26. The Real Line

This chapter deals with some properties of the real line, primarily with questions concerning measure and category. Among others we present the theorem of Solovay establishing the consistency of the statement "every set of reals is Lebesgue measurable."

Random and Cohen reals

Let us consider generic extensions using either the algebra of Borel sets modulo the ideal of null sets or the algebra of Borel sets modulo the ideal of meager sets.

Let $\mathcal{B}$ be the σ-algebra of Borel sets of reals, let I_m and I_c (m for measure, c for category) be the σ-ideals

$$I_m = \{B \in \mathcal{B} : \mu(B) = 0\}, \qquad I_c = \{B \in \mathcal{B} : B \text{ is meager}\}$$

and let

$$\mathcal{B}_m = \mathcal{B}/I_m = \{[B]_m : B \in \mathcal{B}\}, \qquad \mathcal{B}_c = \mathcal{B}/I_c = \{[B]_c : B \in \mathcal{B}\} \tag{26.1}$$

where $[B]_m$ and $[B]_c$ denote equivalence classes mod I_m and mod I_c, respectively. $\mathcal{B}_m$ and $\mathcal{B}_c$ are complete Boolean algebras and if B_n, $n \in \omega$, are Borel sets then (in either $\mathcal{B}_m$ or $\mathcal{B}_c$),

$$\sum_{n=0}^{\infty} [B_n] = \Big[\bigcup_{n=0}^{\infty} B_n\Big].$$

(Also, $-[B] = [\boldsymbol{R} - B]$).

Forcing with $\mathcal{B}_c$ is the same as adjoining a Cohen generic real, see Exercise 26.1.

Let M be a transitive model of ZF + DC. Let us consider Borel sets in M; let $\mathcal{B}$ denote the collection of all Borel sets in M, and let $\mathcal{B}_m$ and $\mathcal{B}_c$ denote the complete Boolean algebras (26.1) in M.

Let B be a Borel set in M. B has a Borel code $c \in M$, $B = A_c$. Let us denote B^* the Borel set in the universe coded by c. This definition does not depend on the choice of $c \in \mathrm{BC}^M$ because by Lemma 25.45, if $A_c = A_d$, then $A_c^* = A_d^*$. We recall that $B = B^* \cap M$, for every $B \in \mathcal{B}$.

Lemma 26.1. *"A_c is null" and "A_c is meager" are properties absolute for all transitive models of* ZF + DC.

Proof. Let M be a transitive model of ZF + DC. Let μ denote the Lebesgue measure. First we claim that if $c \in M$ is a $\mathbf{\Sigma}^0_1$-code, then $\mu^M(A_c^M) = \mu(A_c)$. Let $k_0, k_1, \ldots, k_n, \ldots$ be all the $k \in \boldsymbol{N}$ such that $c(k) = 1$; thus A_c is the union $\bigcup_{n=0}^\infty I_{k_n}$ of open intervals with rational endpoints. For each n, let $X_n = I_{k_n} - (I_{k_0} \cup \ldots \cup I_{k_{n-1}})$; hence $A_c = \bigcup_{n=0}^\infty X_n$ and $\mu(A_c) = \sum_{n=0}^\infty \mu(X_n)$ is absolute. Hence $\mu^M(A_c^M) = \mu(A_c)$.

A similar argument shows that if $c \in M$ is a $\mathbf{\Pi}^0_1$-code, then $\mu^M(A_c^M) = \mu(A_c)$.

Next we claim that if $c \in M$ is a $\mathbf{\Pi}^0_1$-code, then A_c is nowhere dense if and only if $M \vDash A_c$ is nowhere dense. This is because $d = u(c) \in \Sigma_1$ and it is easily verified (using open rational intervals) that "A_d is dense" is absolute.

Now we are ready to prove the lemma. Let us consider first the property "A_c is null." We use the following properties of Lebesgue measure: (1) X is null if and only if for every $n \in \boldsymbol{N}$, there is an open set $G \supset X$ of measure $\leq 1/n$, and (2) $\mu(X) > 0$ if and only if there is a closed set $F \subset X$ of positive measure.

If $M \vDash A_c$ is null, then M satisfies

$$\forall n\, \exists e\, (e \in \Sigma_1 \text{ and } A_e \supset A_c \text{ and } \mu(A_e) \leq 1/n). \tag{26.2}$$

Since the part $(\ldots)$ of (26.2) is absolute, it is clear that (26.2) holds in V, and hence A_c is null.

If $M \vDash A_c$ is not null, then M satisfies

$$\exists e\, (e \in \Pi_1 \text{ and } A_e \subset A_c \text{ and } \mu(A_e) > 0). \tag{26.3}$$

Again, $(\ldots)$ is absolute, thus (26.3) holds in V and hence A_c is not null.

Finally, we consider the property "A_c is meager." If $M \vDash A_c$ is meager, then M satisfies:

(26.4) There exist $c_n \in \Pi_1$, $n = 0, 1, \ldots$, such that each A_{c_n} is nowhere dense, and $A_c \subset \bigcup_{n=0}^\infty A_{c_n}$.

Then (26.4) holds in V and so A_c is meager.

A Borel set B is not meager if and only if there is a nonempty open set G such that $B \triangle G$ is meager. Thus if $M \vDash A_c$ is not meager, then M satisfies

$$\exists d\, \exists e\, (d \in \Sigma_1 \text{ and } A_d \neq \emptyset \text{ and } A_e = A_c \triangle A_d \text{ and } A_e \text{ is meager}). \tag{26.5}$$

Then (26.5) holds in V and hence A_c is meager. □

As before, it is not necessary that the transitive models in Lemma 26.1 satisfy all of ZF. The properties are absolute for all adequate transitive models, in particular for all transitive models of $\mathrm{ZF}^- + \mathrm{DC}$.

Lemma 26.2.

(i) *If G is an M-generic ultrafilter on $\mathcal{B}_m$, then there is a unique real number x_G such that for all $B \in \mathcal{B}$,*

$$x_G \in B^* \leftrightarrow [B]_m \in G. \tag{26.6}$$

The formula (26.6) *determines G and hence $M[G] = M[x_G]$.*

(ii) *If G is an M-generic ultrafilter on $\mathcal{B}_c$, then there is a unique real number x_G such that for all $B \in \mathcal{B}$,*

$$x_G \in B^* \leftrightarrow [B]_c \in G. \tag{26.7}$$

The formula (26.7) *determines G and hence $M[G] = M[x_G]$.*

Definition 26.3. If x is a real number and if $x = x_G$ for some $G \subset \mathcal{B}_m$ generic over M, then x is *random* over M. If $x = x_G$ for some $G \subset \mathcal{B}_c$ generic over M, then x is *Cohen* over M.

Proof. The same proof works for both (i) and (ii); let $[B]$ denote $[B]_m$ in case (i) and $[B]_c$ in case (ii).

First we claim that there is at most one real number x that satisfies

$$x \in B^* \leftrightarrow [B] \in G \qquad \text{(for all } B \in \mathcal{B}\text{)}. \tag{26.8}$$

If x satisfies (26.8), then x belongs to all B^* such that $[B] \in G$. If $x < y$ are two real numbers, let r be a rational number such that $x < r < y$, and let A be the interval $(r, \infty) = \{z \in \boldsymbol{R} : z > r\}$. Either $[A]$ or $[\boldsymbol{R} - A]$ belongs to G but $x \notin A^*$ and $y \notin (\boldsymbol{R} - A)^*$.

In order to show that there exists a real number x that satisfies (26.8), let

$$x = \sup\{r : r \text{ is a rational number and } [(r, \infty)] \in G\}. \tag{26.9}$$

By the genericity of G, there exists r such that $[(r, \infty)] \notin G$, and hence the supremum (26.9) exists. Note also that $x \notin M$ (by the genericity of G). We shall show that x satisfies (26.8). We shall show, by induction on Borel codes in M, that for every $c \in \mathrm{BC}^M$,

$$x \in A_c^* \leftrightarrow [A_c] \in G. \tag{26.10}$$

First we consider $\boldsymbol{\Sigma}_1^0$-codes (in M), and let us start with those $c \in \Sigma_1 \cap M$ that code a rational interval, i.e., such that $c(n) = 1$ for exactly one n; then c codes the interval I_n. Let $I_n = (p, q)$. We have

$$\begin{aligned} x \in A_c^* \quad &\text{if and only if} \quad p < x < q \\ &\text{if and only if} \quad p < \sup\{r : [(r, \infty)] \in G\} < q \\ &\text{if and only if} \quad [(p, \infty)] \in G \text{ and } [(q, \infty)] \notin G \\ &\text{if and only if} \quad [(p, q)] \in G \\ &\text{if and only if} \quad [A_c] \in G. \end{aligned}$$

Now if $c \in \Sigma_1$, then $A_c = \bigcup_{n=0}^{\infty} I_{k_n}$, where $\{k_n : n = 0, 1, \ldots\}$ is the set $\{k : c(k) = 1\}$, and we have

$$\begin{aligned} x \in A_c^* \quad &\text{if and only if} \quad x \in \textstyle\bigcup_{n=0}^{\infty} I_{k_n}^* && \text{if and only if} \quad \exists n\,(x \in I_{k_n}^*) \\ &\text{if and only if} \quad \exists n\,([I_{k_n}] \in G) && \text{if and only if} \quad \textstyle\sum_{n=0}^{\infty} [I_{k_n}] \in G \\ &\text{if and only if} \quad [\textstyle\bigcup_{n=0}^{\infty} I_{k_n}] \in G && \text{if and only if} \quad [A_c] \in G. \end{aligned}$$

Next let $\alpha < \omega_1^M$ and let $c \in \Pi_\alpha \cap M$, and let us assume that (26.10) holds for all $c \in \Sigma_\alpha \cap M$. We may assume that $c(0) = 0$; then $u(c) \in \Sigma_\alpha \cap M$ and $A_{u(c)} = \boldsymbol{R} - A_c$, and we have

$$\begin{aligned} x \in A_c^* \quad &\text{if and only if} \quad x \notin A_{u(c)}^* \\ &\text{if and only if} \quad [A_{u(c)}] \notin G \quad \text{if and only if} \quad [A_c] \in G. \end{aligned}$$

Finally, the induction step for Σ_α is handled in a way similar to the case for $c \in \Sigma_1$. Thus (26.10) holds for every $c \in \mathrm{BC}^M$, and thus x is the unique real number that satisfies (26.6) (in case of $\mathcal{B}_m$) or (26.7) (in case of $\mathcal{B}_c$). $\square$

The following lemma provides a characterization of random and Cohen reals.

Lemma 26.4. *A real number is random over M if and only if it does not belong to any null Borel set coded in M, and is Cohen over M if and only if it does not belong to any meager Borel set coded in M.*

Hence if $R(M)$ and $C(M)$ denote the sets of all random and all Cohen reals over M, we have

$$\begin{aligned} R(M) &= R^* - \textstyle\bigcup\{A_c^* : c \in \mathrm{BC}^M \text{ and } A_c^* \text{ is null}\}, \\ C(M) &= R^* - \textstyle\bigcup\{A_c^* : c \in \mathrm{BC}^M \text{ and } A_c^* \text{ is meager}\}. \end{aligned} \tag{26.11}$$

Note that by Lemma 26.1, A_c is null (in M) if and only if A_c^* is null (in V).

Proof. On the one hand, if x is random over M, let G be an M-generic ultrafilter on $\mathcal{B}_m$ such that $x = x_G$. Then if A_c is null then $[A_c] \notin G$, and by (26.6), $x \notin A_c^*$. Similarly for x that is Cohen over M.

On the other hand, let x be such that $x \notin A_c^*$ whenever A_c is null (and $c \in M$). First we observe that if $[A_c] = [A_d]$ then $A_c \triangle A_d$ is null, hence $A_c^* \triangle A_d^*$ is null and it follows that x belongs to A_c^* if and only if x belongs to A_d^*. Let

$$G = \{[A_c] : x \in A_c^*\}. \tag{26.12}$$

It is easy to see that G is a filter on $\mathcal{B}_m$: If $[A_c] \in G$ and $[A_d] \in G$, then $x \in A_c^* \cap A_d^*$ and hence $[A_c \cap A_d] \in G$; similarly, if $[A_c] \le [A_d]$ and $[A_c] \in G$, then $[A_d] \in G$.

