preserved and hence $\aleph^{M[G]}_{j(\kappa)} = \aleph^M_{\omega_2}$. Thus

$$|P^M(\omega_1^M)| \le \aleph^M_{\omega_2}$$

holds in $M[G]$; and because cardinals above $\aleph_2$ are preserved, this also holds in M.

To prove (iv) or (v), note that if $\aleph_{\omega_1}$ (or $\Phi(\omega_1)$) is strong limit, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1}^{\aleph_1}$ ($2^{\Phi(\omega_1)} = (\Phi(\omega_1))^{\aleph_1}$). Let λ denote $\aleph_{\omega_1}$ in (iv) and $\Phi(\omega_1)$ in (v). It is easy to see that $j(\lambda) > \lambda$. Now $N \vDash \forall \alpha < j(\lambda)\ |\alpha^\kappa| < j(\lambda)$, and because $(\lambda^\kappa)^M \subset (\lambda^\kappa)^N$, we have $M[G] \vDash |(\lambda^\kappa)^M| \le j(\lambda)$.

In case (iv) we have $j(\lambda) \le \aleph^M_{\omega_2}$ as in (iii); and in case (v) we obtain similarly $j(\lambda) \le \Phi(j(\kappa)) \le \Phi(\omega_2)$. The rest of the proof of either (iv) or (v) is as before. □

To conclude this section we prove a generalization of Lemma 22.11; we show in Lemma 22.32 that for all $\lambda \le \kappa^+$, if $V[G]$ is a generic extension by λ-saturated forcing, then a λ-saturated ideal in V generates a λ-saturated ideal in $V[G]$.

Lemma 22.31. *Let I be a κ^+-saturated κ-complete ideal on κ. Let G be a corresponding generic ultrafilter and let $N = \mathrm{Ult}_G(M)$ be the generic ultrapower. Then every $s : \kappa \to M$ in $M[G]$ is in N.*

Proof. Let $\dot{s}$ be a name for s; for each $\alpha < \kappa$, let $\dot{s}_\alpha$ be a name such that $\|\dot{s}(\alpha) = \dot{s}_\alpha\| = 1$. Each $\dot{s}_\alpha$ is represented by a function $f_\alpha \in M$ on κ. Let $h : \kappa \to \kappa$ be the least unbounded function. Let f be the function on κ defined by $f(\alpha) = \langle f_\beta(\alpha) : \beta < h(\alpha)\rangle$. Then f represents $\dot{s}$ in the generic ultrapower. □

Lemma 22.32. *Let B be a complete Boolean algebra, let G be a V-generic ultrafilter on B and let κ be an uncountable regular cardinal. Let $\lambda \le \kappa^+$ be regular and assume that* $\mathrm{sat}(B) \le \lambda$ *and* $\mathrm{sat}(B) < \kappa$. *If I is a λ-saturated κ-complete ideal on κ, then in $V[G]$, I generates a λ-saturated κ-complete ideal.*

Proof. Let $J \in V[G]$ be the ideal generated by I. Since $\mathrm{sat}\, B \le \kappa$, J is κ-complete. Let $\dot{J} \in V^B$ be the canonical name for J, and let $\dot{C} \in V^B$ be the Boolean algebra $\dot{C} = P(\check{\kappa})/\dot{J}$.

We want to show that $V^B \vDash \dot{C}$ is $\check{\lambda}$-saturated; by Lemma 16.5 it suffices to show that $B * \dot{C}$ is λ-saturated because B is λ-saturated. Let $D = P(\kappa)/I$. We shall find in V^D a Boolean algebra $\dot{E}$ such that $V^D \vDash \dot{E}$ is $\check{\lambda}$-saturated, and such that $D * \dot{E}$ is isomorphic to $B * \dot{C}$. Since D is λ-saturated, it will follow that $D * \dot{E}$ is λ-saturated and we shall be done.

In V^D, consider the generic ultrapower $N = \mathrm{Ult}_{\dot{G}}(V)$, where $\dot{G}$ is the canonical ultrafilter on $\check{D}$. Let $j : V \to N$ be the corresponding elementary embedding. Let $\dot{E} = j(B)$.

Let $\mathrm{sat}(B) = \nu < \kappa$. Since j is elementary, we have $N \vDash \mathrm{sat}(j(\check{B})) = j(\nu)$; and since $j(\nu) = \nu$ and by Lemma 22.31 all ν-sequences in V^D are in N, we have $V^D \vDash \mathrm{sat}(\dot{E}) = \nu$. Thus $V^D \vDash \dot{E}$ is λ-saturated.

