23. The Nonstationary Ideal

Stationary sets play a fundamental role in modern set theory. In particular, the analysis of the nonstationary ideal I_{NS} on ω_1 has been used in the study of forcing axioms, large cardinals and determinacy. These will be dealt with in later chapters; this chapter continues the investigations began in Chapters 8 and 22. Throughout this chapter "almost all" means all except nonstationary many.

Some Combinatorial Principles

We begin with combinatorial principles that involve stationary sets. Let us recall Jensen's Principle ($\diamondsuit$): There exist sets $S_\alpha \subset \alpha$ such that for every $X \subset \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is stationary. There are several variants of $\diamondsuit$ (see e.g. Exercise 15.25); most notably the following weak version:

Lemma 23.1. *The following principle is equivalent to $\diamondsuit$: There exists a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of countable sets such that for each $X \subset \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha \in S_\alpha\}$ is stationary.*

Proof. Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a sequence as in the lemma; we shall produce a diamond sequence. First, let f be a one-to-one mapping of ω_1 onto $\omega_1 \times \omega$ such that $f``\alpha = \alpha \times \omega$ for all limit ordinals $\alpha$. For every limit ordinal $\alpha$, let $A_\alpha = \{f``x : x \in S_\alpha\}$ (and $A_\alpha = \emptyset$ otherwise). Note that for each $Y \subset \omega_1 \times \omega$ the set $\{\alpha < \omega_1 : Y \cap (\alpha \times \omega) \in A_\alpha\}$ is stationary.

For each α, let $A_\alpha = \{a_\alpha^n : n \in \omega\}$. It follows that for each $X \subset \omega_1 \times \omega$ there exists some n such that the set $\{\alpha : X \cap (\alpha \times \omega) = a_\alpha^n\}$ is stationary.

For each $\alpha < \omega_1$ and each n, let $S_\alpha^n = \{\xi < \alpha : (\xi, n) \in a_\alpha^n\}$. We complete the proof by showing that for some n, $\langle S_\alpha^n : \alpha < \omega_1\rangle$ is a diamond sequence. If not, there exist sets $X_n \subset \omega_1$ such that $\{\alpha < \omega_1 : X_n \cap \alpha = S_\alpha^n\}$ are nonstationary. Letting $X = \bigcup_{n\in\omega}(X_n \times \{n\})$, it follows that for each n, $\{\alpha < \omega_1 : X \cap (\alpha \times \omega) \neq a_\alpha^n\}$ is nonstationary; a contradiction. $\square$

The Diamond Principle admits a generalization from ω_1 to any regular cardinal κ. Even more generally, let E be a stationary subset of a regular cardinal κ. $\diamondsuit(E)$ is the following principle (and $\diamondsuit_\kappa$ is $\diamondsuit(\kappa)$):

(23.1) There exists a sequence of sets $\langle S_\alpha : \alpha \in E\rangle$ with $S_\alpha \subset \alpha$ such that for every $X \subset \kappa$, the set $\{\alpha \in E : X \cap \alpha = S_\alpha\}$ is a stationary subset of κ.

The proof of Theorem 13.21 generalizes to show that if $V = L$, then $\diamondsuit(E)$ holds for any regular cardinal κ and any stationary set $E \subset \kappa$.

For a successor cardinal κ^+ and a stationary subset E, consider the following:

(23.2) There exists a sequence of sets $\langle S_\alpha : \alpha \in E\rangle$ such that $|S_\alpha| \le \kappa$ for each α, and for every $X \subset \kappa^+$, the set $\{\alpha \in E : X \cap \alpha \in S_\alpha\}$ is stationary.

The proof of Lemma 23.1 generalizes and shows that (23.2) is equivalent to $\diamondsuit(E)$.

While the Diamond Principle holds in L, as well as in $L[U]$ and other inner models for large cardinals, restrictions of $\diamondsuit$ to various stationary sets can be proved just from assumptions on cardinal arithmetic. Let $\lambda < \kappa^+$ be a regular cardinal, and recall (8.4) that $E^{\kappa^+}_\lambda$ is the set of all ordinals $\alpha < \kappa^+$ of cofinality λ.

Theorem 23.2 (Gregory). *If λ is regular such that $\kappa^\lambda = \kappa$ and if $2^\kappa = \kappa^+$, then $\diamondsuit(E^{\kappa^+}_\lambda)$ holds.*

In particular, if $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ then $\diamondsuit(E^{\aleph_2}_{\aleph_0})$ holds.

Proof. We prove the version of $\diamondsuit(E)$ from (23.2) where $E = E^{\kappa^+}_\lambda$; by Lemma 23.1, $\diamondsuit(E)$ follows. Let $\langle x_\alpha : \alpha < \kappa^+\rangle$ enumerate all bounded subsets of κ^+ (this is possible by $2^\kappa = \kappa^+$). For each $\alpha \in E$, we let S_α be the set of all $Y \subset \alpha$ such that Y is the union of at most λ elements of the set $\{x_\beta : \beta < \alpha\}$. Since $\kappa^\lambda = \kappa$, we have $|S_\alpha| \le \kappa$.

We claim that $\langle S_\alpha : \alpha \in E\rangle$ satisfies (23.2). Let $X \subset \kappa^+$; we will show that $X \cap \alpha \in S_\alpha$ for almost all $\alpha \in E$. Let C be the set of all $\alpha < \kappa^+$ such that for every $\beta < \alpha$, $X \cap \beta = x_\gamma$ for some $\gamma < \alpha$. The set C is closed unbounded.

We claim that if $\alpha \in C \cap E$ then $X \cap \alpha \in S_\alpha$. Let $Z \subset \alpha$ be a set cofinal in α such that $|Z| = \lambda$. If for each $\beta \in Z$, $\gamma(\beta) < \alpha$ is such that $X \cap \beta = x_{\gamma(\beta)}$, then $X \cap \alpha = \bigcup\{x_{\gamma(\beta)} : \beta \in Z\}$, and hence $X \cap \alpha \in S_\alpha$. □

A property related to $\diamondsuit$ is *club-guessing*. This has been introduced and investigated in detail by Shelah. Let κ be a regular uncountable cardinal and let E be a stationary subset of κ. If $C \subset \kappa$ is closed unbounded and if each c_α, $\alpha \in E$, is cofinal in α, we say that $\langle c_\alpha : \alpha \in E\rangle$ *guesses* C if for all $\alpha \in E$, C contains an end segment of c_α, i.e., $C \supset c_\alpha - \beta$ for some $\beta < \alpha$.

Theorem 23.3 (Shelah). *Let $\kappa \ge \aleph_3$ be a regular uncountable cardinal, and let λ be a regular uncountable cardinal such that $\lambda^+ < \kappa$. Then there exists a sequence $\langle c_\alpha : \alpha \in E^\kappa_\lambda\rangle$ with each $c_\alpha \subset \alpha$ closed unbounded, such that for every closed unbounded set $C \subset \kappa$, the set $\{\alpha \in E^\kappa_\lambda : c_\alpha \subset C\}$ is stationary.*

Proof. It suffices to find a family $\{c_\alpha : \alpha \in E^\kappa_\lambda\}$ such that each c_α is a closed subset of α, and for every closed unbounded $C \subset \kappa$, the set $\{\alpha \in E^\kappa_\lambda : c_\alpha$ is unbounded in α and $c_\alpha \subset C\}$ is stationary.

Assume that no such $\{c_\alpha : \alpha \in E^\kappa_\lambda\}$ exists. Let $\{c^0_\alpha : \alpha \in E^\kappa_\lambda\}$ be any collection of closed unbounded subsets of the α's of order-type λ. By induction on $\nu < \lambda^+$, we construct closed unbounded sets $C_\nu \subset \kappa$ and collections $\{c^\nu_\alpha : \alpha \in E^\kappa_\lambda\}$ as follows: $c^\nu_\alpha = c^0_\alpha \cap \bigcap_{\xi<\nu} C_\xi$, and C_ν is such that the set $\{\alpha \in E^\kappa_\lambda : c^\nu_\alpha$ is unbounded in α and $c^\nu_\alpha \subset C_\nu\}$ is nonstationary.

