27. Combinatorial Principles in L

The Fine Structure Theory

In his paper [1972], Ronald Jensen embarked on a detailed analysis of the levels of the constructible hierarchy. The resulting theory, the *fine structure theory*, describes precisely how new sets arise in the construction of L, and has significant applications. Historically, the first application of the fine structure theory was Jensen's proof of $\Box_\kappa$ in L, and we shall use that as a motivation for the introduction of fine-structural concepts. We have already described another, later, application of Jensen's theory, the Covering Theorem 18.30. While Magidor's proof presented in Chapter 18 does not use the full force of the fine structure theory, it can serve as a starting point toward the study of fine structure.

We have seen that the constructible hierarchy is Σ_1, in a uniform way, and we have also seen the role played by the condensation arguments. In particular we mention Lemma 18.38, the *Condensation Lemma*, stating that every Σ_1-elementary submodel of L_α is isomorphic to some L_γ, for every infinite ordinal α.

Every L_α (for $\alpha \geq \omega$) has a Σ_1 Skolem function, with a Σ_1 definition independent of α. Precisely, there is a Σ_0 formula Φ such that for every $\alpha \geq \omega$, the (partial) function $h_\alpha : \omega \times L_\alpha \to L_\alpha$ defined by

$$y = h_\alpha(n, x) \leftrightarrow (L_\alpha, \in) \vDash \exists z\, \Phi(n, x, y, z) \tag{27.1}$$

is a Σ_1 *Skolem function* for L_α in the sense that for every $X \subset L_\alpha$,

$$h_\alpha``(\omega \times X) = H_1^\alpha(X) \tag{27.2}$$

is the Σ_1 Skolem hull of X in L_α. This can be deduced by using the Σ_1 well-ordering $<_L$, as in (18.5). (For details, we refer the reader to Devlin's book [1984], in particular Lemma II.6.5.)

In Chapter 18 we introduced Σ_n Skolem functions for $n > 1$ as well, but mentioned (following Definition 18.40) that a Σ_n Skolem function is not necessarily a Σ_n function. In fact, for $n > 1$ there is no uniform Σ_2 Skolem function (in the sense of (27.1)–(27.2)); for details, see Exercises on pages 106–107 in Devlin's book [1984], or Proposition 2 in Friedman's [1997].

To overcome this obstacle, Jensen introduced an elaborate machinery by which arguments about Σ_n predicates on L_α can be reduced to arguments about Σ_1 predicates on a structure (L_ρ, A) which in some sense describes the Σ_n properties on L_α.

The Principle $\square_\kappa$

We recall (cf. (23.4)) that for an uncountable cardinal κ, a square-sequence is a sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that every C_α is closed unbounded in α, $|C_\alpha| < \kappa$ whenever $\operatorname{cf}\alpha < \kappa$, and if $\bar\alpha$ is a limit point of C_α then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$. In [1972], Jensen proved that in L, every uncountable cardinal κ has a square-sequence.

Theorem 27.1 (Jensen). *If $V = L$ then $\square_\kappa$ holds for every uncountable cardinal κ.*

To illustrate how the proof of $\square_\kappa$ uses condensation principles, and to introduce the fine structure theory, we shall now outline the construction of C_α in the most important special case.

First we observe that it suffices to define the sets C_α for a closed unbounded set of $\alpha < \kappa^+$; a square-sequence is then easily produced. Thus we consider only the $\alpha \in \mathrm{Lim}(\kappa^+)$ that satisfy

$$\alpha > \kappa \text{ and } L_\alpha \models \forall\gamma < \alpha\, |\gamma| \le \kappa; \tag{27.3}$$

these α's form a closed unbounded set.

As α is a singular limit ordinal, there is a stage of the constructible hierarchy where that is witnessed. Let $\beta = \beta(\alpha) \ge \alpha$ be the least β such that there is a cofinal subset of α of smaller order-type that is definable over L_β. Let $n = n(\alpha)$ be the least positive integer such that there exists such a subset that is Σ_n over L_β (with parameters in L_β).

We outline the construction of C_α for the special case when β is a limit ordinal and $n = 1$. (In general one has to consider also successor β's and $n > 1$—this is where the fine structure comes in.)

Using our assumption on α one proves that there exists a function g, Σ_1 over L_β, that maps κ onto L_β: Firstly, since there exists a $\Sigma_1(L_\beta)$ subset of α that is not in L_β (by minimality of $\beta(\alpha)$), there exists a $\Sigma_1(L_\beta)$ function that maps α onto L_β (Exercise 27.1). Then, using (27.3), one gets a Σ_1 function on κ.