It remains to show that $B * \dot{C}$ and $D * \dot{E}$ are isomorphic. Let $\dot{c} \in B * \dot{C}$. Then $\dot{c} \in V^B$ and $\|\dot{c} \in \dot{C}\|_B = 1$. Thus there is some $\dot{X} \in V^B$ such that $\|\dot{X} \subset \kappa\|_B = 1$ and that

$$\|\dot{c} \text{ is the equivalence class of } \dot{X} \text{ mod } \dot{J}\|_B = 1.$$

Let $f : \kappa \to B$ be the function $f(\alpha) = \|\alpha \in \dot{X}\|$. Since $f(\alpha) \in B$ for all $\alpha < \kappa$, f represents in $N = \mathrm{Ult}_G(B) \subset V^D$ an element $\dot{e} \in j(\check{B}) = \dot{E}$; and $\dot{e} \in D * \dot{E}$. We let $h(\dot{c}) = \dot{e}$.

The proof is completed by verifying that the definition of $h(\dot{c})$ does not depend on the choice of $\dot{X}$ and that h is an isomorphism. □

Consistency Strength of Precipitousness

Theorem 22.33.

(i) *If κ is a regular uncountable cardinal that carries a precipitous ideal, then κ is measurable in an inner model of* ZFC.

(ii) *If κ is a measurable cardinal, then there exists a generic extension in which $\kappa = \aleph_1$, and κ carries a precipitous ideal.*

The proof of (i) uses the technique of iterated ultrapowers (compare with (20.5)–(20.8)).

Let κ be a regular uncountable cardinal, and let I be a precipitous ideal on κ. Let C be the class of all strong limit cardinals $\nu > 2^\kappa$ such that $\mathrm{cf}\,\nu \geq \mathrm{sat}(I)$. Let $\gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots$ $(n < \omega)$, be elements of C such that $|\gamma_n \cap C| = \gamma_n$, let $A = \{\gamma_n : n = 0, 1, \ldots\}$ and let $\lambda = \sup(A)$.

Lemma 22.34. *There exists an $L[A]$-ultrafilter W on κ such that W is nonprincipal, κ-complete normal and iterable, and every iterated ultrapower $\mathrm{Ult}_W^{(\alpha)}(L[A])$ is well-founded.*

Proof. Since I is precipitous, the generic ultrapower is well-founded, and so the diagonal function $d(\alpha) = \alpha$ represents some ordinal number in $\mathrm{Ult}_G(V)$. Thus there is a set S of positive measure, and an ordinal γ such that

$$S \Vdash \check{d} \text{ represents } \check{\gamma} \text{ in } \mathrm{Ult}_G(V). \tag{22.21}$$

We shall first show that for every $X \in L[A]$, $X \subset \kappa$, either $S \cap X$ or $S - X$ has measure 0, and so

$$U = \{X \in P(\kappa) \cap L[A] : X \cap S \text{ has positive measure}\} \tag{22.22}$$

is an $L[A]$-ultrafilter.

Let $H = H^{L[A]}(\kappa \cup C \cup \{A\})$ be the class of all sets definable in $L[A]$ from elements of $\kappa \cup C \cup \{A\}$ (this is expressible in ZF similarly to the way in which ordinal definability is; or we can use $L_\theta[A]$ for some large θ). Since $|C \cap \gamma_n| = \gamma_n$ for each n, it follows that if π is the transitive collapse of H, then $\pi(A) = A$ and $\pi(H) = L[A]$. Now if $X \subset \kappa$ and $X \in L[A]$, then because π is the identity on κ, we have $X = \pi(Y) = Y \cap \kappa$ for some $Y \in H$, and $Y = \{\xi : L[A] \vDash \varphi(\xi, E, A)\}$. Thus for every $X \in P(\kappa) \cap L[A]$ there is a formula φ and a finite set $E \subset \kappa \cup K$ such that

$$X = \{\xi < \kappa : L[A] \vDash \varphi(\xi, E, A)\}. \tag{22.23}$$

We shall now show that (22.22) defines an $L[A]$-ultrafilter. Recall that for any generic ultrafilter G on κ, j_G is the identity on κ, and moreover, $j_G(\nu) = \nu$ for all $\nu \in C$ (this follows from the definition of C).