Let C be the closed unbounded set $C = \bigcap_{\nu<\lambda^+} C_\nu$, and for each α let $c_\alpha = c^0_\alpha \cap C$. The set $S = \{\alpha \in E^\kappa_\lambda : C \cap \alpha$ is unbounded in $\alpha\}$ is stationary, and for each $\alpha \in S$ there exists a $\nu(\alpha) < \lambda^+$ such that $c_\alpha = c^{\nu(\alpha)}_\alpha$ (because $c^0_\alpha \supset c^1_\alpha \supset \ldots$ of length λ^+).

There exist a $\nu < \lambda^+$ and a stationary set $T \subset S$ such that $c_\alpha = c^\nu_\alpha$ for all $\alpha \in T$. If $\alpha \in T$ then $c^\nu_\alpha = c^{\nu+1}_\alpha = c^\nu_\alpha \cap C_\nu$, and so $c^\nu_\alpha \subset C_\nu$, contrary to the choice of C_ν. $\square$

The sequence $\langle c_\alpha : \alpha \in E^\kappa_\lambda\rangle$ guesses every closed unbounded set at stationary many α's. The same proof shows that for every stationary $E \subset E^\kappa_\lambda$ there exists a sequence $\langle c_\alpha : \alpha \in E\rangle$ that guesses every closed unbounded set at stationary many $\alpha \in E$ (Exercise 23.1). We state, without proof, a further refinement that will be used later in this chapter in the proof of the Gitik-Shelah Theorem 23.17.

Lemma 23.4. *Let κ and λ be regular uncountable cardinals such that $\lambda^+ < \kappa$. For every stationary set $E \subset E^\kappa_\lambda$ there exists a sequence $\langle c_\alpha : \alpha \in E\rangle$ with each $c_\alpha \subset \alpha$ closed unbounded, such that for every closed unbounded $C \subset \kappa$, the set $\{\alpha \in E : c_\alpha \in C\}$ is stationary, and moreover,*

(23.3) *if $\alpha \in E$ is a limit of ordinals of cofinality greater than λ, then all nonlimit elements of c_α have cofinality greater than λ.*

Proof. For proof, see Gitik and Shelah [1997]. $\square$

This cannot be improved much further; see Exercise 23.2.

One of the most fundamental combinatorial principles is Jensen's *Square Principle.* Let κ be an uncountable cardinal; $\square_\kappa$ (*square-kappa*) is as follows:

(23.4) ($\square_\kappa$) There exists a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that
(i) C_α is a closed unbounded subset of α;
(ii) if $\beta \in \mathrm{Lim}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$;
(iii) if $\mathrm{cf}\,\alpha < \kappa$ then $|C_\alpha| < \kappa$.

The sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ is called a *square-sequence.* Note that by (ii) and (iii), the order-type of every C_α is at most κ.

Using the fine structure theory of L, Jensen proved that in L, $\square_\kappa$ holds for every uncountable cardinal κ. (We elaborate on this in Part III). This has been extended to most inner models for large cardinals: the Square Principles hold in $L[U]$, $L[\mathcal{U}]$, an in more general inner models.

Squares are relatively easy to obtain by forcing; as an example, see Exercise 23.3.

Definition 23.5. Let κ be a regular uncountable cardinal and let $\alpha < \kappa$ be a limit ordinal of uncountable cofinality. We say that a stationary set $S \subset \kappa$ *reflects* at α if $S \cap \alpha$ is a stationary subset of α.

Corollary 17.20 states that if κ is a weakly compact cardinal then every stationary subset of κ reflects. We address the subject of reflection of stationary sets later in this chapter. See also Exercises 23.4 and 23.5. In general, squares provide examples of nonreflecting stationary sets:

Lemma 23.6. *$\square_{\omega_1}$ implies that there exists a stationary set $S \subset E^{\aleph_2}_{\aleph_0}$ that does not reflect.*

Proof. Let $\langle C_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ be a square-sequence. For each $\alpha < \omega_2$ of cofinality ω_1, the order-type of C_α is ω_1. It follows that there exists a countable limit ordinal η such that the set $S = \{\gamma \in E^{\aleph_2}_{\aleph_0} : \gamma \text{ is the } \eta\text{th element of some } C_\alpha\}$ is stationary. But for every α of cofinality ω_1, S has at most one element in common with C_α. Hence S does not reflect. $\square$

Stationary Sets in Generic Extensions

If κ is a regular uncountable cardinal then a closed unbounded subset of κ in the ground model remains a closed unbounded subset of κ in a generic extensions (but κ may fail to remain a cardinal or its cofinality may change). It follows that if $S \in V$ is stationary in $V[G]$ then S is stationary in V. A stationary set in V may, however, be no longer stationary in $V[G]$, as there may exist a new closed unbounded set in $V[G]$ that is disjoint from it.

We recall Lemma 22.25 that states that if the forcing satisfies the κ-chain condition then every stationary subset of κ in V is preserved; i.e., remains stationary in $V[G]$. Another condition on preservation of stationary sets is the following:

Lemma 23.7. *Let κ be a regular uncountable cardinal and let P be a notion of forcing. If P is $<\kappa$-closed then every stationary $S \subset \kappa$ remains stationary in $V[G]$.*

Proof. Let $p \Vdash \dot{C}$ is closed unbounded; we find a $\gamma \in S$ and a $q \le p$ such that $q \Vdash \gamma \in \dot{C}$ as follows: We construct an increasing continuous ordinal sequence $\langle \gamma_\alpha : \alpha < \kappa\rangle$ and a decreasing sequence $\langle p_\alpha : \alpha < \kappa\rangle$ of conditions such that $p_{\alpha+1} \Vdash \gamma_{\alpha+1} \in \dot{C}$. If α is a limit ordinal then $\gamma_\alpha = \lim_{\xi<\alpha} \gamma_\xi$ and p_α is a lower bound of $\{p_\xi : \xi < \alpha\}$. There exists a limit ordinal α such that $\gamma_\alpha \in S$. It follows that $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$. $\square$

The basic method for destroying stationary sets by forcing is the following forcing known as "shooting a closed unbounded set."

Theorem 23.8. *Let S be a stationary subset of ω_1. There is a notion of forcing P_S that adds generically a closed unbounded set $C \subset \omega_1$ such that $C \subset S$, and such that P_S adds no new countable sets.*

Since P_S adds no countable sets, $\aleph_1$ is preserved. The set $\omega_1 - S$ is nonstationary in $V[G]$; thus if S is chosen so that its complement is stationary, the forcing destroys some stationary set.

Proof. P_S consists of all bounded closed sets of ordinals p such that $p \subset S$; p is stronger than q if p is an end-extension of q (if $q = p \cap \alpha$ for some α).

If G is a generic filter, let $C = \bigcup G$. Clearly, C is a subset of S, and because for every $\alpha < \omega_1$ the set $\{p \in P : \max(p) \geq \alpha\}$ is dense in P_S, C is an unbounded subset of ω_1. Also, $\sup(C \cap \alpha) \in C$ holds for every $\alpha < \omega_1$, and so C is a closed unbounded set. It remains to prove that $\aleph_1$ is preserved and that there are no new countable sets of ordinals.

Lemma 23.9. *P_S is ω-distributive.*

Proof. Let $p \Vdash \dot{f} : \omega \to Ord$; we shall find a $q \leq p$ and some f so that $q \Vdash \dot{f} = f$.

By induction on α we construct a chain $\{A_\alpha : \alpha < \omega_1\}$ of countable subsets of P_S. Let $A_0 = \{p\}$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if α is a limit ordinal. Given A_α, let $\gamma_\alpha = \sup\{\max(q) : q \in A_\alpha\}$. For each $q \in A_\alpha$ and each n, we choose some $r = r(q,n) \in P_S$ so that $r \leq q$, r decides $\dot{f}(n)$, and $\max(r) > \gamma_\alpha$. Then we let $A_{\alpha+1} = A_\alpha \cup \{r(q,n) : q \in A_\alpha,\ n < \omega\}$.

The sequence $\langle \gamma_\alpha : \alpha < \omega_1 \rangle$ is increasing and continuous. Let $C = \{\lambda :$ if $\alpha < \lambda$ then $\gamma_\alpha < \lambda\}$. As C is closed unbounded, there exists a limit ordinal λ such that $\lambda \in C \cap S$. Let $\langle \alpha_n : n < \omega \rangle$ be an increasing sequence with limit λ; then $\lim_n \gamma_{\alpha_n} = \lambda$ as well.