We shall show that G is M-generic. Since $\mathcal{B}_m$ satisfies the c.c.c., it suffices to show that if $\{A_{c_n} : n \in \omega\} \in M$ is such that $\sum_{n=0}^{\infty}[A_{c_n}] \in G$, then some $[A_{c_n}]$ is in G. But this is true because

$$\sum_{n=0}^{\infty}[A_{c_n}] = \Big[\bigcup_{n=0}^{\infty} A_{c_n}\Big] \quad \text{and} \quad \Big(\bigcup_{n=0}^{\infty} A_{c_n}\Big)^* = \bigcup_{n=0}^{\infty} A^*_{c_n}.$$

Finally, we claim that $x = x_G$. But this follows from (26.12), by the genericity of G. Thus a real number x is random over M if and only if $x \notin A^*_c$ for any null Borel set $A_c \in M$.

The proof is entirely similar for Cohen reals. □

Solovay Sets of Reals

Let M be a transitive model of ZFC. Let S be a set of reals. We say that the set S is *Solovay* over M if there is a formula $\varphi(x)$, with parameters in M, such that for all reals x,

$$x \in S \leftrightarrow M[x] \vDash \varphi(x). \tag{26.13}$$

Lemma 26.5. *Let S be a Solovay set of reals over M. There exist Borel sets A and B such that*

$$S \cap R(M) = A \cap R(M) \quad \textit{and} \quad S \cap C(M) = B \cap C(M).$$

Proof. Let us prove the lemma for random reals. Let us consider the forcing language in M associated with $\mathcal{B}_m$. Let $\dot{G}$ be the canonical name for a generic ultrafilter on $\mathcal{B}_m$, and let $\dot{a}$ be the canonical name for a random real; i.e., let $\dot{a}$ be the $\mathcal{B}_m$-valued name defined in $M^{\mathcal{B}_m}$ from $\dot{G}$, by (26.6): $\|\dot{a} = x_G\| = 1$.

Let $\varphi(x)$ be a formula with parameters in M such that (26.13) holds for all x. Let $A_c \in \mathcal{B}$ be such that $[A_c] = \|\varphi(\dot{a})\|$ and let $A = A^*_c$. The set A is a Borel set (in the universe); we claim that for all $x \in R(M)$, x belongs to S if and only if x belongs to A. But if x is random over M, let G be M-generic on $\mathcal{B}_m$ such that $x = x_G$; then $\dot{a}$ is a name for x and we have

$$x \in S \leftrightarrow M[x] \Vdash \varphi(x) \leftrightarrow M[G] \Vdash \varphi(x) \leftrightarrow \|\varphi(\dot{a})\| \in G \leftrightarrow [A_c] \in G \leftrightarrow x \in A^*_c.$$

□

Corollary 26.6. *Let S be a Solovay set of reals over M.*

(i) *If the set of all reals that are not random over M is null, then S is Lebesgue measurable.*

(ii) *If the set of all reals that are not Cohen over M is meager, then S has the property of Baire.*

Proof. Under the assumptions of the corollary, $S \mathbin{\triangle} A$ is null and $S \mathbin{\triangle} B$ is meager. □

The Lévy Collapse

We review properties of the forcing that collapses uncountable cardinals to $\aleph_0$, and establish the homogeneity of the Lévy collapse.

If λ is an infinite cardinal, let P_λ denote the set of all finite sequences

$$p = \langle p(0), \ldots, p(n-1)\rangle \qquad (n \in \omega) \tag{26.14}$$

of ordinals less than λ and let $\mathrm{Col}(\aleph_0, \lambda) = B(P_\lambda)$.

The following lemma provides a characterization of the collapsing algebra:

Lemma 26.7. *Let $(Q, <)$ be a notion of forcing such that $|Q| = \lambda > \aleph_0$ and such that Q collapses λ onto $\aleph_0$, i.e.,*

$$\|\check{\lambda} \text{ is countable}\|_{B(Q)} = 1.$$

Then $B(Q) = \mathrm{Col}(\aleph_0, \lambda)$.

Proof. Without loss of generality we may assume that $(Q, <)$ is a separative partial ordering. We shall find a dense subset of Q isomorphic to P_λ.

Let $B = B(Q)$, and let $\dot{G}$ be the canonical name for the generic filter on Q. Let $\dot{f} \in V^B$ be such that

$$\|\dot{f} \text{ maps } \check{\omega} \text{ onto } \dot{G}\|_B = 1.$$

For each $p \in P_\lambda$, we shall construct $q(p) \in Q$ such that $D = \{q(p) : p \in P_\lambda\}$ is dense in Q and that $p \mapsto q(p)$ is an isomorphism of P_λ onto D. We construct $q(p)$ by induction on the length of p.

If $p = \langle p(0)\rangle$, we construct $q(p)$ as follows: Since Q collapses λ, there exists an antichain $W \subset Q$ of size λ. Moreover, we may find such W of size λ with the additional property that each $w \in W$ *decides* $\dot{f}(0)$, i.e., there is $q_w \in Q$ such that $w \Vdash \dot{f}(0) = \check{q}_w$. Thus let $W_\emptyset$ be a maximal antichain with these properties, $W_\emptyset = \{w_\xi : \xi < \lambda\}$, and for each $p = \langle p(0)\rangle \in P_\lambda$ we let $q(p) = w_\xi$, where $\xi = p(0)$.

Having constructed $q(p)$, where $p = \langle p(0), \ldots, p(n-1)\rangle$, we construct $q(p^\frown\xi)$, $\xi < \lambda$, as follows: We let $W_p = \{w_\xi : \xi < \lambda\}$ be a maximal antichain below $q(p)$ such that $|W_p| = \lambda$ and that each $w \in W_p$ decides $\dot{f}(n)$. Then we let $q(p^\frown\xi) = w_\xi$, for all $w_\xi \in W_p$.

The set $D = \{q(p) : p \in P_\lambda\}$ is clearly isomorphic to P_λ. Let us show that D is dense in Q. Let $q \in Q$ be arbitrary. Since $q \Vdash \check{q} \in \dot{G}$, and $q \Vdash \check{q} \in \mathrm{ran}(\dot{f})$, there is $r \le q$ and $n < \omega$ such that $r \Vdash q = \dot{f}(n)$. Now there is $p \in P_\lambda$ of length $n+1$ such that $q(p)$ is compatible with r; since $q(p)$ decides $\dot{f}(n)$, we necessarily have $q(p) \Vdash \dot{f}(n) = \check{q}$. Therefore, $q(p) \Vdash \check{q} \in \dot{G}$. Since Q is separative, it follows that $q(p) \le q$. This proves that D is dense in Q. $\square$

Corollary 26.8 (Kripke). *If B is a complete Boolean algebra and $|B| \le \lambda$ then B embeds as a complete subalgebra of $\mathrm{Col}(\aleph_0, \lambda)$.*

Proof. Let B be a complete Boolean algebra, $|B| \le \lambda$. The notion of forcing $Q = B^+ \times P_\lambda$ has cardinality λ and collapses λ. By Lemma 26.7, $B(Q) = \operatorname{Col}(\aleph_0, \lambda)$. In other words, $B \oplus \operatorname{Col}(\aleph_0, \lambda)$ is isomorphic to $\operatorname{Col}(\aleph_0, \lambda)$, and so B is isomorphic to a complete subalgebra of $\operatorname{Col}(\aleph_0, \lambda)$. □

Lemma 26.9. *Let B be a complete Boolean algebra, $|B| = \lambda$. Let C be a complete subalgebra of B such that $|C| < \lambda$, and let h_0 be an embedding of C in $\operatorname{Col}(\aleph_0, \lambda)$. Then there exists an embedding h of B in $\operatorname{Col}(\aleph_0, \lambda)$ such that $h(c) = h_0(c)$ for all $c \in C$.*

Proof. Let D be the image of C under the embedding h_0. Let Col be an abbreviation for $\operatorname{Col}(\aleph_0, \lambda)$; let Col^C and Col^D denote, respectively, the $(\aleph_0, \check{\lambda})$-collapsing algebra in the Boolean valued models V^C and V^D.

First, we find an embedding k of B in $C * \operatorname{Col}^C$: Working in V^C, we observe that $\check{\lambda}$ is a cardinal and that $B : C$ is a complete Boolean algebra that collapses $\check{\lambda}$ onto $\check{\aleph}_0$. Also, since $B : C$ is a quotient of $\check{B}$, $B : C$ has cardinality $\check{\lambda}$. Thus by Corollary 26.8 (in V^C), there is an embedding of $B : C$ in Col^C.

It follows that there is an embedding k of $C * (B : C)$ into $C * \operatorname{Col}^C$ such that $k(c) = c$ for all $c \in C$ (and C is considered a complete subalgebra of both those algebras). Since $C * (B : C) = B$, we have $k : B \to C * \operatorname{Col}^C$ such that $k(c) = c$ for all $c \in C$.

Next we find an isomorphism between Col and $D * \operatorname{Col}^D$: Working in V^D, we observe that $\check{\lambda}$ is a cardinal, and that $\operatorname{Col} : D$ collapses $\check{\lambda}$ onto $\check{\aleph}_0$. Also, since the algebra $\operatorname{Col}^\vee$ has a dense subset $\check{P}_\lambda$ of size $\check{\lambda}$, its quotient $\operatorname{Col} : D$ has a dense subset $\dot{Q}$ of size $\check{\lambda}$. Thus by Lemma 26.7 (in V^D), there is an isomorphism between $\operatorname{Col} : D$ and Col^D.

By the same argument as above, we get an isomorphism π between $\operatorname{Col} = D * (\operatorname{Col} : D)$ and $D * \operatorname{Col}^D$ such that $\pi(d) = d$ for all $d \in D$.

Since C and D are isomorphic, there exists an isomorphism $\sigma : C * \operatorname{Col}^C \to D * \operatorname{Col}^D$ such that $\sigma(c) = h_0(c)$ for all $c \in C$. Thus we define $h : B \to \operatorname{Col}$ as follows: $h(b) = \pi^{-1}(\sigma(k(b)))$, for all $b \in B$:

$$\begin{tikzcd} C * \operatorname{Col}^C \arrow[r, "\sigma"] & D * \operatorname{Col}^D \\ B \arrow[u, "k"] \arrow[r, "h"] & \operatorname{Col} \arrow[u, "\pi"'] \end{tikzcd}$$

Clearly, h is an embedding of B into Col, and $h(c) = h_0(c)$ for all $c \in C$. □

Corollary 26.10. *Let G be a generic filter on P_λ and let X be a set of ordinals in $V[G]$. Then either $V[X] = V[G]$ or there exists a $V[X]$-generic filter H on P_λ such that $V[X][H] = V[G]$.*

Proof. If λ is uncountable in $V[X]$, then $V[G]$ is a generic extension of $V[X]$ by $\operatorname{Col}^{V[X]}(\aleph_0, \kappa)$, where $\kappa = |\lambda|^{V[X]}$. However, P_κ is isomorphic in $V[X]$

to P_λ. If λ is countable in $V[X]$ and $V[X] \neq V[G]$, then $V[G]$ is a generic extension of $V[X]$ by a countable atomless notion of forcing Q. There is only one atomless complete Boolean algebra with a countable dense subset and so $B(Q)$ is isomorphic (in $V[X]$) to $B(P_\lambda)$. $\square$

We now consider the Lévy collapse $\mathrm{Col}(\aleph_0, {<}\lambda)$: Let λ be an inaccessible cardinal. The conditions are functions p on finite subsets of $\lambda \times \omega$ such that $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \mathrm{dom}(p)$; p is stronger than q if $p \supset q$.

Corollary 26.11 (The Factor Lemma). *Let G be a generic filter on the Lévy collapse P, and let X be a countable set of ordinals in $V[G]$. Then there exists a $V[X]$-generic filter H on P such that $V[X][H] = V[G]$.*

Proof. For each $\nu < \lambda$ we have a decomposition of P into $P_\nu \times P^\nu$ where $P_\nu = \{p \in P : \mathrm{dom}\, p \subset \nu \times \omega\}$ and $P^\nu = \{p \in P : \mathrm{dom}\, p \subset (\lambda - \nu) \times \omega\}$. Note that if ν is an infinite cardinal then $|P_{\nu+1}| = \nu$ and so by Lemma 26.7 $B(P_{\nu+1}) = \mathrm{Col}(\aleph_0, \nu)$.