If $X \in L[A]$, and $X \cap S$ has positive measure, then because $X \Vdash \check{X} \in \dot{G}$ and because (22.21) holds, we have

$$X \cap S \Vdash \check{\gamma} \in j_G(\dot{X}).$$

Now using (22.23), and the fact that $j_G(A) = A$ and $j_G(E) = E$, we have

$$X \cap S \Vdash (L[\check{A}] \Vdash \varphi(\check{\gamma}, \check{E}, \check{A})). \tag{22.24}$$

But the formula forced by $X \cap S$ in (22.24) is about V, and thus true. Hence

$$\|\check{\gamma} \in j_G(\check{X})\| = 1,$$

and by (22.21),

$$S \Vdash \check{X} \in \dot{G}.$$

This, however, means that $S - X$ has measure 0.

Since I is κ-complete, it is clear that U is $L[A]$-κ-complete, and moreover the intersection of any countable family of elements of U is nonempty. It is less clear that U is iterable: Let $\langle X_\alpha : \alpha < \kappa \rangle \in L[A]$; it suffices to show that $S \Vdash (\{\alpha : X_\alpha \in U\} \in L[A])$. If G is generic such that $S \in G$, then $\{\alpha < \kappa : X_\alpha \in U\} = \{\alpha < \kappa : \gamma \in j_G(X_\alpha)\}$, but this is in $L[A]$ because $j_G(\langle X_\alpha : \alpha < \kappa \rangle) \in L[j_G(A)]$ and $j_G(A) = A$.

By Exercise 19.10, $\mathrm{Ult}_U(L[A])$ is well-founded; let $f : \kappa \to \kappa$ be the function that represents κ in $\mathrm{Ult}_U(L[A])$. Let $W = f_*(U)$.

It is easy to verify that W is a normal, $L[A]$-κ-complete, iterable $L[A]$-ultrafilter on κ, and that the intersection of any countable family of elements of W is nonempty. By Exercise 19.10, every iterated ultrapower $\mathrm{Ult}_W^{(\alpha)}(L[A])$ is well-founded. □

Proof of Theorem 22.33(i). Let $A = \{\gamma_n : n = 0, 1, \ldots\}$ be as above, let $\lambda = \sup(A)$, and let W be an $L[A]$-ultrafilter as in Lemma 22.34. Let us define in $L[A]$

$$F = \{X \subset \lambda : \exists n_0\ \forall n \geq n_0\ \gamma_n \in X\} \tag{22.25}$$

(compare with (20.6)). We claim that $D = F \cap L[F]$ is a normal measure on λ in $L[D]$.

For each α, let $i_{0,\alpha} : L[A] \to \mathrm{Ult}_W^{(\alpha)}(L[A])$ be the canonical elementary embedding. It follows from the definition of the class C that:

(22.26)
$$\begin{aligned} &\text{if } \alpha < \gamma_n, \text{ then } i_{0,\alpha}(\gamma_n) = \gamma_n; \\ &i_{0,\gamma_n}(\kappa) = \gamma_n; \\ &\text{if } \alpha < \lambda, \text{ then } i_{0,\alpha}(\lambda) = \lambda. \end{aligned}$$

Hence for all $\alpha < \lambda$, $i_{0,\alpha}(L[A]) = L[A]$, $i_{0,\alpha}(F) = F$, and $i_{0,\alpha}(D) = D$.

We shall now prove that D is an ultrafilter in $L[D]$. Otherwise, let $X \subset \lambda$ be the least X (in the canonical well-ordering of $L[D]$) such that $X \notin D$ and $\lambda - X \notin D$. Since $i_{0,\alpha}(D) = D$ for all $\alpha < \lambda$, we have $i_{0,\alpha}(X) = X$ for all $\alpha < \lambda$; in particular, $i_{0,\gamma_n}(X) = X$ for all n. Now for any n, if $\gamma_n \in X$, then $i_{0,\gamma_n}(\kappa) \in i_{0,\gamma_n}(X)$ and hence $\kappa \in X$, and vice versa. Hence either all γ_n are in X or none, and so either $X \in F$ or $\lambda - X \in F$, a contradiction.

The proof that D is λ-complete (in $L[D]$) and normal is similar and is left to the reader.

Thus we have proved that there exists a D in $L[A]$ such that

(22.27)
$$L[D] \vDash D \text{ is a normal measure on } \lambda.$$

The proof will be complete if we find a transitive model M and an elementary embedding $i : M \to L[D]$ such that $i(\kappa) = \lambda$. Then κ is measurable in some transitive model.