There is a sequence of conditions $\langle p_n : n < \omega \rangle$ such that $p_0 = p$ and that for every n, $p_{n+1} \in A_{\alpha_{n+1}}$, $p_{n+1} \leq p_n$, and p_{n+1} decides $\dot{f}(n)$. Since $\gamma_{\alpha_n} < \max(p_{n+1}) \leq \gamma_{\alpha_{n+1}}$, we have $\lim_n \max(p_n) = \lambda$, and because $\lambda \in S$, the closed set $q \in \bigcup_{n=0}^{\infty} p_n \cup \{\lambda\}$ is a condition. Since $q \leq p_n$ for all n, q decides each $\dot{f}(n)$, and so there exists some f such that $q \Vdash \dot{f} = f$. $\square$

The forcing P_S can be generalized for cardinals κ greater than $\aleph_1$ but additional assumptions on S must be made in order to preserve κ. See, e.g., Exercises 23.7 and 23.8.

Precipitousness of the Nonstationary Ideal

In Theorem 22.33(ii) we showed that if κ is a measurable cardinal and P is the Lévy collapse (with finite conditions, making $\kappa = \aleph_1$) then in $V[G]$, there exists a precipitous ideal on $\aleph_1$. We now improve this to making the nonstationary ideal precipitous:

Theorem 23.10 (Magidor). *It is consistent, relative to the existence of a measurable cardinal, that the nonstationary ideal on $\aleph_1$ is precipitous.*

Proof. Let κ be a measurable cardinal, and let us assume that $2^\kappa = \kappa^+$. Let U be a normal measure on κ. Let M be the ultrapower $M = \mathrm{Ult}_U(V)$ with $j : V \to M$ the canonical embedding.

Let P be the Lévy collapse: a condition $p \in P$ is a finite function with $\mathrm{dom}(p) \subset \kappa \times \omega$ such that $p(\alpha, n) < \alpha$ for every $(\alpha, n) \in \mathrm{dom}(p)$.

Let G be a P-generic filter. In $V[G]$ (where $\kappa = \aleph_1$) let I_0 be the ideal generated by the dual of U: $I_0 = \{X \subset \kappa : X \cap Y = \emptyset \text{ for some } Y \in U\}$. The proof of Theorem 22.33(ii) shows that in $V[G]$, for every $X \subset \kappa$, $X \notin I_0$, there exists an I_0-generic ultrafilter D_0 with $X \in D_0$ such that $\mathrm{Ult}_{D_0}(V[G])$ is well-founded. Let $G \times H$ be $j(P)$-generic that contains some condition (p, q) with $(p, q) \Vdash_{j(P)} \kappa \in j(\dot{X})$; then $j : V \to M$ extends in $V[G \times H]$ to an elementary $j : V[G] \to M[G \times H]$ by setting $j(\dot{x}^G) = (j(\dot{x}))^{G\times H}$ for every P-name $\dot{x}$, and $D_0 = \{\dot{X}^G : \kappa \in (j(\dot{X}))^{G\times H}\}$.

Our model will be of the form $V[G, \mathcal{C}]$ where G is P-generic and $\mathcal{C} = \langle C_\alpha : \alpha < \kappa^+\rangle$, with each C_α a closed unbounded subset of κ, is $V[G]$-generic on a set Q_{κ^+} of conditions. The sets Q_α, $\alpha \le \kappa^+$, will be defined by induction on α, together with ideals I_α on κ in $V[G, \mathcal{C}{\restriction}\alpha]$.

Since $2^\kappa = \kappa^+$, we can define a sequence $\langle \dot{A}_\alpha : \alpha < \kappa^+\rangle$ such that for each $\alpha < \kappa^+$, $\dot{A}_\alpha$ is a name for a subset of κ in $V[G, \mathcal{C}{\restriction}\alpha]$ and for all $\alpha < \kappa^+$ every subset of κ in $V[G, \mathcal{C}{\restriction}\alpha]$ has a name $\dot{A}_\gamma$ for some $\gamma \ge \alpha$. We will show that Q_{κ^+} satisfies the κ-chain condition; it will follow that every subset of κ in $V[G, \mathcal{C}]$ is in $V[G, \mathcal{C}{\restriction}\alpha]$ for some $\alpha < \kappa^+$.

The forcing Q_{κ^+} is, in $V[G]$, a countable support iteration of shooting a closed unbounded subset C_α of $\kappa - \dot{A}_\alpha$, if $\dot{A}_\alpha \in I_\alpha$. More precisely:

(23.5) A condition $q \in Q_\alpha$ is a sequence $\langle q_\beta : \beta < \alpha\rangle$ in $V[G]$ such that
- (i) $q_\beta = \emptyset$ for all but countably many $\beta < \alpha$,
- (ii) q_β is a closed countable subset of κ, for all $\beta < \alpha$,
- (iii) $q_\beta{\restriction}\beta \in Q_\beta$ for all $\beta < \alpha$,
- (iv) if $\alpha = \beta + 1$ then either $q{\restriction}\beta \Vdash \dot{A}_\beta \notin I_\beta$ or $q{\restriction}\beta \Vdash q_\beta \cap \dot{A}_\beta = \emptyset$.

(The ideals I_α, $\alpha < \kappa^+$, will be defined in Lemma 23.12 below.) If $q, q' \in Q_\alpha$ then $q \le q'$ if for each $\beta < \alpha$, q_β is an end-extension of q'_β.

In a generic extension of $V[G]$ by Q_{κ^+}, for every $\alpha < \kappa^+$ the union of all q_α, with $q = \langle q_\alpha : \alpha < \kappa^+\rangle$ in the generic filter, is a closed unbounded subset of κ.

Lemma 23.11. *Q_{κ^+} satisfies the κ^+-chain condition.*

Proof. Let W be a maximal antichain. Since $|Q_\alpha| \le \kappa$ for each $\alpha < \kappa^+$ there exists an $\alpha < \kappa^+$ such that for every $q \in W$ there is some $q' \in W \cap Q_\alpha$ with

$q' \leq q{\restriction}\alpha$. But $q \cup q'$ is a condition, so q' and q are compatible, and so $q = q'$. Hence $W \subset Q_\alpha$ and $|W| \leq \kappa$. $\square$

Working in $V[G \times H]$, let us consider again the elementary embedding $j : V[G] \to M[G \times H]$. If $\alpha < \kappa^+$, then $|P(Q_\alpha)^{V[G]}| = \kappa^+ < j(\kappa)$ and therefore $P(Q_\alpha)^{V[G]}$ is countable in $V[G \times H]$, and hence there exists a Q_α-generic set $\mathcal{C} = \langle C_\beta : \beta < \alpha \rangle$. Because each $C_\beta \subset \kappa$ and $|\alpha| \leq \kappa$, it follows that $\mathcal{C} \in M[G \times H]$ (for the proof see Lemma 21.9).

In the following arguments, we consider sequences $\mathcal{C} = \langle C_\beta : \beta < \alpha \rangle$ of closed unbounded subsets of κ, and $q = \langle q_\beta : \beta < \alpha \rangle$ of conditions in Q_α, and use the notation $q \in \mathcal{C}$ to mean that each q_β is an initial segment of C_β.

For any $\alpha < \kappa^+$, we define (in $M[G \times H]$), for any Q_α-generic sequence $\mathcal{C}$, the sequence $q^{\mathcal{C}} = \langle q^{\mathcal{C}}_\gamma : \gamma < j(\alpha) \rangle$ by

$$(23.6) \qquad q^{\mathcal{C}}_\gamma = \begin{cases} C_\beta \cup \{\kappa\} & \text{if } \gamma = j(\beta), \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 23.12. *$q^{\mathcal{C}} \in j(Q_\alpha)$ and $q^{\mathcal{C}} \leq j(q)$ for any $q \in \mathcal{C}$.*

Proof. By induction on α. Simultaneously, we define the ideals I_α for $\alpha > 0$. Assuming that the lemma holds, we define I_α as follows: If $X \in V[G, \mathcal{C}{\restriction}\alpha]$, then $X \in I_\alpha$ if and only if for some $p \in G$ and some $q \in \mathcal{C}{\restriction}\alpha$

$$(23.7) \quad p \Vdash_{j(P)} (\text{for every } \mathcal{C} \ni q,\ Q_\alpha\text{-generic over } M[G],\ q^{\mathcal{C}} \Vdash_{j(Q_\alpha)} \kappa \notin j(\dot{X})).$$

Now assume that the lemma has been proved for all $\beta < \alpha$. When we define $q^{\mathcal{C}}$ by (23.7), we have $q^{\mathcal{C}} \in M[G \times H]$, and once we verify that $q^{\mathcal{C}}$ is a condition in $j(Q_\alpha)$ then the rest of the lemma follows. The only nontrivial verification of $q^{\mathcal{C}} \in j(Q_\alpha)$ is clause (iv) of (23.5).