Let $\nu < \lambda$ be such that $X \in V[G \cap P_{\nu+1}]$. By Corollary 26.10 there exists a $K \subset P_{\nu+1}$ generic over $V[X]$ such that $V[G \cap P_{\nu+1}] = V[X][K]$. Hence $V[G] = V[X][K][G \cap P^{\nu+1}]$; let $H = K \times (G \cap P^{\nu+1})$. $\square$

Theorem 26.12 (The Homogeneity of the Lévy Collapse). *Let $B = \mathrm{Col}(\aleph_0, {<}\lambda)$. If A and A' are isomorphic complete subalgebras of B such that $|A| = |A'| < |B|$ and if π_0 is an isomorphism between A and A', then there exists an automorphism π of B such that $\pi(a) = \pi_0(a)$ for all $a \in A$.*

Proof. First we construct increasing sequences of complete subalgebras $A_0 \subset A_1 \subset \ldots \subset A_n \subset \ldots$, and $A'_0 \subset A'_1 \subset \ldots \subset A'_n \subset \ldots$, as follows: We let $A_0 = A$ and $A'_0 = A'$. There is ν_1 such that $A'_0 \subset B_{\nu_1}$; we let $A'_1 = B_{\nu_1}$. The embedding π_0^{-1} of A'_0 in B can be extended to an embedding π_1^{-1} of A'_1 in B, and we let $A_1 = \pi_1^{-1}(A'_1)$. Then there is $\nu_2 > \nu_1$ such that $A_1 \subset B_{\nu_2}$; we let $A_2 = B_{\nu_2}$. Then $\pi_1 : A_1 \to B$ extends to some $\pi_2 : A_2 \to B$, and we let $A'_2 = \pi_2(A_2)$. We proceed in this manner.

Clearly, $\bigcup_{n=0}^\infty A_n = \bigcup_{n=0}^\infty A'_n = \bigcup_{n=1}^\infty B_{\nu_n}$, and $\bigcup_{n=0}^\infty \pi_n$ is an automorphism of this Boolean algebra. This automorphism extends to a unique automorphism π_ω of $B_\nu = B(P_\nu)$, where $\nu = \lim_{n\to\infty} \nu_n$.

Now $B = B_\nu \oplus B^\nu$ where $B^\nu = B(P^\nu)$, and the automorphism π_ω of B_ν can be extended to an automorphism π of $B_\nu \oplus B^\nu$ by $\pi(u, v) = (\pi_\omega u, v)$. $\square$

Corollary 26.13. *If u and v are elements of $\mathrm{Col}(\aleph_0, {<}\lambda)$ such that $u \neq 0, 1$ and $v \neq 0$, 1, then there exists an automorphism π of B such that $\pi(u) = v$.* $\square$

It follows that for any formula φ and all $x_1, \ldots, x_n$, $\|\varphi(\check{x}_1, \ldots, \check{x}_n)\|_B$ is either 1 or 0.

Solovay's Theorem

Theorem 26.14 (Solovay). *Assume that there exits an inaccessible cardinal.*

(i) *There is a model of* ZF + DC *in which all sets of real numbers are Lebesgue measurable and have the property of Baire, and every uncountable set of reals has a perfect subset.*
(ii) *There is a model of* ZFC *in which every projective set of reals is Lebesgue measurable, has the Baire property, and if uncountable, then it contains a perfect subset.*

Let M be a transitive model of ZFC and let κ be an inaccessible cardinal in M. Let B be the Lévy collapse for κ, i.e., $B = B(P)$ where P is the notion of forcing that collapses each $\alpha < \kappa$ onto $\aleph_0$: The conditions are functions p on subsets of $\kappa \times \omega$ such that each $\operatorname{dom}(p)$ is finite, and $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \operatorname{dom}(p)$.

Let G be an M-generic ultrafilter on B. We shall show that in $M[G]$ every projective set of reals is Lebesgue measurable, has the property of Baire, and if uncountable, then it contains a perfect subset.

In $M[G]$, let S be the class of all infinite sequences of ordinal numbers, $S = Ord^\omega$, and let $N = HOD(S)$ be the class of all sets hereditarily ordinal definable over S. The class N is a model of ZF; in fact, N is a model of ZF + DC (see Lemma 26.15 below), and we shall show that in N every set of reals is Lebesgue measurable, has the Baire property, and if uncountable, then it contains a perfect subset.

Let s be an infinite sequence of ordinals in $M[G]$, let φ be a formula, and let $X \in M[G]$ be a set such that

$$X = \{x : M[G] \vDash \varphi(x, s)\}. \tag{26.15}$$

X is (in $M[G]$) ordinal definable over $S = Ord^\omega$. Conversely, if $X \in OD(S)$, then for some formula ψ and a finite sequence $\langle s_1, \ldots, s_k \rangle$ of elements of S, $X = \{x \in M[G] : \psi(x, \langle s_1, \ldots, s_k \rangle)\}$. Then clearly there exist φ and $s \in S$ such that (26.15) holds. Hence the class $OD(S)$ consists of all sets X of the form (26.15)—*sets definable in $M[G]$ from a sequence of ordinals.*

Note that every projective set of reals is definable from a sequence of ordinals: If A is $\Sigma^1_n(a)$ for some $a \in \mathcal{N}$, then A is definable in HC from a, and therefore $A \in OD(S)$.

Lemma 26.15.

(i) *If $f \in M[G]$ is a function on ω with values in N, then $f \in N$.*
(ii) *The model N satisfies the Principle of Dependent Choices.*

Proof. (i) We show that if f is a function from ω into $OD(S)$ then $f \in OD(S)$. By (26.15), $OD(S) = \bigcup\{OD(s) : s \in S\}$; therefore there is a definable function F on $Ord \times S$ such that for each $s \in S$, the function $F_s(\alpha) = F(\alpha, s)$ maps Ord onto $OD(s)$. Let $f : \omega \to OD(S)$. For each n, we choose α_n and s_n such that $f(n) = F(\alpha_n, s_n)$. Clearly, f is definable from $\langle \alpha_n : n \in \omega \rangle$ and $\langle s_n : n \in \omega \rangle$. It is easy to find a single sequence u of ordinals such that both $\langle \alpha_n : n \in \omega \rangle$ and $\langle s_n : n \in \omega \rangle$ are definable from u. Hence f is definable from u, and so $f \in OD(S)$.

(ii) In N, let ρ be a relation over a nonempty set A such that for every $x \in A$ there is a y such that $y \,\rho\, x$. Since $M[G]$ satisfies the Axiom of Choice, there exists in $M[G]$ a sequence $a_0, a_1, \ldots, a_n, \ldots$ such that $a_{n+1} \,\rho\, a_n$ for all n. However, by part (i) of this lemma, the sequence $\langle a_n : n \in \omega \rangle$ is in N. $\square$

We shall now prove the part of Theorem 26.14 dealing with Lebesgue measure and the Baire property, using Lemma 26.5.

Lemma 26.16. *Let $s \in M[G]$ be an infinite sequence of ordinals. The set of all reals (in $M[G]$) that are not random over $M[s]$ is null; the set of all reals that are not Cohen over $M[s]$ is meager.*

Proof. Since the algebra B is κ-saturated, there exists a subalgebra $D \subset B$ such that $|D| < \kappa$ and $M[s] = M[D \cap G]$. It follows that κ is inaccessible in $M[s]$; and since $\kappa = \aleph_1^{M[G]}$, $M[s]$ has only countably many subsets of ω. Thus there are only countably many Borel codes in $M[s]$; and by (26.11), the complement of the set $R(M[s])$ is the union of countably many null sets and hence null. Similarly, the complement of $C(M[s])$ is meager. $\square$

Lemma 26.17. *Let $X \in M[G]$ be a set of reals that is definable in $M[G]$ from a sequence s of ordinals. Then X is (in $M[G]$) Solovay over $M[s]$.*

Proof. The proof uses the properties of the Lévy collapse discussed above, in particular the Factor Lemma. We shall first prove the following: Given a formula φ, there is a formula $\tilde{\varphi}$ such that for every sequence of ordinals $x \in M[G]$,

$$M[G] \vDash \varphi(x) \quad \text{if and only if} \quad M[x] \vDash \tilde{\varphi}(x). \tag{26.16}$$

The forcing conditions are finite and so the definition of the Lévy collapse P is absolute for all models. We denote M^P the Boolean valued model constructed in M using P and if ψ is a formula and $z \in M^P$, we denote $\|\psi(z)\|^M$ the Boolean value (computed in M using P) of $\psi(z)$. If $a \in M$, then $\check{a} \in M^P$ is the canonical name for a.

Let $\tilde{\varphi}(x)$ be the following formula

$$\|\varphi(\check{x})\|^{M[x]} = 1. \tag{26.17}$$

Let x be a countable sequence of ordinals in $M[G]$; we shall show that $M[G] \vDash \varphi(x)$ if and only if $M[x] \vDash \tilde{\varphi}(x)$. By the Factor Lemma there exists an $M[x]$-generic filter H on P such that $M[G] = M[x][H]$. Arguing in $M[x]$, we invoke the homogeneity of the Lévy collapse: The Boolean value $b = \|\varphi(\check{x})\|^{M[x]}$ is either 0 or 1. Since H is generic on P over $M[x]$, $\varphi(x)$ is true in $M[x][H]$ if $b = 1$ and false if $b = 0$. Hence $\varphi(x)$ is true in $M[G]$ if and only if $\tilde{\varphi}(x)$ is true in $M[x]$.

For a formula φ with two variables there is a formula $\tilde{\varphi}$ such that for all $x, y \in M[G] \cap Ord^\omega$,

$$M[G] \vDash \varphi(x,y) \quad \text{if and only if} \quad M[x,y] \vDash \tilde{\varphi}(x,y).$$

Now let $X \in M[G]$ be a set of reals that is definable in $M[G]$ from a sequence of ordinals s. For some formula φ

$$x \in X \leftrightarrow M[G] \vDash \varphi(x,s)$$

for all reals $x \in M[G]$. Thus we have, for all $x \in \boldsymbol{R}^{M[G]}$,

$$x \in X \leftrightarrow M[s][x] \vDash \tilde{\varphi}(s,x)$$

which shows that X is Solovay over $M[s]$. □

Corollary 26.18. *In $M[G]$ every set of reals definable from a sequence of ordinals (and in particular, every projective set of reals) is Lebesgue measurable and has the property of Baire.*

Proof. This follows from Lemmas 26.5 26.16, and 26.17. □

Corollary 26.19. *In N, every set of reals is Lebesgue measurable and has the property of Baire.*

Proof. Clearly, the model N has the same reals as the model $M[G]$. In particular, N and $M[G]$ have the same Borel codes, and since $A_c^M = A_c^{M[G]} \cap N = A_c^{M[G]}$ for every $c \in \mathrm{BC}^{M[G]}$, the two models have the same Borel sets.

If $X \in N$ is a set of reals, then X is definable in $M[G]$ from a sequence of ordinals and hence $M[G] \vDash$ (X is Lebesgue measurable and has the Baire property). Thus there are (in $M[G]$) Borel sets A, B, H, K such that $X \triangle A \subset H$, $X \triangle B \subset K$, and H is null and K is meager (in $M[G]$). By Lemma 26.1, N satisfies that H is null and K is meager, and hence N satisfies that X is Lebesgue measurable and has the Baire property. □

We shall now finish the proof of Theorem 26.14 by showing that in $M[G]$ every uncountable set of reals definable from a countable sequence of ordinals contains a perfect subset. Then it follows that in N, every uncountable set A of reals has a perfect subset: If A is uncountable in N, then A is uncountable in $M[G]$ (by Lemma 26.15); and since A is definable from a sequence of ordinals, A has a perfect subset F (in $M[G]$); but then $N \vDash F$ is a perfect set.

By Lemma 26.17, every set of reals definable in $M[G]$ from s is Solovay over $M[s]$; thus it suffices to prove that in $M[G]$ every uncountable set of reals, Solovay over $M[s]$, contains a perfect subset. Furthermore, it suffices to give the proof only for sets of reals Solovay over M since the general case (Solovay over $M[s]$) follows from the special case by the Factor Lemma: $M[G] = M[s][H]$ is a generic extension of $M[s]$ by the Lévy collapse. And finally, we can consider subsets of the Cantor space instead of sets of reals.

Thus let A be, in $M[G]$, an uncountable subset of $\{0,1\}^\omega$, and let φ be a formula (with parameters in M) such that for all $x \in \{0,1\}^\omega$ in $M[G]$,

$$x \in A \quad \text{if and only if} \quad M[x] \vDash \varphi(x).$$

Since A is uncountable, there exists an $x \in A$ such that $x \notin M$. There exists (in M) a complete subalgebra $C \subset B$ such that $|C| < \kappa$ and that $x \in M[G \cap C]$. Let us consider the Boolean-valued model M^C and the corresponding forcing relation $\Vdash$. There exists a name $\dot{x} \in M^C$ and a condition $p \in C \cap G$ such that

$$p \Vdash \dot{x} \in \{0,1\}^\omega \text{ and } \dot{x} \notin M \text{ and } (M[\dot{x}] \vDash \varphi(\dot{x})). \tag{26.18}$$

Since $P^M(C)$ is countable in $M[G]$, let D_0, D_1, ..., D_n, ... be an enumeration (in $M[G]$) of all open dense subsets of C in M.