Moreover, we can find such a function g in a canonical way. Since g exists, we have $L_\beta = H_1^\beta(\kappa \cup p)$, the Σ_1-Skolem hull of $\kappa \cup p$ in L_β, where p is some finite subset of L_β, and therefore

$$L_\beta = h_\beta``(\omega \times (\kappa \cup p)), \tag{27.4}$$

where h_β is the canonical Σ_1 Skolem function from (27.1). Disregarding the parameter p (which in general is taken to be the $<_L$-least such p), we obtain from h_β (uniformly) a Σ_1 function g_β mapping κ onto L_β; using (27.1) we find a Σ_0 formula Ψ such that

$$g_\beta(\nu) = y \leftrightarrow (\exists z \in L_\beta)\, \Psi(\nu, y, z). \tag{27.5}$$

Now we construct C_α as an increasing continuous transfinite sequence $\langle \alpha_\xi : \xi < \vartheta \rangle$ of limit ordinals $< \alpha$ with limit α. Simultaneously, we construct ordinals $\mu_\xi < \beta$ and $\nu_\xi < \kappa$, with $\langle \mu_\xi : \xi < \vartheta \rangle$ increasing and continuous, as follows: Given $\alpha_\xi < \alpha$ and $\mu_\xi < \beta$, we let

(27.6) $\nu_\xi =$ least ν such that $\alpha_\xi < g_\beta(\nu) < \alpha$ and $L_{g_\beta(\nu)} \vDash |\alpha_\xi| = \kappa$,

(27.7) $\mu_{\xi+1} =$ least μ such that $\alpha_\xi, \mu_\xi, g_\beta(\nu_\xi) \in H_1^\mu(\kappa)$ and $(\exists z \in L_\mu)$ $\Psi(\nu_\xi, g_\beta(\nu_\xi), z)$.

It follows from the assumptions on α that the least ordinal ϑ such that $\lim_{\xi\to\vartheta} \mu_\xi = \beta(\alpha)$ is the least ordinal with $\lim_{\xi\to\vartheta} \alpha_\xi = \alpha$, producing the set $C_\alpha = \{\alpha_\xi : \xi < \vartheta\}$, with $\vartheta \le \kappa$. The canonical Σ_1 definition (27.5) of g_β is the key to the coherence property (23.4)(ii) of the C_α's. Let $\bar\alpha < \alpha$ be a limit point of C_α, $\bar\alpha = \alpha_\lambda$ where λ is limit. Let $\bar\mu = \mu_\lambda$, and let $L_{\bar\beta}$ be the transitive collapse of $H_1^{\bar\mu}(\kappa)$. Let $e : L_{\bar\beta} \to L_\beta$ be the inverse of the transitive collapse; e is Σ_1-elementary. Using condensation arguments, one proves that $\bar\alpha \subset H_1^{\bar\mu}(\kappa)$ (and therefore $e{\restriction}\bar\alpha$ is the identity), $\bar\beta = \beta(\bar\alpha)$, $e(\bar\alpha) = \alpha$, and finally that the definition of $C_{\bar\alpha} = \{\bar\alpha_\xi : \xi < \lambda\}$ agrees with the definition of C_α up to λ. In other words, $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

This completes the outline for the special case. When $n(\alpha) = 1$ and $\beta(\alpha)$ is a successor ordinal, it can be shown that $\operatorname{cf}\alpha = \omega$ and this case is sufficiently exceptional to allow to choose C_α a sequence of order-type ω, without limit points. When $n(\alpha) > 1$, the proof requires the machinery of the fine structure theory: the model $(L_{\beta(\alpha)}, \in)$ is replaced by $(L_\rho, \in, A)$ where ρ is $(n-1)$-*projectum* of β, allowing the use of canonical Σ_1-Skolem functions for models $(L_\rho, \in, A)$.

A complete proof of Theorem 27.1 can be found in Jensen's paper [1972] or in Devlin's book [1984]. There have been several attempts at simplification of the proof; among the more recent published proofs we mention Friedman [1997] and Friedman and Koepke [1997]. $\square$

As Jensen pointed out in [1972], his proof of $\square_\kappa$ in L shows that if κ^+ is not Mahlo in L then $\square_\kappa$ holds. As a consequence the consistency strength of the failure of Square is at least that of a Mahlo cardinal. By a result of Solovay (Exercise 27.2), the consistency strength of $\neg\square_{\omega_1}$ is that of a Mahlo cardinal.

We also note a result of Solovay from [1974] that the existence of supercompact cardinals implies the failure of Square (Exercise 27.3).