Thus let $\alpha = \beta + 1$ and assume that $q^{\mathcal{C}}{\restriction}j(\beta)$ does not force $j(\dot{A}_\beta) \notin j(I_\beta)$; we want $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash q^{\mathcal{C}}_{j(\beta)} \cap j(\dot{A}_\beta) = \emptyset$.

Let $\xi \leq \kappa = \max(q^{\mathcal{C}}_{j(\beta)})$, and first consider the case $\xi < \kappa$. Assume that (in some extension of $M[G \cap H]$ by a generic filter containing $q^{\mathcal{C}}{\restriction}j(\beta)$), $\xi \in q^{\mathcal{C}}_{j(\beta)} \cap j(\dot{A}_\beta)$. Since $\xi = j(\xi)$, we have $\xi \in C_\beta$, and there is some $q \in Q_\alpha$ such that $q \in \mathcal{C}$, $\xi \in q_\beta$, and $q{\restriction}\beta \Vdash \xi \in \dot{A}_\beta$. Hence $q{\restriction}\beta \Vdash \dot{A}_\beta \notin I_\beta$, therefore $j(q{\restriction}\beta) \Vdash j(\dot{A}_\beta) \notin I_{j(\beta)}$ and because $q^{\mathcal{C}}{\restriction}j(\beta) \leq j(q{\restriction}\beta)$, we have $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash j(\dot{A}_\beta) \notin I_{j(\beta)}$, a contradiction.

Now consider the case $\xi = \kappa$. Let $q \in \mathcal{C}{\restriction}\beta$ be such that q decides $\dot{A}_\beta \in I_\beta$. The assumption $q \Vdash \dot{A}_\beta \notin I_\beta$ leads to a contradiction as in the preceding case; thus assume that $q \Vdash \dot{A}_\beta \in I_\beta$. Then, by definition of I_β, we have $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash \kappa \notin j(\dot{A}_\beta)$, and so $q^{\mathcal{C}}{\restriction}j(\beta) \Vdash q^{\mathcal{C}}_{j(\beta)} \cap j(\dot{A}_\beta) = \emptyset$.

Hence $q^{\mathcal{C}}$ satisfies (23.5)(iv). $\square$

Lemma 23.13. *If $\beta < \alpha$ then $I_\beta = I_\alpha \cap V[G, \mathcal{C}{\restriction}\beta]$.*

Proof. It is clear that $I_\beta \subset I_\alpha$. Thus let $\dot X$ be a $P * Q_\beta$-name for a subset of κ, and let $p \in P$ and $q \in Q_\alpha$ be such that (p, q) forces $\dot X \in I_\alpha$ and $\dot X \notin I_\beta$. By the latter,

$$\begin{aligned} p \Vdash_{j(P)} &\exists Q_\beta\text{-generic } \mathcal{C}' \text{ over } M[G] \text{ with } q{\restriction}\beta \in \mathcal{C} \text{ and}\\ &\exists q' \le q^{\mathcal{C}'} \text{ such that } q' \Vdash_{j(Q_\beta)} \kappa \in j(\dot X). \end{aligned}$$

In $M[G \times H]$, Q_α is countable, so $\mathcal{C}' \in M[G, H{\restriction}\delta]$ for some $\delta < j(\kappa)$. In $M[G \times H]$, find a Q_α-generic $\mathcal{C}''$ over $M[G]$ such that $\mathcal{C}{\restriction}\beta = \mathcal{C}'$ and $q \in \mathcal{C}''$. Then $q' \cup q^{\mathcal{C}''}$ is stronger than $q^{\mathcal{C}''}$ and forces $\kappa \in j(\dot X)$. It follows that (p, q) does not force $\dot X \in I_\alpha$, a contradiction. ◻

We let $I = \bigcup_{\alpha<\kappa^+} I_\alpha$.

Lemma 23.14. *I is a normal ideal.*

Proof. Let $f \in V[G, \mathcal{C}]$ be a function $f : \kappa \to \kappa$, and assume that

(23.8) $\quad (p, q) \Vdash \{\alpha : f(\alpha) < \alpha\} \notin I \quad \text{and} \quad (\forall\gamma < \kappa)\ \{\alpha : f(\alpha) = \gamma\} \in I.$

By Lemma 23.11, $f \in V[G, \mathcal{C}{\restriction}\alpha]$ for some $\alpha < \kappa^+$, and (23.8) holds for I_α in place of I. Let $G \times H$ be $j(P)$-generic with $p \in G$, and work in $M[G \times H]$. There exists a Q_α-generic $\mathcal{C}'$ with $q \in \mathcal{C}'$ such that some $q' < q^{\mathcal{C}'}$ forces $j(f)(\kappa) < \kappa$. Then some $q'' < q'$ forces $j(f)(\kappa) = \gamma$, for some γ, and hence (p, q) does not force $\{\alpha : f(\alpha) = \gamma\} \in I_\alpha$, a contradiction. ◻

Lemma 23.15. *I is, in $V[G, \mathcal{C}]$, the nonstationary ideal on $\aleph_1 = \kappa$.*

Proof. That $\kappa = \aleph_1$ in $V[G, \mathcal{C}]$ follows from the normality of I. Each C_α is a closed unbounded subset of ω_1, and since $C_\alpha \cap A_\alpha = \emptyset$ if $A_\alpha \in I$, every set in I is nonstationary. On the other hand let $\dot C$ be a name for a subset of κ and let $q \in \mathcal{C}$ be such that $q \Vdash \dot C$ is a closed unbounded set. Then for every $\mathcal{C}' \ni q$,

$$q^{\mathcal{C}'} \Vdash j(\dot C) \text{ is closed and } j(\dot C) \cap \kappa \text{ is unbounded in } \kappa$$

and so $q^{\mathcal{C}'} \Vdash \kappa \in j(\dot C)$. It follows that $q \Vdash \kappa - \dot C \in I_\alpha$ and so every nonstationary set is in I. ◻

It remains to show that I is precipitous.

Let $R(I)$ denote the forcing with I-positive sets; a generic filter on $R(I)$ is an ultrafilter that extends the dual of I. Let (p_1, q_1) be a condition in $P * Q$ and let $\dot X$ be a name for a subset of κ, such that $(p_1, q_1) \Vdash \dot X \notin I$. We want to find generic G and $\mathcal{C}$ with $p_1 \in G$ and $q_1 \in \mathcal{C}$, and an $R(I)$-generic D with $\dot X \in D$ such that $\operatorname{Ult}_D V[G, \mathcal{C}]$ is well-founded.

Since (p_1, q_1) forces $\dot X \notin I$, there exist $p_1' \in j(P)$ and $\alpha < \kappa^+$ such that $p_1' < p_1$, $p_1' \Vdash \dot X \in V[G, \mathcal{C}{\restriction}\alpha]$ and

$$\begin{aligned} p_1' \Vdash_{j(P)} &\exists Q_\alpha\text{-generic } \mathcal{C}' \text{ over } M[G] \text{ with } q_1 \in \mathcal{C}'\\ &\text{such that } q^{\mathcal{C}'} \Vdash_{j(Q_\alpha)} \kappa \notin j(\dot X). \end{aligned} \tag{23.9}$$

Let $G \times H$ be $j(P)$-generic over V with $p_1' \in G \times H$. Let $\mathcal{C}' \in M[G \times H]$ be as in (23.9), and pick $q_1' \in j(Q_\alpha)$ such that $q_1' \le q^{\mathcal{C}'}$ and $q_1' \Vdash \kappa \in j(\dot X)$.

We shall find $\mathcal{C}$ and $\mathcal{C}^*$ so that j extends to $j : V[G,\mathcal{C}] \to M[G\times H,\mathcal{C}^*]$. We require that $j(q) \in \mathcal{C}^*$ whenever $q \in \mathcal{C}$, or $\mathcal{C} = \{q \in Q : j(q) \in \mathcal{C}^*\}$.