We shall construct conditions $p_s \in C$, for all finite 0–1 sequences s, as follows:

Let $p_\emptyset \le p$ be such that $p_\emptyset \in D_0$. Given p_s, there exists $n_s \in \omega$ such that p_s does not decide $\dot{x}(n_s)$ (because $p \Vdash \dot{x} \notin M$), and we let $p_{s^\frown 0}$ and $p_{s^\frown 1}$ be such that $p_{s^\frown 0} \Vdash \dot{x}(n_s) = 0$ and $p_{s^\frown 1} \Vdash \dot{x}(n_s) = 1$; moreover, we choose $p_{s^\frown 0}$ and $p_{s^\frown 1}$ so that both are in the open dense set D_k where k is the length of s.

For every $z \in \{0,1\}^\omega$, let $G_z = \{p \in C : p \ge p_s \text{ for some } s \subset z\}$. Clearly, $G_z \cap D_n \ne \emptyset$ for every n, and hence G_z is an M-generic ultrafilter on C. Let $f(z) = \dot{x}^{G_z}$ be the interpretation of $\dot{x}$ by G_z. Since G_z is generic, and by (26.18), we have $f(z) \in A$. Thus f is a function from $\{0,1\}^\omega$ into A.

It follows from the construction of f that f is one-to-one and continuous. Thus $f(\{0,1\}^\omega)$, the one-to-one continuous image of a perfect compact set, is a perfect subset of A. $\square$

Lebesgue Measurability of Σ^1_2 Sets

Lemma 26.5 and its Corollary 26.6 provide the following equivalences:

Theorem 26.20 (Solovay). *Let $a \in \mathcal{N}$.*

(i) *Every $\Sigma^1_2(a)$ set of reals is Lebesgue measurable if and only if almost all reals are random over $L[a]$.*

(ii) *Every* $\Sigma^1_2(a)$ *set of reals has the Baire property if and only if the set* $\{x : x$ *is not a Cohen real over* $L[a]\}$ *is meager.*

Proof. We prove only part (i) as part (ii) is similar.

First we note that every $\Sigma^1_2(a)$ set is Solovay over $L[a]$: Let A be $\Sigma^1_2(a)$, and let $T \in L[a]$ be a tree on $\omega \times \omega_1$ such that for all $x \in \mathcal{N}$,

$$x \in A \quad \text{if and only if} \quad T(x) \text{ is ill-founded.}$$

By absoluteness of well-foundedness we have

$$x \in A \quad \text{if and only if} \quad L[a][x] \vDash T(x) \text{ is ill-founded,}$$

and hence A is Solovay over $L[a]$.

If almost all reals are random over $L[a]$ then every $\Sigma^1_2(a)$ set is Lebesgue measurable by Corollary 26.18.

Thus assume that every $\Sigma^1_2[a]$ set is Lebesgue measurable; we shall prove that the union

$$B = \bigcup\{A_c : c \in \mathrm{BC},\ c \in L[a] \text{ and } A_c \text{ is null}\}$$

of all null Borel sets coded in $L[a]$ is null. Let

$$\begin{aligned} &C(x, c) \leftrightarrow c \in \mathrm{BC} \wedge A_c \text{ is null} \wedge x \in A_c,\\ &D(x, c) \leftrightarrow C(x, c) \wedge c \in L[a] \wedge \forall d\, (d <_{L[a]} c \rightarrow \neg C(x, d)),\end{aligned}$$

and for $x, y \in B$,

$$x \preccurlyeq y \leftrightarrow \exists c\, \exists d\, (D(x, c) \wedge D(x, d) \wedge c \leq_{L[a]} d).$$

The set B as well as the relations C, D and $\preccurlyeq$ are $\Sigma^1_2(a)$, and $\preccurlyeq$ is a prewellordering of B. Under the assumption of Lebesgue measurability of $\Sigma^1_2(a)$ sets, B is Lebesgue measurable and $\preccurlyeq$ is a measurable subset of $\mathcal{N} \times \mathcal{N}$.

The order-type of $\mathcal{N}$ in $<_{L[a]}$ is $\omega_1^{L[a]} \leq \omega_1$. Hence for every $y \in B$, the set $\{x : x \preccurlyeq y\}$ is a countable union of null sets and therefore null. Thus $\preccurlyeq$ is a null set, by Fubini's Theorem. By the same argument, the complement of $\preccurlyeq$ in $B \times B$ is null as well, and hence $B \times B$ is null. Therefore B is a null set. □

Corollary 26.21. *If* $\omega_1^{L[a]} < \omega_1$, *then every* $\Sigma^1_2(a)$ *set of reals is Lebesgue measurable and has the Baire property.*

Proof. Under the assumption, each $L[a]$ has only countably many reals and hence only countably many Borel codes, and it follows that almost all reals are random over $L[a]$. Similarly for Cohen reals. □

Ramsey Sets of Reals and Mathias Forcing

For an infinite set $A \subset \omega$, let $[A]^\omega$ denote the set of all infinite subsets of A. Let us consider the following partition property for $[\omega]^\omega$: If $S \subset [\omega]^\omega$, we call an infinite set $H \subset \omega$ *homogeneous* for S if either $[H]^\omega \subset S$ or $[H]^\omega \cap S = \emptyset$. A set $S \subset [\omega]^\omega$ is a *Ramsey* set if there exists an infinite homogeneous set H for S.

A consequence of the Axiom of Choice is that not every set $S \subset [\omega]^\omega$ is Ramsey (Exercise 26.3). We prove that the Axiom of Choice is necessary, and that all analytic sets are Ramsey.

Identifying subsets of ω with their characteristic functions, we consider $[\omega]^\omega$ as a G_δ subspace of the Cantor space. We prove the following theorems:

Theorem 26.22 (Galvin-Prikry, Silver). *Every analytic subset of $[\omega]^\omega$ is Ramsey.*

Theorem 26.23 (Mathias). *Let $M[G]$ and N be the models from Theorem* 26.14.

(i) *In N, every subset of $[\omega]^\omega$ is Ramsey.*
(ii) *In $M[G]$, every projective subset of $[\omega]^\omega$ is Ramsey.*

The method of proof of both theorems uses a notion of forcing introduced by Mathias, and a topology based on the Mathias forcing.

Definition 26.24 (Mathias Forcing). A forcing condition is a pair (s, A) where s is a finite subset of ω and A is an infinite subset of ω such that $\max s < \min A$. A condition (s, A) is stronger than a condition (t, B) if

(26.19)
(i) t is an initial segment of s;
(ii) $A \subset B$;
(iii) $s - t \subset B$.

(Compare this with the Prikry forcing (21.15).) For the rest of this section, (s, A) will denote a Mathias forcing condition.

For $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, let $A \setminus s = A - (\max(s)+1) = \{n \in A : n > k$ for all $k \in s\}$, and

$$[s, A]^\omega = \{X \in [\omega]^\omega : s \subset X \text{ and } X \setminus s \subset A\}. \tag{26.20}$$

Note that $[\emptyset, A]^\omega = [A]^\omega$, and $[s, A]^\omega \subset [t, B]^\omega$ if and only if (s, A) is stronger than (t, B).

Definition 26.25. The *Ellentuck topology* on $[\omega]^\omega$ has as basic open sets the sets of the form $[s, A]^\omega$ where $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$.

Note that every open set in the usual topology is open in the Ellentuck topology.

Definition 26.26 (Galvin-Prikry).

(i) A set $S \subset [\omega]^\omega$ is *completely Ramsey* if for every (s, A) there exists an infinite $H \subset A$ such that either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$.
(ii) A set $N \subset [\omega]^\omega$ is *Ramsey null* if for every (s, A) there exists an infinite $H \subset A$ such that $[s, H]^\omega \cap S = \emptyset$.

We first observe that every Ramsey null set is nowhere dense in the Ellentuck topology: S is nowhere dense if and only if for every basic open set there exists a basic open subset disjoint from S, i.e.,

$$\forall (s, A)\, \exists (t, B) < (s, A)\, [t, B]^\omega \cap S = \emptyset.$$

Let S be completely Ramsey, let $\mathrm{int}(S)$ denote the interior of S (in the Ellentuck topology), and let $N = S - \mathrm{int}(S)$. For every (s, A) there exists an $H \subset A$ such that either $[s, H]^\omega \subset S$, and since $[s, H]^\omega$ is open, we have $[s, H]^\omega \subset \mathrm{int}(S)$; or $[s, H]^\omega \cap S = \emptyset$, and in either case $[s, H]^\omega \cap N = \emptyset$. Hence N is Ramsey null, and therefore nowhere dense. It follows that $S = \mathrm{int}(S) \cup N$ has the Baire property.

We shall prove the following (for the Ellentuck topology):

Lemma 26.27.

(i) *A set S is completely Ramsey if and only if it has the Baire property.*
(ii) *A set N is Ramsey null if and only if it is nowhere dense if and only if it is meager.*

Toward the proof of Lemma 26.27, let S be a given subset of $[\omega]^\omega$. Given (s, A) we say that A *accepts* s if $[s, A]^\omega \subset S$; we say that A *rejects* s if no $X \subset A$ accepts s.

Lemma 26.28. *There is an X that accepts or rejects each of its finite subset.*

Proof. Let X_0 be such that X_0 either accepts or rejects $\emptyset$ (if no X accepts $\emptyset$ then $X_0 = \omega$ rejects $\emptyset$). Let a_0 be the least element of X_0. Let $X_1 \subset X$ be such that X_1 either accepts or rejects each subset of $\{a_0\}$. Let a_1 be the least element of $X_1 \setminus \{a_0\}$, and let $X_2 \subset X_1$ be such that X_2 accepts or rejects each subset of $\{a_0, a_1\}$. We continue in this fashion and construct a set $X = \{a_0, a_1, a_2, \ldots\}$. This X accepts or rejects each of its finite subsets. $\square$

Lemma 26.29. *There is a Y that either accepts $\emptyset$ or rejects each of its finite subsets.*

Proof. Let X be as in Lemma 26.28, and assume that it rejects $\emptyset$. We construct $Y = \{a_0, a_1, \ldots\} \subset X$ as follows: Assume we have constructed $a_0, \ldots, a_{n-1}$ such that X rejects each subset of $\{a_0, \ldots, a_{n-1}\}$. For every $s \subset \{a_0, \ldots, a_{n-1}\}$ there are only finitely many $z \in X$ such that X accepts

$s \cup \{z\}$ (otherwise there is an infinite $Z \subset X$ such that X accepts $s \cup \{z\}$ for each $z \in Z$; then Z accepts s and hence X does not reject s). Therefore we can find $a_n \in X \smallsetminus \{a_0, \ldots, a_{n-1}\}$ such that X rejects each subset of $\{a_0, \ldots, a_n\}$. ☐

Lemma 26.30. *Every open set is Ramsey.*

Proof. Let S be open, and let X be as in Lemma 26.29. If X accepts $\emptyset$ then $[X]^\omega = [\emptyset, X]^\omega \subset S$.