Let us recall that for each α, $i_{0,\alpha}$ is the elementary embedding $i_{0,\alpha} : L[A] \to \mathrm{Ult}_W^{(\alpha)}(L[A])$. As we have seen, if $\alpha < \lambda$, then $i_{0,\alpha}(\lambda) = \lambda$ and $i_{0,\alpha}(L[A]) = L[A]$. Let C_1 be a proper class of ordinals, greater than λ such that $i_{0,\lambda}(\nu) = \nu$ for all $\nu \in C_1$.

Let $H = H^{L[D]}(\kappa \cup \{\lambda\} \cup C_1)$ be the class of all sets definable in $L[D]$ from elements of $\kappa \cup \{\lambda\} \cup C_1$. (As before, the problem of expressibility of H in ZF can be overcome by replacing $L[D]$ by a suitable large segment $L_\theta[D]$.) H is an elementary submodel of $L[D]$.

If $\alpha < \lambda$, then $i_{0,\alpha}(\nu) = \nu$ for all $\nu \in \kappa \cup \{\lambda\} \cup C_1$; it follows that $i_{0,\alpha}(x) = x$ for all $x \in H$. Observing that for every ν such that $\kappa \le \nu < \lambda$ there exists $\alpha < \lambda$ such that $i_{0,\alpha}(\nu) > \nu$, we conclude that H contains no ordinal ν such that $\kappa \le \nu < \lambda$. Hence if π is the transitive collapse of H, and $M = \pi(H)$, then $\pi(\lambda) = \kappa$; thus $i = \pi^{-1}$ is an elementary embedding of some transitive model M into $L[D]$, and $i(\kappa) = \lambda$. □

The proof of (ii) uses the notion of forcing which collapses all $\alpha < \kappa$ onto ω and makes $\kappa = \aleph_1$ (the Lévy collapse).

Proof of (ii). Let κ be a measurable cardinal. We shall show that if $V[G]$ is the generic extension by the Lévy collapse such that κ becomes $\aleph_1$, then $V[G]$ has a precipitous ideal on $\aleph_1$.

Let P be the set of all functions p such that $\operatorname{dom}(p)$ is a finite subset of $\kappa \times \omega$ and such that $p(\alpha, n) < \alpha$ for all $(\alpha, n) \in \operatorname{dom}(p)$; p is stronger than q if $p \supset q$. Let G be a V-generic filter on P. In $V[G]$, κ is $\aleph_1$.

Let D be a normal measure on κ, let $M = \operatorname{Ult}_D(V)$ and let $j : V \to M$ be the elementary embedding $j = j_D$. In $V[G]$, let I be the ideal on κ generated by the dual of D; i.e.,

$$X \in I \quad \text{if and only if} \quad X \cap Y = \emptyset \text{ for some } Y \in D. \tag{22.28}$$

A routine argument (using $\operatorname{sat} P = \kappa$) shows that I is in $V[G]$ a countably complete ideal containing all singletons. It can be proved that I is precipitous; instead, we shall prove a weaker (but sufficient) property, namely that there exists an $S \subset \aleph_1$, $S \notin I$, such that $I \restriction S = \{X \subset \aleph_1 : X \cap S \in I\}$ is a precipitous ideal.

For that, it suffices to show that there exists an $S \notin I$ such that (when forcing with sets $X \notin I$) S forces that the generic ultrapower is well-founded. In turn, it suffices to construct an extension of $V[G]$ in which there exists a $V[G]$-ultrafilter W on κ, generic over $V[G]$ (with respect to forcing with sets $X \notin I$) such that the generic ultrapower $\operatorname{Ult}_W(V[G])$ is well-founded.

For every ν, let P_ν be the set of all $p \in P$ such that $\alpha < \nu$ whenever $(\alpha, n) \in \operatorname{dom}(p)$, and let $P^\nu = \{p \in P : \alpha \geq \nu \text{ for all } (\alpha, n) \in \operatorname{dom}(p)\}$; P is isomorphic to the product $P_\nu \times P^\nu$.

Let us consider the notion of forcing $j(P)$. Clearly, $(j(P))_\kappa = P$, and thus $j(P)$ is isomorphic to $P \times Q$ where $Q = (j(P))^\kappa$. Every $q \in Q$ is represented in the ultrapower M by a function $\langle q_\alpha : \alpha < \kappa\rangle$ such that $q_\alpha \in P^\alpha$ for all $\alpha < \kappa$.