Let Q^* be the following subordering of $j(Q)$ in $V[G\times H]$. For each $q \in j(Q)$ let $\mathcal{C}_q = \{q' \in Q : j(q') \geq q\}$, and let

$$Q^* = \{q \in j(Q) : (\exists \alpha < \kappa^+)\ \mathcal{C}_q \subset Q_\alpha \text{ and } \mathcal{C}_q \text{ is } Q_\alpha\text{-generic over } V[G]\}. \tag{23.10}$$

Then $q_1' \in Q^*$, so we can find a set $\mathcal{C}^*$ that is Q^*-generic over $V[G\times H]$ with $q_1^* \in \mathcal{C}^*$, and let $\mathcal{C} = j_{-1}(\mathcal{C}^*)$.

Lemma 23.16. *$\mathcal{C}$ is Q-generic over $V[G]$ and $\mathcal{C}^*$ is $j(Q)$-generic over $M[G\times H]$.*

Proof. We first show that for all $\alpha < \kappa^+$, the set

$$B_\alpha = \{q \in Q^* : \mathcal{C}_q{\restriction}\alpha \text{ is } Q_\alpha\text{-generic over } V[G]\}$$

is dense in Q^*. Let $q \in Q^*$. If $q \notin B_\alpha$ then by (23.10) there exists some $\beta < \alpha$ such that $C_q \subset Q_\beta$ and C_q is Q_β-generic over $V[G]$. But $P(Q_\alpha)\cap V[G]$ is countable in $M[G\times H]$ so there exists a $\mathcal{C}'$ Q_α-generic over $V[G]$ such that $\mathcal{C}'{\restriction}\beta = C_q$. Since $|Q_\alpha| = \kappa$, this $\mathcal{C}'$ is in $M[G\times H]$ and we can take $q' = q \cup q^{\mathcal{C}'}$. Then $q' \leq q$ and $q' \in Q^* \cap B_\alpha$.

Now let A be an open dense subset of Q in $V[G]$. Since Q satisfies the κ^+-chain condition in $V[G]$, A contains a maximal antichain of cardinality κ. Thus for some $\alpha < \kappa^+$, $A\cap Q_\alpha$ is dense in Q_α. Since B_α is dense in Q^*, there is a $q \in \mathcal{C}^*$ such that $\mathcal{C}_q{\restriction}\alpha$ is Q_α-generic over $V[G]$ and hence $\mathcal{C}\cap A \supset \mathcal{C}_q \cap A \neq \emptyset$, and so $\mathcal{C}$ is Q-generic over $V[G]$.

Similarly, if $A \in M[G\times H]$ is open dense in $j(Q)$ then (because $j(Q)$ satisfies the $j(\kappa^+)$-chain condition in $M[G\times H]$ and $j(\kappa^+) = \bigcup_{\alpha<\kappa^+} j(\alpha)$) $A \cap j(Q_\alpha)$ is dense in $j(Q_\alpha)$ for some $\alpha < \kappa^+$. Since B_α is dense in Q^*, $\mathcal{C}^*{\restriction}j(\alpha)$ is $j(Q_\alpha)$-generic over $V[G\times H]$ and hence $A\cap \mathcal{C}^* \neq \emptyset$, and so $\mathcal{C}^*$ is $j(Q)$-generic over $M[G\times H]$. □

Hence j extends to an elementary embedding $j : V[G,\mathcal{C}] \to M[G\times H,\mathcal{C}^*]$. Let

$$D = \{z \in P(\kappa)\cap V[G,\mathcal{C}] : \kappa \in j(z)\};$$

D is an ultrafilter extending the dual of I, and $\mathrm{Ult}_D\, V[G,\mathcal{C}]$ is well-founded. Also, $\dot{X} \in D$ because $(p_1', q_1') \in G\times H\times \mathcal{C}^*$; it remains to show that D is $R(I)$-generic over $V[G,\mathcal{C}]$.

Toward a contradiction, let W be a subset of $R(I)$ in $V[G,\mathcal{C}]$ such that $W\cap D = \emptyset$; we will show that W is not dense in $R(I)$. Since W is disjoint from D, there exist $p_2 \in G\times H$ and $q_2 \in \mathcal{C}^*$, and some $\dot{A}$ such that $(p_2,q_2) \leq (p_1,q_1)$ and

$$p_2 \Vdash_{j(P)} q_2 \Vdash_{Q^*} \dot{A} \in W \text{ and } \kappa \notin j(\dot{A}). \tag{23.11}$$

Since $(p_2, q_2) \in M = \mathrm{Ult}_U(V)$, there is a function $f : \kappa \to P * Q$ such that $(p_2, q_2) = j(f)(\kappa)$. Let $T = \{\alpha : f(\alpha) \in G \times \mathcal{C}\}$. Then since $\kappa \in j(T)$ if and only if $(p_2, q_2) \in G \times H \times \mathcal{C}^*$, we can rewrite (23.11) as

$$\Vdash_{j(P)} \Vdash_{Q^*} (\forall A \in W)\ \kappa \notin j(A \cap T). \tag{23.12}$$

For any $A \in W$, let α_A be such that $q_2 \in j(Q_{\alpha_A})$ and $A \in V[G, \mathcal{C} \restriction \alpha_A]$. By (23.12) and (23.10) we have

$$\Vdash_{j(P)} \text{ for every } Q_{\alpha_A}\text{-generic } \mathcal{C}' \text{ over } V[G],\ q^{\mathcal{C}'} \Vdash_{j(Q)} \kappa \notin j(A \cap T).$$

This says that $A \cap T \in I$ for all $A \in W$. But $T \in D$, and hence $T \notin I$. This contradicts W being dense. $\square$

Thus the consistency strength of "I_{NS} on ω_1 is precipitous" is exactly the existence of a measurable cardinal. For cardinals greater than ω_1 the consistency is considerably stronger. For instance, "I_{NS} on ω_2 is precipitous" is equiconsistent with a measurable cardinal of order 2 (Gitik); for larger cardinals it is much stronger. Most of the best results to date are due to Gitik.

Saturation of the Nonstationary Ideal

By Solovay's Theorem 8.10, the nonstationary ideal I_{NS} on κ is nowhere κ-saturated. For $\kappa = \aleph_1$ it is consistent that I_{NS} is κ^+-saturated; its consistency strength is roughly that of a Woodin cardinal. We shall return to this subject in Part III.

For κ greater than $\aleph_1$, the nonstationary ideal is not κ^+-saturated:

Theorem 23.17 (Gitik-Shelah). *For every regular cardinal $\kappa \geq \aleph_2$, the ideal I_{NS} on κ is not κ^+-saturated.*

The proof of Theorem 23.17 appears in Gitik and Shelah [1997]. Most special cases were proved earlier by Shelah, and we present this proof first, as it is somewhat easier. The complete proof will follow.

The results presented here are somewhat more general as they apply to other normal ideals. If I is a normal ideal, I^+ denotes the collection $\{S \subset \kappa : S \notin I\}$ of sets of positive I-measure. For $S \in I^+$, $I \restriction S$ denotes the ideal $\{X \subset \kappa : X \cap S \in I\}$; we say that $I \restriction S$ *concentrates* on S.

We shall use the method of generic ultrapowers, and start with several observations. Let I be a normal κ^+-saturated κ-complete ideal on a regular uncountable cardinal κ. The generic ultrapower $M = \mathrm{Ult}_G(V)$ is well-founded, and since the forcing with sets of positive I-measure satisfies the κ^+-chain condition, κ^+ is a cardinal in $V[G]$, and hence in M.

Lemma 23.18. *Let I be a normal κ^+-saturated κ-complete ideal on κ, let $R(I)$ be the forcing with I-positive sets, let G be the $R(I)$-generic ultrafilter and let $M = \mathrm{Ult}_G(V)$. Then $P^M(\kappa) = P^{V[G]}(\kappa)$, and all cardinals (and cofinalities) $< \kappa$ are preserved in $V[G]$.*

Proof. The Boolean algebra $B = P(\kappa)/I$ is complete (see Exercise 22.9). If $\dot{A}$ is a name for a subset $A = \dot{A}^G$ of κ in $V[G]$, let $S_\alpha \in I^+$ be, for each $\alpha < \kappa$, such that $\|\alpha \in \dot{A}\| = [S_\alpha]$. If $j : V \to M$ is the canonical embedding, we have, for each α, $\alpha \in A$ if and only if $S_\alpha \in G$ if and only if $\kappa \in j(S_\alpha)$, and so the set $A = \{\alpha \in \kappa : \kappa \in j(S_\alpha)\}$ is in M.