If X rejects each of its finite subsets, we claim that $[X]^\omega \cap S = \emptyset$. Otherwise, there is an infinite $Y \subset X$ such that $Y \in S$. Since S is open, there is an open neighborhood of Y included in S; i.e., there exists a finite $s \subset Y$ such that $[s, Y \smallsetminus s]^\omega \subset S$. Hence Y accepts s, contrary to the assumption that X rejects s. ☐

Lemma 26.31. *Every open set is completely Ramsey.*

Proof. Let S be open and let (s, A) be arbitrary. Let $f : \omega \to A$ be a one-to-one increasing enumeration of A, and for each $X \in [\omega]^\omega$, let $f^*(X) = s \cup f``X$. The function $f^*$ is a continuous function from $[\omega]^\omega$ into $[\omega]^\omega$. Let $T = \{X : f^*(X) \in S\}$; $T$ is open and hence Ramsey. If $K$ is a homogeneous set for $T$, then $H = f``K$ satisfies either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$. ☐

Lemma 26.32. *Every nowhere dense set is Ramsey null.*

Proof. Let S be nowhere dense; we may also assume that S is closed. Let (s, A) be arbitrary. By Lemma 26.31 there is an $H \subset A$ such that either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$. But $[s, H]^\omega \subset S$ is impossible since S is nowhere dense. ☐

Lemma 26.33. *If $S = \bigcup_{n=0}^\infty S_n$ and each S_n is Ramsey null then S is Ramsey null.*

Proof. Let (s, A) be arbitrary. We construct an infinite $H = \{a_0, a_1, a_2, \ldots\} \subset A$ as follows: Let $X_0 \subset A$ be such that $[s, X_0]^\omega \cap S_0 = \emptyset$, and let a_0 be the least element of X_0. Find an $X_1 \subset X_0 \smallsetminus \{a_0\}$ such that for every t with $s \subset t \subset s \cup \{a_0\}$, $[t, X_1]^\omega \subset S_1$. Having constructed $X_0 \supset X_1 \supset \ldots \supset X_n$, let $a_n = \min(X_n)$ and find an $X_{n+1} \subset X_n \smallsetminus \{a_0, \ldots, a_n\}$ such that for every t with $s \subset t \subset s \cup \{a_0, \ldots, a_n\}$, $[t, X_{n+1}]^\omega \subset S_{n+1}$. It follows that $[s, H]^\omega \cap S = \emptyset$. ☐

Proof of Lemma 26.27. By Lemmas 26.32 and 26.33, every meager set is Ramsey null, proving (ii). To prove (i), let S be a set with the Baire property; we have $S = G \triangle M$ where G is open and M is meager. Let (s, A) be arbitrary. By (ii) there is some $X \subset A$ such that $[s, X]^\omega \cap M = \emptyset$. By Lemma 26.31 there is some $H \subset X$ such that either $[s, H]^\omega \subset G$ or $[s, H]^\omega \cap G = \emptyset$. It follows that either $[s, H]^\omega \subset S$ or $[s, H]^\omega \cap S = \emptyset$. ☐

Proof of Theorem 26.22. Every analytic set (in the usual topology) is the result of the Suslin operation $\mathcal{A}$ applied to closed sets. Every closed set is closed in the Ellentuck topology and therefore has the Baire property (in the Ellentuck topology). It can be proved (as in Theorem 11.18) that the Baire property in the Ellentuck topology is preserved under the operation $\mathcal{A}$. Hence every analytic set is completely Ramsey, by Lemma 26.27(i). $\square$

The combinatorial content of Lemma 26.27 is this property of Mathias forcing (compare with Lemma 21.12):

Lemma 26.34. *Let σ be a sentence of the forcing language and let (s, A) be a condition. Then there exists an infinite set $B \subset A$ such that (s, B) decides σ.*

Proof. Let $Q^+ = \{p : p \Vdash \sigma\}$, $Q^- = \{p : p \Vdash \neg\sigma\}$, $S^+ = \bigcup\{[t, X]^\omega : (t, X) \in Q^+\}$ and $S^- = \bigcup\{[t, X]^\omega : (t, X) \in Q^-\}$. Since the complement of $S^+ \cup S^-$ is nowhere dense, there exists, by Lemma 26.27, an infinite $B \subset A$ such that $[s, B]^\omega \subset S^+$ or $[s, B]^\omega \subset S^-$. We claim that in the former case $(s, B) \Vdash \sigma$ and in the latter case $(s, B) \Vdash \neg\sigma$. This is because for every $(t, X) < (s, B)$ there exists some $(u, Y) < (t, X)$ which is in Q^+ (or Q^-). $\square$

If G is a generic filter on the Mathias forcing (over a ground model M), let x_G be the infinite set

$$x_G = \bigcup\{s : (s, A) \in G \text{ for some } A\}; \tag{26.21}$$

x_G is called a *Mathias real* (over M). The filter G is determined by $x = x_G$, as

$$G = G_x = \{(s, A) : s \subset x \subset s \cup A\}. \tag{26.22}$$

Mathias reals admit the following characterization, analogous to Theorem 21.14:

Theorem 26.35 (Mathias). *Let M be a transitive model of* ZFC. *An infinite set $x \subset \omega$ is a Mathias real over M if and only if for every maximal almost disjoint family $\mathcal{A} \in M$ of subsets of ω, there exists an $X \in \mathcal{A}$ such that $x - X$ is finite.*

Proof. The condition is necessary: If $\mathcal{A}$ is a maximal almost disjoint family then $D = \{(s, A \setminus s) : s \in [\omega]^{<\omega}, A \in \mathcal{A}\}$ is a predense set of forcing conditions, and it follows that if x is a Mathias real then $G_x \cap D \neq \emptyset$.

For the proof of sufficiency, let D be an open dense set of Mathias forcing conditions (in the ground model). We need a more detailed analysis of Mathias forcing. If $X \subset \omega$ is infinite and $\max s < \min X$ we say that X *captures* (s, D) if for every infinite $Y \subset X$ there exists an initial segment t of Y such that $(s \cup t, X) \in D$.

Lemma 26.36. *For every infinite set $A \subset \omega$ and for every finite $s \subset \omega$ there exists an infinite set $X \subset A \smallsetminus s$ such that X captures (s, D).*

Proof. We construct a sequence $Y_0 \supset Y_1 \supset \ldots \supset Y_n \supset \ldots$ of infinite sets and a sequence $m_0 < m_1 < \ldots < m_n < \ldots$ such that $m_n = \min Y_n$, as follows: Let $Y_0 = A \smallsetminus s$. Given Y_n, we can find $Y_{n+1} \subset Y_n \smallsetminus \{m_n\}$ with the property that for every $t \subset \{m_0, \ldots, m_n\}$, if there exists a $Y \subset Y_n$ such that $(s \cup t, Y) \in D$, then $(s \cup t, Y_{n+1}) \in D$ (we use the fact that D is an open set of conditions).

Let $Y = \{m_0, m_1, \ldots, m_n, \ldots\}$. As the set $U = \bigcup\{[t, S]^\omega : (t, S) \in D\}$ is a dense open subset of $[\omega]^\omega$ (in the Ellentuck topology) it follows from Lemma 26.27(ii) that there exists an infinite set $X \subset Y$ such that $[s, X]^\omega \subset U$. We claim that X captures (s, D).

If $Z \subset X$ is infinite then because $s \cup Z \in U$, there exist an initial segment t of Z and an infinite $S \subset \omega$ such that $(s \cup t, S) \in D$ and $s \cup Z \in [s \cup t, S]^\omega$. It follows that $(s \cup t, Z \smallsetminus t) \in D$, and if $\max t = m_n$, we have $(s \cup t, Y_{n+1}) \in D$. It follows that $(s \cup t, X \smallsetminus t) \in D$. ◻

Lemma 26.37. *For every infinite $A \subset \omega$ there exists an $X \subset A$ such that for every s, $X \smallsetminus s$ captures (s, D).*

Proof. By Lemma 26.36 there exist sets $X_s \subset A$ such that for each s, X_s captures (s, D). We construct $X_0 \supset X_1 \supset \ldots \supset X_n \supset \ldots$ and $m_0 < m_1 < \ldots < m_n < \ldots$ such that $m_n = \min X_n$, as follows: Let $X_0 = X_\emptyset$. Given X_n, we find an X_{n+1} such that for every s with $\max s = m_n$, X_{n+1} captures (s, D) (here we use the fact that if X captures and $X' \subset X$, then X' also captures). Let $X = \{m_0, m_1, \ldots, m_n, \ldots\}$. It follows that $X \smallsetminus s$ captures (s, D) for every s. ◻

We now finish the proof of Theorem 26.35. Let $x \subset \omega$ be infinite and assume that for every maximal almost disjoint $\mathcal{A} \in M$ there exists an $X \in \mathcal{A}$ such that $x - X$ is finite. By Lemma 26.37 there exists (in M) a maximal almost disjoint family $\mathcal{A}$ such that for every $X \in \mathcal{A}$ and every s, $X \smallsetminus s$ captures (s, D). Let $X \in \mathcal{A}$ be such that $x - X$ is finite, and let s be an initial segment of x such that $x \subset s \cup X$. As $X \smallsetminus s$ captures (s, D), we have (in M)

$$\forall \text{ infinite } Y \subset X \smallsetminus s \ \exists \text{ initial segment } t \subset Y \text{ such that } (s \cup t, X \smallsetminus t) \in D. \tag{26.23}$$

Consider the set of finite sets $W = \{t \subset X \smallsetminus s : (s \cup t, X \smallsetminus t) \notin D\}$ partially ordered by the relation $t \preccurlyeq t'$ if and only if t' is an initial segment of t. Then (26.23) states that $(W, \prec)$ is well-founded in M. By absoluteness, $(W, \prec)$ is well-founded in any larger universe, and so (26.23) holds in any $V \supset M$. In particular, letting $Y = x \smallsetminus s$, we obtain an initial segment t of $x \smallsetminus s$ such that $(s \cup t, X \smallsetminus t) \in D$, and since $s \cup t$ is an initial segment of x and $x \subset s \cup t \cup X$, the filter G_x from (26.22) meets D. Since D was an arbitrary open dense set in M, G_x is generic, and x is a Mathias real over M. ◻

Corollary 26.38. *If x is a Mathias real over M and $y \subset x$ is infinite, then y is Mathias over M.*

Proof of Theorem 26.23. Let $M[G]$ be a generic extension of M by the Lévy collapse. We shall prove that every set of reals in $M[G]$ that is definable from a countable sequence of ordinals is Ramsey. Thus let u be a countable sequence of ordinals in $M[G]$ and let $X \in M[G]$ be a subset of $[\omega]^\omega$ definable from u. By Lemma 26.17 X is a Solovay set over $M[u]$ and so for some formula φ,

$$x \in X \quad \text{if and only if} \quad M[u, x] \models \varphi(x)$$

for all $x \in [\omega]^\omega \cap M[G]$.

Let us consider the Mathias forcing in $M[u]$, and let $\dot{x}$ be the canonical name for a Mathias generic real. By Lemma 26.34 there exists an infinite set $A \in M[u]$ such that $(\emptyset, A)$ decides $\varphi(\dot{x})$. Assume that $(\emptyset, A) \Vdash \varphi(\dot{x})$ as the other case is similar.

Since $\aleph_1^{M[G]}$ is inaccessible in $M[u]$, there exists a Mathias generic filter in $M[G]$ containing $(\emptyset, A)$; therefore there exists a Mathias real x over $M[u]$ such that $x \subset A$. We complete the proof by verifying that $[x]^\omega \subset X$.

If y is an infinite subset of x then by Corollary 26.38, y is a Mathias real over $M[u]$. Since $y \subset A$ and $(\emptyset, A) \Vdash \varphi(\dot{x})$, we have $M[u][y] \models \varphi(y)$ and so $y \in X$. ◻

Measure and Category

Lebesgue measure and Baire property have been the most thoroughly investigated properties of sets of reals, both in the classical descriptive set theory, and in the modern era of independence results. We shall touch briefly on the subject, with emphasis on the role of Martin's Axiom and combinatorial "cardinal invariants." We start with the following application of Martin's Axiom:

Theorem 26.39 (Martin-Solovay). *If Martin's Axiom holds, then the union of fewer than $2^{\aleph_0}$ null sets is null, and the union of fewer than $2^{\aleph_0}$ meager sets is meager.*

Proof. First we prove that the union of fewer than $2^{\aleph_0}$ null sets is mull. Let $\kappa < 2^{\aleph_0}$ and let A_α, $\alpha < \kappa$, be null sets of reals. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. In order to prove that A is null, it suffices to find, for each $\varepsilon > 0$, an open set $U \supset A$ such that $\mu(U) \le \varepsilon$. Let $\varepsilon > 0$.

We apply Martin's Axiom as follows. Let P be the set of all open sets of measure $< \varepsilon$, and let $p \in P$ be stronger than $q \in P$ if $p \supset q$. We claim that the notion of forcing $(P, \supset)$ satisfies the countable chain condition.

It suffices to show that if W is an uncountable subset of P, then there are $p, q \in W$, $p \neq q$, such that $\mu(p \cup q) < \varepsilon$. Let S be the countable set of all unions of finitely many open intervals with rational endpoints. If $W \subset P$ is uncountable, then there exist an $n \in \boldsymbol{N}$ and an uncountable $Z \subset W$ such that $\mu(p) < \varepsilon - 1/n$ for all $p \in Z$. For each $p \in Z$, let $p^* \in S$ be such that $p^* \subset p$ and $\mu(p - p^*) < 1/n$. Since S is countable, there exist $p, q \in Z$, $p \neq q$, such that $p^* = q^*$. Then $\mu(p \cup q) < \varepsilon$.

For each $\alpha < \kappa$, let $D_\alpha = \{p \in P : A_\alpha \subset p\}$. Each D_α is a dense subset of P: If $p \in P$, then since A_α is null, there exists an open set $q \supset A_\alpha$ such that $\mu(p) + \mu(q) < \varepsilon$, and hence $p \cup q \in D_\alpha$ and $p \cup q \supset p$.