The Jensen Hierarchy

One of the technical obstacles in the analysis how constructible sets arise in the hierarchy L_α is that the sets L_α are not closed under the formation of ordered pairs. This can be overcome by modifying the constructible hierarchy in an inessential way. The resulting hierarchy J_α has become the preferred tool for studying the fine structure of L and of more general inner models.

Definition 27.2 (Rudimentary Functions).

(i) $F(x_1, \dots, x_n) = x_i \qquad (i = 1, \dots, n)$,
$F(x_1, \dots, x_n) = \{x_i, x_j\} \qquad (i, j = 1, \dots, n)$,
$F(x_1, \dots, x_n) = x_i - x_j \qquad (i, j = 1, \dots, n)$
are rudimentary.

(ii) If G is rudimentary, then so is

$$F(y, x_1, \dots, x_{n-1}) = \bigcup_{z \in y} g(z, x_1, \dots, x_{n-1}).$$

(iii) A composition of rudimentary functions is rudimentary.

The *rudimentary closure* of a set X is the smallest $Y \supset X$ closed under all rudimentary functions. If X is transitive then so is its rudimentary closure, and for every transitive set M, let

$$\text{rud}(M) = \text{the rudimentary closure of } M \cup \{M\}. \tag{27.8}$$

It can be shown that for every transitive set M,

$$\text{rud}(M) \cap P(M) = \text{def}(M) \tag{27.9}$$

(compare with Corollary 13.8).

Definition 27.3 (The Jensen Hierarchy).

(i) $J_0 = \emptyset$, $J_{\alpha+1} = \text{rud}(J_\alpha)$,
(ii) $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ if α is a limit ordinal.

Each J_α is transitive, the hierarchy is cumulative, and for each α,

$$J_\alpha \subset V_{\omega\alpha} \quad \text{and} \quad J_\alpha \cap Ord = \omega\alpha.$$

From (27.9) it follows that

$$J_{\alpha+1} \cap P(J_\alpha) = \text{def}(J_\alpha).$$

The exact relationship between the J_α's and the L_α's is not important, but we have

$$J_\alpha = L_\alpha \text{ for all } \alpha \text{ such that } \alpha = \omega\alpha. \tag{27.10}$$

Every J_α is closed under $\{x,y\}$, $\bigcup x$, $x\times y$, and if A is a Σ_0 subset of J_α then $A\cap x\in J_\alpha$ for every $x\in J_\alpha$. This has the effect that

$$\langle J_\xi : \xi < \alpha\rangle$$

is uniformly Σ_1 over J_α, and there is a well-ordering $<_J$ of L such that its restriction to J_α is (uniformly) Σ_1 over J_α. Also, there is a (uniform) Σ_1 function over J_α that maps $\omega\alpha$ onto J_α. Similarly as for the L_α, every J_α has a canonical Σ_1 Skolem function h_α (analogous to (27.1) and (27.2)).

The fine structure theory capitalizes on the fact that the existence of a uniform Σ_1 Skolem function relativizes to models (J_α, A) where A is a one-place predicate as long as

(27.11) $$A\cap u\in J_\alpha \text{ for all } u\in J_\alpha;$$

such models (J_α, A) are called *amenable*. There is a Σ_0 formula Φ of the language $(\in, A)$ such that for every α and every amenable model (J_α, A), the (partial) function $h_{\alpha,A} : \omega\times J_\alpha \to J_\alpha$ defined by

(27.12) $$y = h_{\alpha,A}(n,x) \leftrightarrow (J_\alpha, \in, A) \vDash \exists z\, \Phi(n,x,y,z)$$

is a Σ_1 Skolem function for (J_α, A).

Projecta, Standard Codes and Standard Parameters

Definition 27.4. For $n>0$, the Σ_n-*projectum* ρ^n_α of α is the smallest ordinal $\rho\le\alpha$ such that there exists a $\Sigma_n(J_\alpha)$ function f such that $f``J_\rho = J_\alpha$; for $n=0$, let $\rho^0_\alpha=\alpha$.

An argument similar to Exercise 27.1 is used to prove that ρ^n_α is the smallest ρ such that there exists a $\Sigma_n(J_\alpha)$ subset of $\omega\rho$ not in J_α.