If $\lambda < \kappa$ is a cardinal then since κ is the critical point of j, λ is a cardinal in M. Since $P^{V[G]}(\lambda) = P^M(\lambda)$, λ is a cardinal in $V[G]$. □

We shall use a combinatorial lemma due to Shelah. Let λ be a cardinal and let $\alpha < \lambda^+$ be a limit ordinal. A family $\{X_\xi : \xi < \lambda^+\}$ of subsets of α is *strongly almost disjoint* if every $X_\xi \subset \alpha$ is unbounded, and if for every $\vartheta < \lambda^+$ there exist ordinals $\delta_\xi < \alpha$, for $\xi < \vartheta$, such that the sets $X_\xi - \delta_\xi$, $\xi < \vartheta$, are pairwise disjoint. If κ is a regular cardinal then there exists a strongly almost disjoint family of κ^+ subsets of κ (see Exercise 23.10).

Lemma 23.19. *If $\alpha < \lambda^+$ and $\mathrm{cf}\,\alpha \neq \mathrm{cf}\,\lambda$ then there exists no strongly almost disjoint family of subsets of α.*

Proof. Assume to the contrary that $\{X_\xi : \xi < \lambda^+\}$ is a strongly almost disjoint family of subsets of α. We may assume that each X_ξ has order-type $\mathrm{cf}\,\alpha$. Let f be a function that maps λ onto α. Since $\mathrm{cf}\,\lambda \neq \mathrm{cf}\,\alpha$ there exists for each ξ some $\gamma_\xi < \lambda$ such that $X_\xi \cap f``\gamma_\xi$ is cofinal in α. There exist some γ and a set $W \subset \lambda^+$ of size λ such that $\gamma_\xi = \gamma$ for all $\xi \in W$. Let $\vartheta > \sup W$. By the assumption on the X_ξ there exist ordinals $\delta_\xi < \alpha$, $\xi < \vartheta$, such that the $X_\xi - \delta_\xi$ are pairwise disjoint. Thus $f^{-1}(X_\xi - \delta_\xi)$, $\xi \in W$, are λ pairwise disjoint nonempty subsets of γ. A contradiction. □

Corollary 23.20. *If κ is a regular cardinal and if a notion of forcing P makes $\mathrm{cf}\,\kappa \neq \mathrm{cf}\,|\kappa|$, then P collapses κ^+.*

Proof. Assume that κ^+ is preserved; thus in $V[G]$, $(\kappa^+)^V = \lambda^+$ where $\lambda = |\kappa|$. In V there is a strongly almost disjoint family $\{X_\xi : \xi < (\kappa^+)^V\}$, and it remains strongly almost disjoint in $V[G]$, and has size λ^+. Since $\mathrm{cf}\,\kappa \neq \mathrm{cf}\,\lambda$ (in $V[G]$), this contradicts Lemma 23.19. □

Corollary 23.21. *If $\kappa = \lambda^+$, if $\nu \neq \mathrm{cf}\,\lambda$ is a regular cardinal, and if I is a normal κ-complete κ^+-saturated ideal on κ, then $E^\kappa_\nu = \{\alpha < \kappa : \mathrm{cf}\,\alpha = \nu\} \in I$.*

Proof. Assume that $E^\kappa_\nu \in I^+$, and let G be a generic ultrafilter on $P(\kappa)/I$. By Lemma 23.18, all cardinals $\leq \lambda$, as well as κ^+, are preserved in $V[G]$. If $E^\kappa_\nu \in G$, then in M, $\mathrm{cf}\,\kappa = \nu$, and so (by Lemma 23.18) $\mathrm{cf}\,\kappa = \nu$ in $V[G]$. Thus we have, in $V[G]$, $\mathrm{cf}\,\kappa = \nu$ and $|\kappa| = \lambda$ while κ^+ is preserved, contradictory Corollary 23.20. □

It follows that if κ is a successor cardinal greater than $\aleph_1$ then the nonstationary ideal on κ is not κ^+-saturated: In fact $I_{\rm NS}{\restriction}E^\kappa_\nu$ is not κ^+-saturated, for all regular $\nu \neq \operatorname{cf}\lambda$ where λ is the predecessor of κ.

We complete the proof of Theorem 23.17 using Lemma 23.4 on club-guessing. We shall show that for every regular $\kappa \geq \aleph_3$ and every uncountable regular λ such that $\lambda^+ < \kappa$, the ideal $I_{\rm NS}{\restriction}E^\kappa_\lambda$ is not κ^+-saturated.

Thus let κ and λ be regular uncountable such that $\lambda^+ < \kappa$. Let E be a stationary subset of E^κ_λ. By Lemma 23.4 there exists a sequence $\langle c_\alpha : \alpha \in E\rangle$ with each c_α cofinal in α, that satisfies (23.3) and such that for every closed unbounded C, the set

$$\mathcal{G}(C) = \{\alpha \in E : (\exists\beta < \alpha)\ C \supset c_\alpha - \beta\}$$

is stationary.

Lemma 23.22. *If $I_{\rm NS}{\restriction}E^\kappa_\lambda$ is κ^+-saturated then there exists a stationary set $\tilde{E} \subset E$ such that for every closed unbounded C, $\tilde{E} - \mathcal{G}(C)$ is nonstationary (C is guessed at almost every $\alpha \in \tilde{E}$).*

Proof. If not, then for every stationary $S \subset E$ there exists a closed unbounded set C such that $S - \mathcal{G}(C)$ is stationary. By the κ^+-saturation, there exists a collection $\{(S_i, C_i) : i < \kappa\}$ such that $W = \{S_i - \mathcal{G}(C_i) : i < \kappa\}$ is a maximal antichain in $P(\kappa)/I_{\rm NS}$ below E. Let $C = \triangle_{i<\kappa} C_i$. For every $i < \kappa$, C_i contains an end-segment of C, and hence $\mathcal{G}(C_i)$ contains an end-segment of $\mathcal{G}(C)$. As $\mathcal{G}(C)$ is stationary, this contradicts the maximality of W. $\square$

Now we use the κ^+-saturation again, and using Lemma 23.22 obtain a maximal antichain $\{S_i : i < \kappa\}$ of pairwise disjoint stationary subsets of E^κ_λ, and for each i a sequence $\langle c_\alpha : \alpha \in S_i\rangle$ of cofinal c_α satisfying (23.3) such that every closed unbounded C is guessed at almost every $\alpha \in S_i$. Then $\langle c_\alpha : \alpha \in \bigcup_{i<\kappa} S_i\rangle$ guesses every C almost everywhere, contrary to Exercise 23.2.

This completes the proof of Theorem 23.17. $\square$

The question whether various restrictions of the nonstationary ideal can be κ^+-saturated has been studied extensively. For instance, it is proved in Jech and Woodin [1985] that it is consistent, relative to a measurable cardinal, that κ is a Mahlo cardinal and $I_{\rm NS}{\restriction}\mathrm{Reg}$ is κ^+-saturated, where $\mathrm{Reg} = \{\alpha < \kappa : \alpha \text{ is a regular cardinal}\}$. It is open whether (for instance) $I_{\rm NS}{\restriction}E^{\aleph_2}_{\aleph_1}$ can be $\aleph_3$-saturated.

Reflection

There has been a large number of results on reflecting stationary sets. Let us recall that a stationary set S *reflects* at α if $S \cap \alpha$ is a stationary subset of α. In this section we investigate the simplest case, namely $\kappa = \aleph_2$.

There are two kinds of limit ordinals below ω_2: those of cofinality $\aleph_0$ and those of cofinality $\aleph_1$; the sets $E^{\aleph_2}_{\aleph_0}$ and $E^{\aleph_2}_{\aleph_1}$. By Exercise 23.4, the set $E^{\aleph_2}_{\aleph_1}$ does not reflect (at any ordinal $\alpha < \omega_2$). By Exercise 23.5, the set $E^{\aleph_2}_{\aleph_0}$ reflects at every $\alpha \in E^{\aleph_2}_{\aleph_1}$; the question is whether every stationary $S \subset E^{\aleph_2}_{\aleph_0}$ can reflect. By Lemma 23.6, if every $S \subset E^{\aleph_2}_{\aleph_0}$ reflects then $\square_{\omega_1}$ fails, and this is known (due to Jensen) to imply that $\aleph_2$ is a Mahlo cardinal in L. On the other hand, it is consistent relative to the existence of a Mahlo cardinal, that every stationary $S \subset E^{\aleph_2}_{\aleph_1}$ reflects (Harrington and Shelah [1985]).