By Martin's Axiom, there exists a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Let $U = \bigcup\{p : p \in G\}$. It is clear that $A \subset U$ and it remains to show that $\mu(U) \leq \varepsilon$. We use the well-known fact (easy to verify) that if $U = \bigcup\{p : p \in G\}$, then there is a countable $H \subset G$ such that $U = \bigcup\{p : p \in H\}$. Thus if $\mu(U) > \varepsilon$, there exist $p_1, \ldots, p_n \in H$ such that $\mu(p_1 \cup \ldots \cup p_n) > \varepsilon$. But this is impossible: Since G is a filter on P, we have $p \cup q \in G$ whenever $p \in G$ and $q \in G$; thus $p_1 \cup \ldots \cup p_n \in G$ and hence $\mu(p_1 \cup \ldots \cup p_n) < \varepsilon$.

This completes the proof that the union of $< 2^{\aleph_0}$ null sets is null if MA holds.

In order to show that the union of less than $2^{\aleph_0}$ meager sets is meager, it suffices to show that the union of less than $2^{\aleph_0}$ closed nowhere dense sets is meager. The following lemma will complete the proof:

Lemma 26.40. *Assume Martin's Axiom. Let $\kappa < 2^{\aleph_0}$ and let A_α, $\alpha < \kappa$, be closed nowhere dense sets of reals. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. Then there exists a countable family of dense open sets H_i, $i = 0, 1, 2, \ldots$, such that A is disjoint from $\bigcap_{i=0}^\infty H_i$.*

Proof. We apply Martin's Axiom as follows. Let P be the set of all finite sequences of pairs

$$p = \langle (U_0, E_0), (U_1, E_1), \ldots, (U_n, E_n) \rangle$$

such that

(26.24) (i) each U_i is the union of finitely many open intervals with rational endpoints;
(ii) each E_i is a finite subset of κ; and
(iii) for each i, U_i is disjoint from $\bigcup_{\alpha \in E_i} A_\alpha$.

A condition $p' = \langle (U'_0, E'_0), \ldots, (U'_m, E'_m) \rangle$ is stronger than a condition $p = \langle (U_0, E_0), \ldots, (U_n, E_n) \rangle$ if

(26.25) (i) $m \geq n$; and
(ii) for each $i \leq n$, $U'_i \supset U_i$ and $E'_i \supset E_i$.

It is clear that this notion of forcing satisfies the countable chain condition: If two conditions have the same $U_0, \ldots, U_n$, then they are compatible, and there are only countably many sequences $\langle U_0, \ldots, U_n \rangle$.

Let I_k, $k = 0, 1, 2, \ldots$, be an enumeration of all open intervals with rational endpoints. We let, for each $\alpha < \kappa$ and all $i, k = 0, 1, 2, \ldots$,

$$\begin{aligned} D_\alpha &= \{p : p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle \text{ and } \alpha \in E_i \text{ for some } i \le n\}, \\ E_{i,k} &= \{p : p = \langle (U_0, E_0), \ldots, (U_n, E_n)\rangle \text{ and } U_i \cap I_k \ne \emptyset\}. \end{aligned} \tag{26.26}$$

Since each A_α, $\alpha < \kappa$, is nowhere dense, it is clear that for all i an k, every condition can be extended to a condition $p \in E_{i,k}$, and hence each $E_{i,k}$ is dense in P. Also, each D_α is dense in P.

By Martin's Axiom, there exists a filter $G \subset P$ such that $G \cap D_\alpha \ne \emptyset$ for all $\alpha < \kappa$, and $G \cap E_{i,k} \ne \emptyset$ for all $i, k \in \omega$. For each $i = 0, 1, 2, \ldots$, we let

$$H_i = \bigcup \{U_i : (\exists p \in G)\, p = \langle \ldots, (U_i, E_i), \ldots \rangle\}.$$

Since $E_{i,k}$ is a dense set of conditions, for all k, H_i is a dense open set of reals.

Now if $\alpha < \kappa$, then because D_α is dense, there exists $i \in \omega$ such that H_i is disjoint from A_α, and hence A_α is disjoint from $\bigcap_{i=0}^\infty H_i$. Therefore A is disjoint from $\bigcap_{i=0}^\infty H_i$. □

Corollary 26.41. *If* MA *holds, then both the algebra of Lebesgue measurable sets and the algebra of sets with the Baire property are* $2^{\aleph_0}$*-complete, and moreover, Lebesgue measure is* $2^{\aleph_0}$*-additive, i.e., if* $\kappa < 2^{\aleph_0}$ *and* A_α, $\alpha < \kappa$, *are pairwise disjoint, then*

$$\mu\Big(\bigcup_{\alpha<\kappa} A_\alpha\Big) = \sum_{\alpha<\kappa} \mu(A_\alpha). \tag{26.27}$$

Proof. We prove by induction on $\kappa < 2^{\aleph_0}$ that if A_α, $\alpha < \kappa$, are Lebesgue measurable, then $A = \bigcup_{\alpha<\kappa} A_\alpha$ is Lebesgue measurable. Given A_α, $\alpha < \kappa$, let $B_\alpha = A_\alpha - \bigcup_{\beta<\alpha} A_\beta$, for each $\alpha < \kappa$. The sets B_α are Lebesgue measurable (by the induction hypothesis), and being pairwise disjoint, all but countably many are null. It follows from the theorem that $A = \bigcup_{\alpha<\kappa} B_\alpha$ is Lebesgue measurable. The same argument proves (26.27) (see also Lemma 10.6), and the property of Baire is analogous. □

Corollary 26.42. *If* MA *holds and if* $2^{\aleph_0} > \aleph_1$, *then every* $\mathbf{\Sigma}^1_2$ *set is Lebesgue measurable and has the property of Baire.*

Proof. If A is $\Sigma^1_2(a)$, then since $(2^{\aleph_0})^{L[a]} = \aleph_1^{L[a]} \le \aleph_1$, the set of all reals that are not random over $L[a]$ is the union of at most $\aleph_1$ null sets, hence null (by Theorem 26.39). By Theorem 26.20, A is Lebesgue measurable. The Baire property is analogous. □

The proof of Theorem 26.39 yields a slightly better result: It shows that for every $\kappa \le 2^{\aleph_0}$, MA_κ implies the κ-additivity of the ideals of null and meager sets. The study of additivity of measure and category initiated by

Theorem 26.39 developed into an extensive theory that established a detailed relationship between various properties of measure and category. We refer the reader to Bartoszyński's article [∞] in the Handbook of Set Theory.

Definition 26.43. (i) Additivity:

add(LM) = the least cardinal κ such that the union of some family of κ null sets is not null,

add(BP) = the least cardinal κ such that the union of some family of κ meager sets is not meager.

(ii) Covering:

cov(LM) = the least cardinal κ for which $\boldsymbol{R}$ is the union of κ null sets,

cov(BP) = the least cardinal κ for which $\boldsymbol{R}$ is the union of κ meager sets.

(iii) Uniformity:

unif(LM) = the least cardinal κ such that there exists a set of cardinality κ that is not null,

unif(BP) = the least cardinal κ such that there exists a set of cardinality κ that is not meager.

(iv) Cofinality:

cof(LM) = the least cardinality of a family $\mathcal{F}$ of null sets such that every null set is included in a set from $\mathcal{F}$,

cof(BP) = the least cardinality of a family $\mathcal{F}$ of meager sets such that every meager set is included in a set from $\mathcal{F}$.

The proof of Theorem 26.39 shows that MA_κ implies $\mathrm{add}(\mathrm{LM}) \geq \kappa$ and $\mathrm{add}(\mathrm{BP}) \geq \kappa$. In a series of results a complete picture of inequalities emerged between these properties. First, it is obvious that add $\leq$ cov $\leq$ cof and add $\leq$ unif $\leq$ cof, both for measure and category (Exercise 26.5). Secondly, two of the inequalities have been known classically; see Exercise 26.7.

Before we proceed we introduce two cardinal invariants that are not only relevant in this context but appear frequently in results in set-theoretic topology. First some notation:

$\forall^\infty$ means for all but finitely many $n \in \omega$,

$\exists^\infty$ means for infinitely many $n \in \omega$.

A family $F \subset \omega^\omega$ is a *dominating* family if

$$\forall g \in \omega^\omega\, \exists f \in F\, \forall^\infty n\, g(n) < f(n); \tag{26.28}$$

F us an *unbounded* family if

$$\forall g \in \omega^\omega\, \exists f \in F\, \exists^\infty n\, g(n) \leq f(n). \tag{26.29}$$

Definition 26.44. The *dominating* number

$$\mathfrak{d} = \text{the least cardinality of a dominating family;}$$

the *bounding* number

$$\mathfrak{b} = \text{the least cardinality of an unbounded family.}$$

It is clear that $\mathfrak{b} \le \mathfrak{d}$, and $\aleph_1 \le \mathfrak{b} \le \mathfrak{d} \le \mathfrak{c}$. Martin's Axiom implies that $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$; see Exercise 26.8.

To see the significance of $\mathfrak{b}$ and $\mathfrak{d}$ for Baire category, notice that for each $f \in \omega^\omega$ and each k, the set $\{g \in \omega^\omega : \forall n \ge k\ g(n) < f(n)\}$ is nowhere dense (in the space $\mathcal{N}$). Hence if $g < f$ means $\forall^\infty n\ g(n) < f(n)$, each set $\{g : g < f\}$ is meager, and it follows that $\mathfrak{b} \le \text{unif(BP)}$ and $\text{cov(BP)} \le \mathfrak{d}$; see Exercise 26.10.

The relationship between the invariants defined in 26.43 and 26.44 can be illustrated by the following diagram:

$$\begin{tikzcd}
\text{cov(LM)} \arrow[r, dash] \arrow[dd, dash] & \text{unif(BP)} \arrow[r, dash] \arrow[d, dash] & \text{cof(BP)} \arrow[r, dash] \arrow[d, dash] & \text{cof(LM)} \arrow[dd, dash] \\
 & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & \\
\text{add(LM)} \arrow[r, dash] & \text{add(BP)} \arrow[r, dash] & \text{cov(BP)} \arrow[r, dash] & \text{unif(LM)}
\end{tikzcd}$$

The cardinals become larger as one moves right and up. Exercises 26.5, 26.7, and 26.10 give proofs of the easy inequalities. The remaining inequalities are given by these theorems (that we state without proofs):

Theorem 26.45 ((i) Truss, Miller; (ii) Fremlin).

(i) $\text{add(BP)} = \min\{\mathfrak{b}, \text{cov(BP)}\}$.
(ii) $\text{cof(BP)} = \max\{\mathfrak{d}, \text{unif(BP)}\}$.

Theorem 26.46 (Bartoszyński, Raisonnier-Stern).

(i) $\text{add(LM)} \le \text{add(BP)}$.
(ii) $\text{cof(BP)} \le \text{cof(LM)}$.

It is no accident that each result is accompanied by a dual version: There is a general theory that explains this duality. (For details, see Bartoszyński's Handbook article.) For instance, consider Theorem 26.46. Both (i) and (ii) can be proved from this general result (see Exercise 26.11):

Theorem 26.47 (Pawlikowski). *Let I_c and I_m be the ideal of all meager sets and the ideal of null sets. There exists a function $\varphi : I_c \to I_m$ with the property that for every family $\mathcal{F} \subset I_m$, if $\bigcup \mathcal{F}$ is null then $\bigcup \varphi^{-1}(\mathcal{F})$ is meager.* □

A significant part of the theory of invariants of measure and category is the characterization of invariants in terms of functions from ω to ω. The cardinals cov(BP) and unif(BP) were so described first by Miller, with the final form due to Bartoszyński:

Theorem 26.48. (i) cov(BP) *is the least cardinality of a family $F \subset \omega^\omega$ such that*

$$\forall g \in \omega^\omega\, \exists f \in F\, \forall^\infty n\, f(n) \neq g(n). \tag{26.30}$$

(ii) unif(BP) *is the least cardinality of a family $F \subset \omega^\omega$ such that*

$$\forall g \in \omega^\omega\, \exists f \in F\, \exists^\infty n\, f(n) = g(n). \tag{26.31}$$ □

For the easy direction of (i) and (ii) see Exercise 26.12.

The key ingredient of Theorem 26.46(i) is the following characterization of add(LM):

Theorem 26.49. add(LM) *is the least cardinality of a family $F \subset \omega^\omega$ such that*

$$\forall \varphi \in S\, \exists f \in F\, \exists^\infty n\, f(n) \notin \varphi(n), \tag{26.32}$$

where S is the set of all functions $\varphi : \omega \to [\omega]^{<\omega}$ such that $|\varphi(n)| = n$ for all n. □

See Exercise 26.13 for the proof of add(LM) $\leq$ cov(BP).

The diagram, along with Theorem 26.45, gives a complete relationship among these invariants. Nothing else can be proved in ZFC, and models have been constructed verifying all independence results based on the diagram.

We conclude this chapter with two of the earliest independence results concerning measure and category. We give an example of a model where $2^{\aleph_0}$ is large and the set $\boldsymbol{R} \cap L$ is not Lebesgue measurable, and another example where $2^{\aleph_0}$ is large and $\boldsymbol{R} \cap L$ does not have the Baire property.