Let H be a $V[G]$-generic filter on Q; thus $G \times H$ is V-generic on $P \times Q$. As in Theorem 21.3 we define in $V[G \times H]$ a $V[G]$-ultrafilter W on κ as follows:

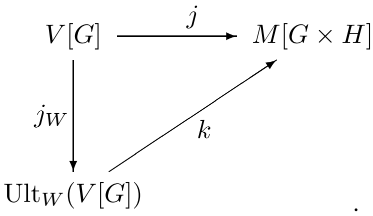
$$X \in W \quad \text{if and only if} \quad \kappa \in (j(\dot{X}))^{G \times H}. \tag{22.29}$$

The definition (22.29) does not depend on the choice of the name $\dot{X}$ because $p \in G$ implies $j(p) \in G \times H$. Let $\dot{W}$ be the canonical name for W. As in (21.8) we have for any $p \in P$, $q \in Q$,

$$(p, q) \Vdash \dot{X} \in \dot{W} \quad \text{if and only if} \quad \text{for almost all } \alpha,\ p \cup q_\alpha \Vdash \check{\alpha} \in \dot{X} \tag{22.30}$$

(here $\dot{X}$ is a P-valued name and $\langle q_\alpha : \alpha < \kappa\rangle$ represents q in M; "almost all" refers to the normal measure D).

First we observe that the ultrapower $\mathrm{Ult}_W(V[G])$ is well-founded. This is because the following commutative diagram holds:



In the diagram, j is the extension of $j : V \to M$ defined by

$$j(x) = (j(\dot{x}))^{G\times H}$$

and k is defined as follows: If $f \in V[G]$ is a function on κ representing $[f]$ in $\mathrm{Ult}_W(V[G])$, then

$$k([f]) = (j(f))(\kappa).$$

Both j and k are elementary and the diagram commutes.

It remains to show that W is $V[G]$-generic with respect to forcing with sets $X \notin I$. It suffices to show that if $\mathcal{X} = \{X_i : i < \theta\}$ is an I-partition of κ, then $X_i \in W$ for some i. Let $\dot{\mathcal{X}} \in V^P$ be a name for $\mathcal{X}$ and let $\dot{X}_i$, $i < \theta$, be names for the X_i. Let us assume that there are conditions $p \in G$ and $q \in H$ such that

$$p \Vdash \dot{\mathcal{X}} \text{ is an } \dot{I}\text{-partition of } \check{\kappa}$$

and for each $i < \theta$,

$$(p, q) \Vdash \dot{X}_i \notin \dot{W}.$$

We shall derive a contradiction.

Let q be represented in M by $\langle q_\alpha : \alpha < \kappa\rangle$. By (22.30) there is for each i a set $A_i \in D$ such that for all $\alpha \in A_i$,

(22.31) $$p \cup q_\alpha \Vdash \alpha \notin \dot{X}_i.$$

Let us define (in $V[G]$),

(22.32) $$T = \{\alpha < \kappa : q_\alpha \in G\}.$$

We shall prove that $T \notin I$ and that $T \cap X_i \in I$ for all $i < \theta$, thus reaching a contradiction since $\mathcal{X}$ is an I-partition.

For each $i < \theta$, if $\alpha \in T$ and $\alpha \in A_i$, then $p \cup q_\alpha \in G$ and hence, by (22.31), $\alpha \notin X_i$. It follows that $T \cap X_i \cap A_i = \emptyset$, and so by (22.28), $T \cap X_i \in I$.

Let us finally show that $T \notin I$. It suffices to show that $T \cap Z \neq \emptyset$ whenever $Z \in D$. Thus let $Z \in D$, and let us prove that $q_\alpha \in G$ for some $\alpha \in Z$. Let

$$E = \{r \in P : r \leq q_\alpha \text{ for some } \alpha \in Z\}.$$

It is easy to see that E is dense in P because Z is unbounded and $q_\alpha \in P^\alpha$ for each $\alpha < \kappa$. Thus $E \cap G \neq \emptyset$ and hence $T \cap Z \neq \emptyset$. □

Exercises

22.1. Let I be a κ-complete ideal and let $\lambda \le \kappa$. If I is not λ-saturated then there exists a family $\{Z_\alpha : \alpha < \lambda\}$ of pairwise disjoint sets of positive I-measure.