The following theorem shows that a stronger version of reflection is consistent, if fact equiconsistent with weak compactness:

Theorem 23.23 (Magidor). *The following are equiconsistent*:

(i) *the existence of a weakly compact cardinal,*
(ii) *every stationary set $S \subset E^{\aleph_2}_{\aleph_0}$ reflects at almost all $\alpha \in E^{\aleph_2}_{\aleph_1}$.*

This result does not generalize to cardinals greater than $\aleph_2$; see Exercise 23.12. Reflection for stationary subsets of $\kappa > \aleph_2$ is considerably more complicated.

We shall prove that (ii) implies that $\aleph_2$ is weakly compact in L, and then give a brief account of the consistency proof of (ii). If every stationary set $S \subset E^{\aleph_2}_{\aleph_0}$ reflects then $\aleph_2$ is a Mahlo cardinal in L. Using Jensen's Theorem 27.1 we prove a somewhat weaker statement.

Lemma 23.24. *If every stationary $S \subset E^{\aleph_2}_{\aleph_0}$ reflects then $\aleph_2$ is inaccessible in L.*

Proof. Let $\kappa = \aleph_2$. Assume that κ is in L the successor of some λ, $\kappa = (\lambda^+)^L$. In L, there exists a square-sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$, and the order-type of each C_α is at most λ. By Fodor's Theorem, there exists a stationary set $A \subset E^{\aleph_2}_{\aleph_1}$ such that all C_α, $\alpha \in A$, have the same order-type.

The set $\bigcup\{C_\alpha : \alpha \in A\}$ is stationary, and it follows that there exists a limit ordinal η such that the set $S = \{\gamma \in E^{\aleph_2}_{\aleph_0} : \gamma$ is the ηth element of some $C_\alpha\}$ is stationary. As in Lemma 23.6, S does not reflect. $\square$

Note that if every stationary $S \subset E^{\aleph_2}_{\aleph_0}$ reflects at almost every $\alpha \in E^{\aleph_2}_{\aleph_1}$ then every two stationary sets $S_1, S_2 \subset E^{\aleph_2}_{\aleph_1}$ reflect at the same α. The following lemma completes the proof:

Lemma 23.25. *If for any stationary sets $S_1, S_2 \subset E^{\aleph_2}_{\aleph_0}$ there exists an $\delta \in E^{\aleph_2}_{\aleph_1}$ such that both $S_1 \cap \delta$ and $S_2 \cap \delta$ are stationary, then $\aleph_2$ is Π^1_1-indescribable in L.*

Proof. Let $\varphi(X)$ be a second order formula with only first order quantifiers and assume that for each $\alpha < \omega_2$ there exists some $X_\alpha \in L$, $X_\alpha \subset \alpha$, such that $L_\alpha \vDash \varphi(X_\alpha)$. We shall find an $X \in L$, $X \subset \omega_2$, such that $L_{\omega_2} \vDash \varphi(X)$.

Let X_α be the least such X_α in L. There exists a $\beta < (\alpha^+)^L$ such that $X_\alpha \in L_\beta$, and let β be the least such β. Let $Z_\alpha \in L$ be such that $Z_\alpha \in \{0,1\}^\alpha$ and that Z_α codes the model $(L_\beta, \in, X_\alpha)$.

For every $\delta < \omega_2$ of cofinality of ω_1, let

$$C_\delta = \{\alpha < \delta : Z_\alpha = Z_\delta{\restriction}\alpha \text{ and } X_\alpha = X_\delta{\restriction}\alpha\}.$$

The set C_δ is a closed unbounded subset of δ.

For each $\gamma < \omega_2$ and each $t \in L$ such that $t \in \{0,1\}^\delta$, let

$$S_t = \{\alpha \in E^{\aleph_2}_{\aleph_0} : t \subset Z_\alpha\}.$$

Since $\aleph_2$ is inaccessible in L, there exists for each $\gamma < \omega_2$ some $t \in \{0,1\}^\gamma$ such that S_t is stationary. Now let $\gamma_1 \le \gamma_2$ and $t_i \in \{0,1\}^{\gamma_i}$ ($i = 1, 2$), and assume that both S_{t_1} and S_{t_2} are stationary. By the assumption of the lemma, there exists a $\delta < \omega_2$ of cofinality ω_1 such that both $S_{t_1} \cap \delta$ and $S_{t_2} \cap \delta$ are stationary. Let $\alpha_1, \alpha_2 \in C_\delta$ be such that $\alpha_i \in S_{t_i}$ ($i = 1, 2$). Since $t_i \subset Z_{\alpha_i} \subset Z_\delta$, it follows that $t_1 \subset t_2$.

Hence for each $\gamma < \kappa$ there is a unique t_γ such that S_{t_γ} contains almost all ordinals in $E^{\aleph_2}_{\aleph_0}$; $S_{t_\gamma} \supset E^{\aleph_2}_{\aleph_0} \cap D_\gamma$ with D_γ closed unbounded. Let $D = \triangle_\gamma D_\gamma$; then for every $\alpha \in E^{\aleph_2}_{\aleph_0} \cap D$ we have $t_\alpha = Z_\alpha$. Now let $Z = \bigcup\{t_\gamma : \gamma < \omega_2\}$. The set Z codes some model $(L_\eta, \in, X)$ with $X \subset \omega_2$ and $X \in L$. It follows that $X \cap \alpha = X_\alpha$ for almost all $\alpha \in E^{\aleph_2}_{\aleph_0}$.

We finish the proof by verifying $L_{\omega_2} \vDash \varphi(X_\alpha)$. This holds because $L_\alpha \vDash \varphi(X_\alpha)$ for all α and therefore $L_{\omega_2} \vDash \varphi(X \cap \alpha)$ for almost all $\alpha \in E^{\aleph_2}_{\aleph_0}$. $\square$

This completes the proof that the existence of a weakly compact cardinal is necessary for the consistency of (ii). We shall not present the consistency proof of (ii) and instead give a brief description of the methods involved.

One starts with a ground model where κ is a weakly compact cardinal, and GCH holds. First one uses the Lévy collapse Q with countable conditions that makes $\kappa = \aleph_2$ (all cardinals between $\aleph_1$ and κ are collapsed). In V^Q, one constructs a forcing iteration P of length κ^+, with $\aleph_1$-support. At every stage α of the iteration, one considers (in V^Q) a P_α-name for a stationary set $S \subset E^\kappa_\omega$ and shoots a closed unbounded set through the set $T = \mathrm{Tr}(S) \cup E^\kappa_\omega$. Forcing conditions are closed bounded subsets of T. It is not difficult to verify that such forcing is ω-closed, and that the iteration satisfies the κ-chain condition. Thus one can arrange the iteration so that every potential stationary set $S \subset E^\kappa_\omega$ is considered.

The main point of the proof is to show that $\aleph_1$ is preserved by the iteration, and that at each stage, if $S \subset E^\kappa_\omega$ is stationary then $\mathrm{Tr}(S) \cap E^\kappa_{\omega_1}$ is unbounded. This is proved using arguments similar to those used in Theorem 23.10.

Weak compactness of κ is used as follows: At a given stage α of the iteration, there is a transitive model $M \supset \alpha$ of size κ of a sufficiently large fragment of ZFC, and (by weak compactness) there is an elementary $j : M \to N$, cf. Lemma 17.17. This j extends to $j : M^Q \to N^{j(Q)}$.

For details, consult Magidor [1982].

Exercises

23.1. Let κ and λ be regular, $\lambda \geq \aleph_1$ and $\lambda^+ < \kappa$. For every stationary $E \subset E^\kappa_\lambda$ there exists a sequence $\langle C_\alpha : \alpha \in E\rangle$ of closed unbounded subsets of the α's such that for every closed unbounded $C \subset \kappa$, the set $\{\alpha \in E : c_\alpha \subset C\}$ is stationary.

23.2. Let κ and λ be regular, $\lambda \geq \aleph_1$ and $\lambda^+ < \kappa$. There exists no sequence $\langle C_\alpha : \alpha \in E^\kappa_\lambda\rangle$ with each $c_\alpha \subset \alpha$ closed unbounded, that guesses every closed unbounded $C \subset \kappa$ almost everywhere (i.e., C contains an end-segment of c_α for almost all $\alpha \in E^\kappa_\lambda$) and satisfies (23.3).