Lemma 26.50. *If there exists a nonconstructible real, then*:

(i) *$\boldsymbol{R} \cap L$ is either null or not Lebesgue measurable.*
(ii) *$\boldsymbol{R} \cap L$ is either meager or does not have the Baire property.*

Proof. (i) Let S be the set of all constructible reals in the unit interval $[0, 1]$. Let a be a nonconstructible real. For each $n > 0$, let $S_n = \{x + (a/n) : x \in S\}$. The sets S_n are pairwise disjoint, $\mu(S_n) = \mu(S)$ for all n and $\bigcup_{n=0}^{\infty} S_n$ is a bounded set. Therefore if S is measurable, then $\mu(S) > 0$ is impossible.

(ii) Let S be the set of all constructible reals. First we prove that $\boldsymbol{R} - S$ is not meager. Let a be a nonconstructible real and let $S_a = \{x + a : x \in S\}$; clearly, $S \cap S_a = \emptyset$. Thus $\boldsymbol{R} = (\boldsymbol{R} - S) \cup (\boldsymbol{R} - S_a) \cup (S \cap S_a) = (\boldsymbol{R} - S) \cup (\boldsymbol{R} - S_a)$, and if $\boldsymbol{R} - S$ were meager, then $\boldsymbol{R} - S_a$ would also be meager, a contradiction.

It follows that for any nonempty interval I, $I - S$ is not meager: For each rational r, let $A_r = \{x + r : x \in I - S\}$; if $I - S$ is meager, then each A_r is meager, and $\boldsymbol{R} - S = \bigcup\{A_r : r \text{ is rational}\}$.

If S has the Baire property, then because $U - S$ is not meager for any nonempty open set U, S is meager. □

Example 26.51 (A model where $2^{\aleph_0} > \aleph_1$ and the set of all constructible reals is not Lebesgue measurable). Let λ be a regular uncountable cardinal and let B be the following measure algebra: Let $(S, \mathcal{F}, m)$ be the product measure space, where S is the product of $\lambda \times \omega$ copies of $\{0, 1\}$, $\mathcal{F}$ is the least σ-complete field of subsets of S containing all $\{t \in S : t(\alpha, n) = 0\}$, and m is the product measure, and let B be the measure algebra $B = \mathcal{F}/\text{sets of measure } 0$.

Let us consider the generic extension of the constructible universe by the measure algebra B. The generic extension $L[G]$ satisfies $2^{\aleph_0} = \lambda$. We shall show that in $L[G]$ the set of all constructible reals is not Lebesgue measurable.

In view of Lemma 26.50, it suffices to show that the set of all constructible reals is not null. Thus assume that it is null and let I_k, $k = 0, 1, 2, \ldots$, be an enumeration (in L) of all intervals with rational endpoints. Let μ denote Lebesgue measure.

Assuming that $L[G] \vDash \mu(\boldsymbol{R} \cap L) = 0$, there is a B-valued name $\dot{X}$ for a subset of ω, and a rational $\varepsilon > 0$ such that the Boolean value

(26.33) $\|\bigcup\{I_k : k \in \dot{X}\}$ contains all constructible reals, and has Lebesgue measure $\le \varepsilon\|$

is in G. We may assume, without loss of generality, that the Boolean value (26.33) is 1.

For each $k \in \boldsymbol{N}$, let $A_k \in \mathcal{F}$ be such that $\|k \in \dot{X}\| = [A_k]$. Let us consider (in L) the product measure space $(\boldsymbol{R}, \mu) \times (S, \mathcal{F}, m)$ with the product measure $\nu = \mu \times m$. Let $E \subset \boldsymbol{R} \times S$ be the set

$$E = \bigcup_{k=0}^{\infty} (I_k \times A_k).$$

We claim that $\nu(E) \le \varepsilon$. It suffices to show that $\nu(\bigcup_{k<k_0}(I_k \times A_k)) \le \varepsilon$ for every k_0. Let $k_0 \in \boldsymbol{N}$. By (26.33), for every condition $a = [A]$ there exist a stronger condition $c = [C]$ and a set $Y \subset k_0$ such that

$$c \Vdash \dot{X} \cap \check{k}_0 = \dot{Y}$$

and that $\mu(\bigcup_{k \in Y} I_k) \le \varepsilon$. Clearly, $[C] \le [A_k]$ if $k \in Y$, and $[C] \cdot [A_k] = 0$ if $k \in k_0 - Y$ and hence

$$\nu\Big(\bigcup_{k<k_0} I_k \times (A_k \cap C)\Big) = \mu\Big(\bigcup_{k \in Y} I_k\Big) \cdot m(C) \le \varepsilon \cdot m(C). \tag{26.34}$$

Thus the set of all $[C]$ for which (26.34) holds is dense in the algebra B and hence
$$\nu\Big(\bigcup_{k<k_0} I_k \times A_k\Big) \le \varepsilon.$$

Since $\nu(E) \le \varepsilon$, the complement of E has positive measure and hence there exists, by Fubini's Theorem, a number $x \in \boldsymbol{R}$ such that
$$m(\{t \in S : (x,t) \notin E\}) > 0.$$

It follows that there exists $A \in \mathcal{F}$ of positive measure such that

$$(x,t) \notin E \qquad \text{for all } t \in A. \tag{26.35}$$

We shall show that

$$[A] \Vdash x \notin \bigcup\{I_k : k \in \dot{X}\}, \tag{26.36}$$

completing the proof.

If (26.36) were not true, there would exist some $k \in \boldsymbol{N}$ and some $C \subset A$ of positive measure such that $x \in I_k$ and $[C] \Vdash k \in \dot{X}$. But then $[C] \le [A_k]$ and hence there is some $t \in C$ such that $(x,t) \in E$, contrary to (26.35). □

Example 26.52 (A model where $2^{\aleph_0} > \aleph_1$ and the set of all constructible reals does not have the property of Baire). Let λ be a regular uncountable ordinal and let P be the notion of forcing that adjoins λ Cohen reals: A condition is a finite 0–1 function whose domain is a subset of λ.

Let us consider the generic extension of the constructible universe by P. In $L[G]$, $2^{\aleph_0} = \lambda$. We shall show that in $L[G]$ the set of all constructible reals does not have the Baire property.

In view of Lemma 26.50, it suffices to show that the set of all constructible reals is not meager. For every $S \subset \lambda$ (in L), let $P_S = \{p \in P : \operatorname{dom}(p) \subset S\}$, and let $G_S = G \cap P_S$.

Lemma 26.53. *If $L[G] \vDash \boldsymbol{R} \cap L$ is meager, then there exists a countable $S \subset \lambda$ (in L) such that $L[G_S] \vDash \boldsymbol{R} \cap L$ is meager.*

Proof. Let I_k, $k \in \boldsymbol{N}$, be an enumeration of all open intervals with rational endpoints. If $L[G] \vDash \boldsymbol{R} \cap L$ is meager, then there exists a sequence $\langle U_n : n \in \boldsymbol{N}\rangle \in L[G]$ such that for every $n \in \boldsymbol{N}$, $L[G] \vDash U_n$ is dense open, and that $\boldsymbol{R} \cap L \subset \bigcup_{n=0}^{\infty}(\boldsymbol{R} - U_n)$. Let $A = \{(n,k) : I_k \subset U_n\}$, and let $\dot{A}$ be a name for A. Since P satisfies the countable chain condition, there is a countable $S \subset \lambda$ such that $\dot{A}$ is P_S-valued. It is easy to verify that if $U'_n = U_n \cap L[G_S]$, then for each $n \in \boldsymbol{N}$, $L[G_S] \vDash U'_n$ is dense open, and that $\boldsymbol{R} \cap L \subset \bigcup_{n=0}^{\infty}(\boldsymbol{R} - U'_n)$. Thus $L[G_S] \vDash \boldsymbol{R} \cap L$ is meager. □

Since P_S is countable, it suffices to prove the following lemma:

Lemma 26.54. *If P is a countable notion of forcing in L and if G is an L-generic filter on P, then $L[G] \vDash \boldsymbol{R} \cap L$ is not meager.*

Proof. It suffices to show that if $\langle U_n : n \in \boldsymbol{N} \rangle$ is (in $L[G]$) a sequence of dense open sets of reals, then there is a constructible real a such that $a \in \bigcap_{n=0}^{\infty} U_n$. Let $\dot{U}_n$ be a name for U_n and let us assume, without less of generality, that every condition forces that each $\dot{U}_n$ is dense open. It is enough to find (in L) a real number x such that for each n and each $p \in P$, there is a $q \le p$ such that $q \Vdash x \in \dot{U}_n$.

Let t_k, $k = 0, 1, \ldots$, be an enumeration of all pairs $t = (n, p)$ where $n \in \boldsymbol{N}$ and $p \in P$. Let us construct a sequence $I_0 \supset I_1 \supset \ldots \supset I_k \supset \ldots$ of closed bounded intervals as follows: Let $t_k = (n, p)$. Since $p \Vdash \dot{U}_n$ is dense open, there is an open interval $J \subset I_{k-1}$ with rational endpoints such that some $q \le p$ forces $J \subset \dot{U}_n$. Let $I_k \subset J$. The intersection $\bigcap_{k=0}^{\infty} I_k$ is nonempty; and if x is in it, then for each n and each $p \in P$ there is $q \le p$ such that $q \Vdash x \in \dot{U}_n$. $\square$

Exercises

26.1. The algebra $\mathcal{B}_c$ is the unique atomless complete Boolean algebra that has a countable dense subset.

[If B is a meager Borel set, then there is a nonempty open set U such that $U \triangle B$ is meager; hence there is a rational interval I such that $[I]_c \le [B]_c$.]

26.2. Every $\Delta^1_2(a)$ set of reals is Lebesgue measurable if and only if there exists a random real over $L[a]$. Every $\Delta^1_2(a)$ set of reals has the Baire property if and only if there exists a Cohen real over $L[a]$.

[If there are no random reals over $L[a]$ then the prewellordering $\preccurlyeq$ in the proof of Theorem 26.20 is $\Delta^1_2(a)$.

Assume that there is a random real over L, and let A be a Δ^1_2 set, $A = \{x : P(x)\} = \{x : \neg Q(x)\}$ with P and Q being Σ^1_2. In L, force with Borel sets mod measure 0, and let $\dot{r}$ be a name for the random real. Show that the set

$$D = \{p : p \Vdash P^{L[\dot{r}]}(\dot{r}) \text{ or } p \Vdash Q^{L[\dot{r}]}(\dot{r})\}$$

is dense, and let $W \subset D$ be a (countable) maximal antichain. Let

$$Z_P = \bigcup \{A_c : c \in \mathrm{BC} \cap L,\ A_c^L \in W \text{ and } A_c^L \Vdash P^{L[\dot{r}]}(\dot{r})\},$$

and Z_Q similarly. Show that $Z_P - P$ and $Z_Q - Q$ are null and conclude that $Z_P \triangle A$ is null. (For details, see Judah and Shelah [1989] or Theorem 14.6 in Kanamori [1994]).]

26.3. For every infinite $X \subset \omega$, let X^* be a chosen representative of the class of all $Y \subset \omega$ such that $X \triangle Y$ is finite. Show that the set

$$S = \{X \in [\omega]^{\omega} : |X \triangle X^*| \text{ is even}\}$$

is not Ramsey.

26.4. "Every clopen set is Ramsey" implies Ramsey's Theorem.

26.5. (i) $\mathrm{add}(\mathrm{LM}) \le \mathrm{cov}(\mathrm{LM}) \le \mathrm{cof}(\mathrm{LM})$, $\mathrm{add}(\mathrm{LM}) \le \mathrm{unif}(\mathrm{LM}) \le \mathrm{cof}(\mathrm{LM})$.
(ii) $\mathrm{add}(\mathrm{BP}) \le \mathrm{cov}(\mathrm{BP}) \le \mathrm{cof}(\mathrm{BP})$, $\mathrm{add}(\mathrm{BP}) \le \mathrm{unif}(\mathrm{BP}) \le \mathrm{cof}(\mathrm{BP})$.

26.6. There exists a decomposition $\boldsymbol{R} = M \cup N$ into a meager set M and a null set N.

26.7. (i) $\mathrm{cov}(\mathrm{LM}) \le \mathrm{unif}(\mathrm{BP})$.
(ii) $\mathrm{cov}(\mathrm{BP}) \le \mathrm{unif}(\mathrm{LM})$.