[If $\{X_\alpha : \alpha < \lambda\}$ is such that $X_\alpha \cap X_\beta \in I$ whenever $\alpha \ne \beta$, let $Z_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$.]

22.2. Let I be a κ-complete σ-saturated ideal on κ. If g is a minimal unbounded function then $g_*(I)$ is a normal κ-complete σ-saturated ideal.

22.3. Let $\dot{x}_\alpha$ be as in (22.5). Show that $x_\alpha \notin M$.

[Let $a \in M$. Show that for each k, $\|\dot{x}_\alpha \cap k = \check{a} \cap k\| = [D_k]$ where $\mu(D_k) = 1/2^k$.]

22.4. Let I be a σ-saturated κ-complete ideal on κ, and let $\{Y_\xi : \xi < \omega_1\}$ be a family of sets of positive measure. Then there is an uncountable $W \subset \omega_1$ such that $\bigcap_{\xi \in W} Y_\xi$ is nonempty.

[Assume that $\{Y_\xi : \xi < \omega_1\}$ is a counterexample. For each $\nu < \omega_1$, let $Z_\nu = \bigcap_{\xi \ge \nu} (\kappa - Y_\xi)$. Show that $Z_0 \subset Z_1 \subset \ldots \subset Z_\nu \subset \ldots$ and that $\bigcup_{\nu<\omega_1} Z_\nu = \kappa$. Hence there is $\gamma < \omega_1$ such that $Z = \bigcup_{\nu<\gamma} Z_\nu$, then $\kappa - Z \in I$. This is a contradiction since $Y_\gamma \subset \kappa - Z$.]

22.5. If I is a σ-saturated κ-complete ideal on κ and P is a σ-saturated notion of forcing then in $V[G]$, the ideal generated by I is a σ-saturated κ-complete ideal.

[Proceed as in Lemma 22.11 and use Exercise 22.4 to show that J is σ-saturated.]

If κ is measurable and if we adjoin $\lambda \ge \kappa$ Cohen reals, then κ carries a σ-saturated κ-complete ideal but is not real-valued measurable:

22.6. Show that in $V[G]$ there are functions $f_\alpha : \omega \to \omega$, $\alpha < \lambda$, such that whenever $g : \omega \to \omega$, then for at most countably many α's we have $f_\alpha(n) \le g(n)$ for all n.

[$V[G]$ is also obtained by forcing with the product of λ copies of the notion of forcing that adjoins a generic function $f : \omega \to \omega$, thus $V[G] = V[\langle f_\alpha : \alpha < \lambda\rangle]$. Show that if $g : \omega \to \omega$, then there is a countable $A \subset \lambda$ such that $g \in V[\langle f_\alpha : \alpha < A\rangle]$; if $\beta \notin A$, use the genericity of f_β over $V[\langle f_\alpha : \alpha < A\rangle]$ to show that $f_\beta(n) > g(n)$ for some n.]

22.7. In $V[G]$, κ is not real-valued measurable.

[Use Exercise 22.6 and the proof of Lemma 10.16.]

22.8. If I is κ^+-saturated, then $P(\kappa)/I$ is a complete Boolean algebra.

[By Exercise 7.33 it suffices to show that B is κ^+-complete. Show that $\sum W$ exists in B for every incompatible $W \subset B$. Extend W to a partition Z of B; $Z = \{[X_\alpha] : \alpha < \kappa\}$. Let $Y_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$, and $Y = \bigcup\{Y_\alpha : [X_\alpha] \in W\}$. Show that $[Y] = \sum W$ in B.]

22.9. If the GCH holds and $B = P(\kappa)/I$ is complete, then I is κ^+-saturated.

[If B is not κ^+-saturated, let W be an incompatible subset of B of size κ^+. For each $X \subset W$ let $u_x = \sum X$. It follows that $|B| \ge 2^{\kappa^+}$, but clearly $|B| \le 2^\kappa$; a contradiction.]

22.10. If I is normal, then $P(\kappa)/I$ is κ^+-complete.

[Let X_α, $\alpha < \kappa$, be disjoint subsets of κ such that $X_\alpha \notin I$ for all α. For each $\alpha < \kappa$ let Y_α be X_α without the least element of X_α; let $Y = \bigcup_{\alpha<\kappa} Y_\alpha$. On the one hand, $[Y] \ge [X_\alpha]$ for all α; on the other hand, if $Z \subset Y$ and $Z \notin I$, let f be the function on Z defined such that for all $x \in Y_\alpha$, $f(x) =$ the least element of X_α. Since f is regressive, and I is normal, f is constant on some $S \notin I$, and hence $Z \cap Y_\alpha \notin I$ for some α. Thus $[Y] = \sum_{\alpha<\kappa}[X_\alpha]$.]