[Assume $\langle c_\alpha : \alpha \in E^\kappa_\lambda\rangle$ is such. Let $E = \{\xi < \kappa : \operatorname{cf}\xi > \lambda\}$ and let $C_0 = E'$. For each n, let $C_{n+1} \subset C'_n$ be closed unbounded such that C'_n contains an end-segment of c_α, for all $\alpha \in E^\kappa_\lambda \cap C_{n+1}$. Let $C = \bigcap_{n<\omega} C_n$ and let α be the least element of $C \cap E^\kappa_\lambda$; C contains an end-segment of C_α. There is a $\beta \in C \cap c_\alpha$ such that $\operatorname{cf}\beta > \lambda$. It follows that there exists some $\gamma \in C \cap \beta \cap E^\kappa_\lambda$, contradicting the minimality of α.]

23.3. There exists an $\aleph_0$-closed, $\aleph_1$-distributive notion of forcing such that $V[G]$ satisfies $\square_{\omega_1}$.

[A forcing condition is a sequence $p = \langle C_\alpha : \alpha \leq \gamma\rangle$, where $\gamma < \omega_2$ is a limit ordinal, and the C_α satisfy (23.4). A condition $\langle C_\alpha : \alpha \leq \gamma\rangle$ is stronger than $\langle C'_\alpha : \alpha \leq \gamma'\rangle$ if $\gamma \geq \gamma'$ and $C_\alpha = C'_\alpha$ for all $\alpha < \gamma'$. To verify $\aleph_1$-distributivity, let $\dot f$ be a name for a function on ω_1 and let p_0 be a condition. Construct an ω_1-chain of conditions $p_0 \subset p_1 \subset \ldots \subset p_\alpha \subset \ldots$, $\alpha < \omega_1$, such that each $p_{\alpha+1}$ decides $\dot f(\alpha)$ and that each limit ordinal $\alpha < \omega_1$, if $\gamma_\alpha = \lim_{\xi\to\alpha}(\operatorname{dom} p_\xi)$, then $\gamma_\alpha \in \operatorname{dom} p_\alpha$, and for each limit ordinal $\beta < \alpha$, C_{γ_β} is an initial segment of C_{γ_α}. Then if $\gamma = \lim_{\alpha\to\omega_1}\gamma_\alpha$, let $C_\gamma = \bigcup_{\alpha<\omega_1} C_{\gamma_\alpha}$ and $p = \langle C_\xi : \xi \leq \gamma\rangle$; p is a condition and decides $\dot f(\alpha)$ for all $\alpha < \omega_1$.]

23.4. Let κ be regular uncountable, $\alpha < \kappa$ and $\operatorname{cf}\alpha > \omega$. If $S \subset \kappa$ is stationary and if $\operatorname{cf}\beta \geq \operatorname{cf}\alpha$ for all $\beta \in S$, then S does not reflect at α.

[There is a closed unbounded $C \subset \alpha$ such that $\operatorname{cf}\beta < \operatorname{cf}\alpha$ for all $\beta \in C$.]

23.5. Let κ and α be as above, let $\lambda < \kappa$ be regular and $\lambda < \operatorname{cf}\alpha$. Then E^κ_λ reflects at α.

23.6. Let P_S be the forcing (in Theorem 23.8) for shooting a closed unbounded subset of S. Show that every stationary subset of S (in V) remains stationary.

[Let $T \subset S$ be stationary and let $p \Vdash \dot C$ is closed unbounded; find a $q \leq p$ and some $\lambda \in T$ such that $q \Vdash \lambda \in \dot C$: As in Lemma 23.9, construct a chain $\{A_\alpha\}_\alpha$ of countable subsets of P_S and an increasing continuous sequence $\langle \gamma_\alpha : \alpha < \omega_1\rangle$, such that for each $q \in A_\alpha$ there exist some stronger $r(q) \in A_{\alpha+1}$ and $\beta(q) > \gamma_\alpha$ with $r(q) \Vdash \beta \in \dot C$. Then find $\lambda \in T$, and a sequence $\langle p_n : n \in \omega\rangle$ of conditions such that $\lim_n \max(p_n) = \lim_n \beta(p_n) = \lambda$.]

23.7. Let S be a stationary subset of ω_2 such that $S \supset E^{\aleph_2}_{\aleph_0}$ and that $S \cap E^{\aleph_2}_{\aleph_1}$ is stationary. Let P_S be the set of all bounded closed subsets of S (ordered by end-extension). Then P_S preserves $\aleph_2$.

23.8. Let κ be inaccessible and let $S \subset \kappa$ be such that S contains every singular limit ordinal $\alpha < \kappa$. Then P_S is *essentially* $<\kappa$*-closed*, i.e., for every regular $\lambda < \kappa$, P_S has a dense subset that is λ-closed. Hence P_S preserves κ (and adds no λ-sequences for $\lambda < \kappa$).

[For each $\lambda < \kappa$, consider $\{p \in P_S : \max(p) > \lambda\}$.]

23.9. If I_{NS} on ω_1 is $\aleph_2$-saturated then every nontrivial normal κ-complete ideal on ω_1 is $\aleph_2$-saturated.

[Use Exercise 22.11, and that every $S \in I^+$ is stationary.]

23.10. If κ is a regular cardinal then there exists a strongly almost disjoint family $\{X_\xi : \xi < \kappa^+\}$ of subsets of κ.

23.11. It is consistent that $\mathrm{sat}(I_{\mathrm{NS}}) < 2^{\aleph_1}$.

[Assume GCH and add more than $\aleph_4$ Cohen reals. Let $\{S_i : i < \omega_4\} \in V[G]$ be a family of stationary sets such that each $S_i \cap S_j$ is nonstationary. Let $i \neq j$. There exists a nonstationary set $A_{i,j} \supset S_i \cap S_j$ in V. Since the forcing notion is c.c.c., there exists an $A_{i,j} \in I_{\mathrm{NS}}$ such that $\Vdash S_i \cap S_j \subset A_{i,j}$. Apply the Erdős-Rado Theorem (namely $\aleph_4 \to (\aleph_3)^2_{\aleph_2}$) to find some set $H \subset \omega_4$ of size $\aleph_3$ and some $A \in I_{\mathrm{NS}}$ such that $\Vdash S_i \cap S_j \subset A$ for all $i, j \in H$. Get $\aleph_3$ disjoint subsets $S_i - A$ of ω_1 in $V[G]$, a contradiction.]

23.12. There exist stationary sets $S \subset E^{\aleph_3}_{\aleph_0}$ and $A \subset E^{\aleph_3}_{\aleph_1}$ such that S does not reflect at any $\alpha \in A$.

[Let S_i, $i < \omega_2$, be pairwise disjoint stationary subsets of $E^{\aleph_3}_{\aleph_0}$. For each $\alpha \in E^{\aleph_3}_{\aleph_1}$, let $C_\alpha \subset \alpha$ be closed unbounded of size $\aleph_1$. For every α there exists an i_α such that $S_i \cap C_\alpha = \emptyset$ for all $i \geq i_\alpha$. There exists a stationary set $A \subset E^{\aleph_3}_{\aleph_1}$ such that i_α is constant on A, $i_\alpha = i$. The set S_i does not reflect at any $\alpha \in A$.]

Historical Notes

The equivalence in Lemma 23.1 is due to Kunen. Theorem 23.2 is due to Gregory [1976]. Club-guessing principles were introduced by Shelah; see Gitik and Shelah [1997] for details. Lemma 23.6 is due to Jensen.

The construction in Theorem 23.8 (shooting a closed unbounded set) appears in Baumgartner et al. [1976]. Theorem 23.10 uses a construction of Magidor, see Jech et al. [1980]. There is a sequence of results on the strength of precipitousness of I_{NS} on cardinals $\kappa > \aleph_1$: Jech [1984], Gitik [1984, 1995, 1997] See the detailed discussion in Jech [∞].

Theorem 23.17 uses the work of Shelah [1982] (Lemma 23.19 and Corollaries 23.20 and 23.21) and Gitik and Shelah [1997]. The paper Jech and Woodin [1985] investigates saturation of $I_{\mathrm{NS}} \restriction \mathrm{Reg}$ for inaccessible cardinals.

Theorem 23.23 appears in Magidor [1982].

Exercise 23.2: Gitik and Shelah [1997].

Exercise 23.9: Baumgartner et al. [1977],

Exercise 23.11: Baumgartner.

Exercise 23.12: Shelah.