[Let $\boldsymbol{R} = M \cup N$ where M is meager and N is null. To prove (i) it suffices to show that if X is a nonmeager set then $\boldsymbol{R} = \bigcup\{N + x : x \in X\}$. By contradiction, assume that some r is not of the form $z + x$ where $z \in N$, and $x \in X$. It follows that $(X - r) \cap \{-z : z \in N\} = \emptyset$, hence $X - r \subset \{-z : z \in M\}$ and so X is meager.]

26.8. MA_κ implies $\mathfrak{b} \ge \kappa$.

[Let $\lambda < \kappa$ and let $\{f_\alpha : \alpha < \lambda\} \subset \omega^\omega$. A forcing condition is a pair (s, E) where s is a finite sequence in ω and F is a finite subset of λ; (s, E) is stronger than (t, F) if $s \supset t$ and $(\forall \alpha \in F)(\forall n \in \mathrm{dom}(s) - \mathrm{dom}(t))\, s(n) > f_\alpha(n)$. This forcing is c.c.c. and every $D_\xi = \{(s, E) : \alpha \in E\}$ is dense. MA_κ produces a function g such that $\forall^\infty n\, f_\alpha(n) < g(n)$ for all $\alpha < \lambda$.]

26.9. $\mathfrak{b} \le \mathrm{cf}(\mathfrak{d})$.

[Find a dominating family $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{d}\}$ such that whenever $\alpha < \beta$ then $\exists^\infty n\, f_\alpha(n) < f_\beta(n)$. If $\{\alpha_\nu : \nu < \mathrm{cf}\,\mathfrak{d}\}$ is cofinal in $\mathfrak{d}$ then $\{f_{\alpha_\nu} : \nu < \mathrm{cf}\,\mathfrak{d}\}$ is an unbounded family.]

26.10. $\mathfrak{b} \le \mathrm{unif}(\mathrm{BP})$ and $\mathrm{cov}(\mathrm{BP}) \le \mathfrak{d}$.

[If $F \subset \omega^\omega$ is not meager then F is an unbounded family. If F is a dominating family, then $\omega^\omega = \bigcup_{f \in F}\{g : g < f\}$.]

26.11. Using Theorem 26.47, show that $\mathrm{add}(\mathrm{LM}) \le \mathrm{add}(\mathrm{BP})$ and $\mathrm{cof}(\mathrm{BP}) \le \mathrm{cof}(\mathrm{LM})$.

[Let $\psi : I_m \to I_c$ be as follows: For each $X \in I_m$ let $\psi(X) = \bigcup\{Z : \varphi(Z) \subset X\}$. If F is a family of fewer than $\mathrm{add}(\mathrm{LM})$ meager sets, let X be the null set $\bigcup\{\varphi(Z) : Z \in F\}$. Then $\bigcup F \subset \psi(X)$ is meager. If $F \subset I_m$ generates I_m then $\{\psi(X) : X \in F\}$ generates I_c.]

26.12. (i) If F satisfies (26.30) and has size κ then $\mathcal{N}$ is the union of κ meager sets.
(ii) If F is not meager then it satisfies (26.31).

[For every f, the set $\{g : \forall^\infty n\, f(n) \ne g(n)\}$ is meager.]

26.13. Use Theorems 26.49 and 26.48(i) to verify $\mathrm{add}(\mathrm{LM}) \le \mathrm{cov}(\mathrm{BP})$.

[Let $\kappa < \mathrm{add}(\mathrm{LM})$ and let $F \subset \omega^\omega$ be such that $|F| = \kappa$. Let I_n be pairwise disjoint subsets of ω, $|I_n| = n$. Apply (26.32) to the family $\{f' : f \in F\}$, where $f'(n) = f{\restriction}I_n$, to find a φ such that $\forall f \in F\, \forall^\infty n\, f{\restriction}I_n \in \varphi(n)$. Now let $g : \omega \to \omega$ be as follows: If a is the kth element of I_n, let $g(a) = s_k(a)$ where s_k is the kth element of $\varphi(n)$. For every $f \in F$ we have $\exists^\infty n\, g(n) = f(n)$, contradicting (26.30); hence $\kappa < \mathrm{cov}(\mathrm{BP})$.]

A set of reals A has *strong measure* 0 if for every sequence $a_0 \ge a_1 \ge \ldots \ge a_n \ge \ldots$ of positive reals, there exists a sequence of open intervals I_n, $n = 0, 1, \ldots$, such that $\mathrm{length}(I_n) \le a_n$ and $A \subset \bigcup_{n=0}^\infty I_n$. It is clear that every set of strong measure 0 is null, but not every null set has necessarily strong measure 0:

26.14. The Cantor set does not have strong measure 0.

[$\boldsymbol{C}$ cannot be covered by open intervals of lengths $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, ...]

26.15. If $A \subset \boldsymbol{R}$ contains a perfect subset, then it does not have strong measure zero.

[Use the fact that A contains a subset homeomorphic to the Cantor set and that a uniformly continuous image of a set of strong measure 0 has strong measure 0.]

26.16. Martin's Axiom implies that every set $A \subset \boldsymbol{R}$ of size $< 2^{\aleph_0}$ has strong measure 0.

[May assume that $2^{\aleph_0} > \aleph_1$. Consider the forcing notion $(P, <) = (Seq, \supset)$ which adjoins a Cohen generic $x \in \omega^\omega$. Let $a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots$ be positive reals. Since $|A| < 2^{\aleph_0}$, MA implies that there exists $x \in \omega^\omega$, P-generic over every $L[a, \langle a_n : n \in \omega\rangle]$, $a \in A$. Let $r_0, r_1, \ldots, r_n, \ldots$ be an enumeration of all rational numbers; let for each n, I_n be the interval with center $r_{x(n)}$ and diameter a_n. Use the genericity of x to show that $a \in \bigcup_{n=0}^\infty I_n$, for each $a \in A$.]

26.17. If $2^{\aleph_0} = \aleph_1$, then there exists an uncountable set $E \subset \boldsymbol{R}$ such that for every nowhere dense set F, $E \cap F$ is at most countable. (E is called a *Luzin set.*) More generally, MA implies that there is a set F of size $2^{\aleph_0}$ whose intersection with every nowhere dense set has size $< 2^{\aleph_0}$.

[Let $F_0, F_1, \ldots, F_\alpha, \ldots, \alpha < 2^{\aleph_0}$, be an enumeration of all closed nowhere dense sets. Let $E = \{e_\alpha : \alpha < 2^{\aleph_0}\}$, where for each α, $e_\alpha \notin \bigcup_{\beta<\alpha} F_\beta$. Each e_α exists; the MA case uses Theorem 26.39.]

26.18. Martin's Axiom (and in particular the Continuum Hypothesis) implies that there is an uncountable set of reals of strong measure 0.

[Let E be the set from Exercise 26.17. Let $a_0 \geq a_1 \geq \ldots a_n \geq \ldots$ be given positive reals. For each n, let I_{2n} be the interval of length a_{2n} around the nth rational. The set $U = \bigcup_{n=1}^\infty I_{2n}$ is open dense and hence $E - U$ has size $< 2^{\aleph_0}$. By Exercise 26.16 there are intervals I_{2n+1} of length a_{2n+1} such that $E - U \subset \bigcup_{n=1}^\infty I_{2n+1}$.]

The smallest cardinality of a set which does not have strong measure zero also admits a combinatorial characterization:

26.19. Let κ be the least cardinality of a bounded family $F \subset \omega^\omega$ that satisfies (26.30). Show that every set $A \subset 2^\omega$ of size $< \kappa$ has strong measure 0.

[Given $\langle \varepsilon_n : n \in \omega\rangle$ let $h \in \omega^\omega$ be such that $1/2^{h(n)} \leq \varepsilon_n$. For each $a \in A$ let $f_a(n) = a{\restriction}h(n)$. The family $\{f_a : a \in A\}$ can be coded as a bounded family. Let $g \in \omega^\omega$ be such that $\forall a \in A\, \exists^\infty n\, f_a(n) = g(n)$; use g to produce the intervals covering A.]

The converse is also true, and κ is the least size of a set that fails to have strong measure zero.

26.20. Martin's Axiom implies that every dense subset of $\mathcal{B}_m$ has size $2^{\aleph_0}$.

[Let $\kappa = 2^{\aleph_0}$. Let x_α, $\alpha < \kappa$, be an enumeration of all reals. MA implies that for every α, $\{x_\beta : \beta \geq \alpha\}$ has positive measure; let K_α be a compact subset of $\{x_\beta : \beta \geq \alpha\}$ such that $\mu(K_\alpha) > 0$. If $\mathcal{B}_m$ has a dense subset of size $< \kappa$, then since κ is regular, there exist a $W \subset \kappa$ of size κ and a set X of positive measure such that $X - K_\alpha$ is null for all $\alpha \in W$. Hence every finite subset of $\{K_\alpha : \alpha \in W\}$ has nonempty intersection and so $\bigcap_{\alpha\in W} K_\alpha$ is nonempty; a contradiction.]

26.21. If $\mathfrak{d} = \mathfrak{c}$ then there exists a p-point.

[Use the proof of Theorem 16.27.]

For subsets of ω, let $X \subset^* Y$ mean that $X - Y$ is finite. A family $\{X_\alpha : \alpha < \kappa\}$ of infinite subsets of ω is a *tower* if $X_\alpha \supset^* X_\beta$ whenever $\alpha < \beta$ and there is no X such that $X_\alpha \supset^* X$ for all $\alpha < \kappa$; let $\mathfrak{t}$ be the least cardinality of a tower.

26.22. $\mathfrak{t} \le \mathfrak{b}$.

[Let $\kappa < \mathfrak{t}$, and let $F = \{f_\alpha : \alpha < \kappa\} \subset \omega^\omega$. For $X \in [\omega]^\omega$ let g_X be the increasing enumeration of X. Construct a sequence $\langle X_\alpha : \alpha \le \kappa \rangle$ of infinite sets such that $X_\alpha \supset^* X_\beta$ for $\beta < \alpha$ and such that for every α, $\forall^\infty n\, f_\alpha(n) < g_{X_{\alpha+1}}(n)$. The function g_{X_κ} eventually dominates each $f \in F$.]

Let $\mathfrak{u}$ be the least cardinality of a family of subsets of ω that generates an ultrafilter.

26.23. $\mathfrak{b} \le \mathfrak{u}$.

[For $X \in [\omega]^\omega$ let g_X be the increasing enumeration of X. For an increasing $f \in \omega^\omega$ let $S_f \subset \omega$ be the union of the intervals $[f^{2n}(0), f^{2n+1}(0))$, $n < \omega$. Show that if an increasing f eventually dominates g_X than both $S \cap X$ and $S - X$ are infinite.]

Historical Notes

The model in which all sets of reals are Lebesgue measurable is due to Solovay [1970], as is the concept of random reals, as well as Lemmas 26.1, 26.2, 26.4, 26.5, the Factor Lemma (Corollary 26.11), 26.16, and Theorem 26.20. Corollary 26.8 is due to Kripke [1967], and Theorem 26.12 is due to Jensen.

Galvin and Prikry proved in [1973] that every Borel set is Ramsey; this was extended by Silver in [1970b] to analytic sets, and Ellentuck [1974] gave the proof of Theorem 26.22 that we reproduce here. Theorem 26.23 is due to Mathias [1977].

Theorem 26.39 is due to Martin and Solovay [1970]. A systematic study of the properties of measure and category from Definition 26.43 was started by Miller in [1981], although the two results in Exercise 26.7 were proved by Rothberger in [1938]. Similarly, there had been various scattered results on what is now known as cardinal invariants (such as $\mathfrak{b}$, $\mathfrak{d}$, etc.) but the first comprehensive account appeared in van Douwen's [1984]. A most recent survey of the results stated here is Bartoszyński's chapter [∞] in the Handbook of Set Theory. Theorem 26.45(i) is due to Truss [1977] and Miller [1981]. Theorem 26.46 was proved independently by Bartoszyński [1984] and Raisonnier and Stern [1985]; Pawlikowski's Theorem 26.47 followed in [1985]. Theorems 26.48 and 26.49 Bartoszyński [1987] and [1984].

Example 26.51 is due to Solovay, and Example 26.52 is due to Vopěnka and Hrbáček [1967].

Exercise 26.2: Judah and Shelah [1989].

Exercise 26.3: Erdős and Rado [1952].

Exercise 26.10: Rothberger [1941].

Exercise 26.13: Bartoszyński [1984].

Strong measure zero sets were introduced by Borel [1919].

Exercise 26.15: Marczewski [1930b].

Exercise 26.16: Kunen.

Exercise 26.17: Luzin [1914].

Exercise 26.18: Sierpiński [1928].

Exercise 26.19: Rothberger [1941].
Exercise 26.20: the argument is due to Erdős.
Exercise 26.22: Rothberger.