22.11. Let I be a normal κ-complete ideal on κ. If I is not κ^+-saturated, then there exists an almost disjoint family of κ^+ sets of positive measure.

[Let X_i, $i < \kappa^+$, be sets of positive measure such that $X_i \cap X_j$ has measure zero. For each $i < \kappa^+$, enumerate $\{X_j : j < i\}$ by $\{Z_\alpha : \alpha < \kappa\}$, and let Y_i be the diagonal intersection of $\{X_i - Z_\alpha : \alpha < \kappa\}$. Now Y_i contains almost all elements of X_i, and $Y_i \cap Z_\alpha \subset \alpha + 1$ for every $\alpha < \kappa$. Thus any Y_i, Y_j are almost disjoint.]

22.12. If I is a κ-complete ideal on κ with the property that every regressive function is bounded almost everywhere (i.e., if $f(\alpha) < \alpha$ for almost all α, then there is $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for almost all α), then I is κ-saturated (and normal).

[Otherwise, let X_α, $\alpha < \kappa$, be a partition of κ into disjoint sets of positive measure. For $\alpha > 0$, let $Y_\alpha = X_\alpha - \{a_\alpha\}$ where $a_\alpha = \min X_\alpha$, and let $Y_0 = X_0 \cup \{a_\alpha : \alpha > 0\}$. The function f that has value a_α on each Y_α is regressive almost everywhere but is not bounded almost everywhere.]

22.13. If I is an atomless κ-complete κ-saturated ideal on an inaccessible cardinal κ, then κ is not weakly compact.

[Show that κ does not have the tree property. Use I to construct a tree $(T, \supset)$ whose elements are sets of positive measure. At successor steps, split each X on the top level into two disjoint sets of positive measure. At limit steps, take all those intersections along branches that have positive measure. Since I is κ-saturated, each level has size $< \kappa$; each level $\alpha < \kappa$ is nonempty because κ is inaccessible and I is κ-complete. Then use $\operatorname{sat}(I) \le \kappa$ to show that T has no branch of length κ.]

22.14. If I is a precipitous ideal on κ, then there exists a minimal unbounded function.

[There is a set X of positive measure and a function f on X such that X forces that f represents κ in the generic ultrapower.]

Historical Notes

Saturated ideals, a concept introduced by Tarski in [1945], were brought to prominence in Solovay's work [1971]. Solovay introduced the technique of generic ultrapowers and proved Theorems 22.1 and 22.26 (as well as Theorem 8.10).

Theorem 22.2 is due to Prikry [1975], and so is the model in Example 22.10 in which κ carries a σ-saturated ideal [1970]. Theorem 22.16 is due to Jech and Prikry ([1976] and [1979]).

Precipitous ideals were introduced by Jech and Prikry in [1976]; their paper [1979] investigates generic ultrapowers. Lemma 22.21: Jech [1976]. Lemma 22.28: Solovay [1971]. Lemmas 22.29 and 22.30: Jech and Prikry [1976, 1979].

Kunen's paper [1978] contains a number of results on saturated ideals. Kunen constructs several generic extensions with saturated ideals, including a model (using a huge cardinal) in which $\aleph_1$ carries an $\aleph_2$-saturated ideal. In [1970], Kunen proves that if κ carries a κ^+-saturated ideal then there is an inner model with a measurable cardinal; in [1971a] Kunen shows that if moreover κ is a successor cardinal then there is an inner model with many measurable cardinals. Mitchell [1983] improved this to measurable cardinals of order κ^+. Part (i) of Theorem 22.33 is due to Jech and Prikry and part (ii) was proved by Mitchell; see Jech et al. [1980].

Exercise 22.4: Silver.

Exercises 22.5, 22.6 and 22.7: Prikry [1970].

Exercise 22.8: Smith and Tarski.

Exercise 22.9: Solovay [1971].
Exercise 22.11: Baumgartner, Hajnal, and Máté [1975].
Exercise 22.12: Kanamori [1976]
Exercise 22.13: Lévy, Silver.