The main feature of the fine structure is that a predicate definable over J_α can be reduced to a Σ_1 predicate over an amenable structure (J_ρ, A) where ρ is a projectum of α. For each α and each $n>0$ there exists a set $A^n_\alpha\subset J_{\rho^n_\alpha}$ that is Σ_n over J_α such that $(J_{\rho^n_\alpha}, A^n_\alpha)$ is amenable, and such that

(27.13) $$\Sigma_1(J_{\rho^n_\alpha}, A^n_\alpha) = P(J_{\rho^n_\alpha})\cap\Sigma_{n+1}(J_\alpha).$$

For $n=0$, we let $A^0_\alpha=\emptyset$. The sets A^n_α are called *standard codes.*

If P is a Σ_{n+1} predicate over J_α, let f be a $\Sigma_n(J_\alpha)$ function that maps $J_{\rho^n_\alpha}$ onto J_α. Then $f_{-1}(P)$ is a $\Sigma_{n+1}(J_\alpha)$ subset of $J_{\rho^n_\alpha}$ and therefore, by (27.13), Σ_1 over the amenable model $(J_{\rho^n_\alpha}, A^n_\alpha)$. This reduction is canonical, as both the standard codes, and the Σ_n functions $f: J_{\rho^n_\alpha}\to J_\alpha$ are canonical. Precisely, we define standard codes along with *standard parameters* p^n_α, by induction on n: $p^0_\alpha=\emptyset$ and

(27.14) p^{n+1}_α is the $<_J$-least $p\in J_{\rho^n_\alpha}$ such that $J_{\rho^n_\alpha}$ is the Σ_1-Skolem hull of $J_{\rho^{n+1}_\alpha}\cup p$ in $J_{\rho^n_\alpha}$;

(27.15) $A^{n+1}_\alpha = \{(k,x) : (J_{\rho^n_\alpha}, A^n_\alpha) \vDash \varphi_k(x, p^{n+1}_\alpha)\}$

where φ_k, $k \in \omega$, is a recursive enumeration of the Σ_1 formulas.

Then a $\Sigma_n(J_\alpha)$ function from $J_{\rho^n_\alpha}$ onto J_α can be produced from the canonical Σ_1 Skolem functions and the standard parameters via (27.14). The fundamental property of standard codes is the following Condensation Lemma:

Lemma 27.5. *Let (J_γ, A) be amenable and let*

$$e : (J_\gamma, A) \to (J_{\rho^n_\alpha}, A^n_\alpha)$$

be a Σ_0-elementary embedding. There exists a unique $\bar\alpha$ such that $\gamma = \rho^n_{\bar\alpha}$ and $A = A^n_{\bar\alpha}$. The embedding e extends to a unique Σ_n-elementary embedding

$$\bar e : J_{\bar\alpha} \to J_\alpha$$

such that $\bar e(p^i_{\bar\alpha}) = p^i_\alpha$ for all $i = 1, \ldots, n$. Moreover, if e is Σ_m-elementary then $\bar e$ is Σ_{n+m}-elementary. □

A detailed account of the fine structure theory can be found in Jensen's paper [1972], or in Devlin's book [1984].

Diamond Principles

Let κ be a regular uncountable cardinal and let E be a stationary subset of κ. $\diamondsuit(E)$, or (more precisely) $\diamondsuit_\kappa(E)$, is the following principle (23.1):

(27.16) There exists a sequence of sets $\langle S_\alpha : \alpha \in E\rangle$ with $S_\alpha \subset \alpha$ such that for every $X \subset \kappa$, the set $\{\alpha \in E : X \cap \alpha = S_\alpha\}$ is a stationary subset of κ.

When $E = \kappa$, $\diamondsuit_\kappa(\kappa)$ is denoted by $\diamondsuit_\kappa$. $\diamondsuit_\kappa$ is a generalization of $\diamondsuit$ from Theorem 13.21, and can be proved under $V = L$ by a similar argument (Exercise 27.4).

Gregory's Theorem 23.2 shows that under GCH, $\diamondsuit_{\kappa^+}$ holds for every successor cardinal κ^+, in fact proving $\diamondsuit(E^{\kappa^+}_\lambda)$ whenever $\lambda < \operatorname{cf}\kappa$. This was extended by Shelah in [1979] by showing, under GCH, that $\diamondsuit(E^{\kappa^+}_\lambda)$ holds whenever $\lambda \neq \operatorname{cf}\kappa$, and if κ is singular, then GCH and $\Box_\kappa$ together imply $\diamondsuit(E^{\kappa^+}_{\operatorname{cf}\kappa})$. See also Devlin [1984], Lemma IV.2.8. For $\kappa = \aleph_1$, GCH yields a weak version of $\diamondsuit$. In [1978], Devlin and Shelah formulate and prove, under the assumption $2^{\aleph_0} < 2^{\aleph_1}$ the following statement:

(27.17) For every $F : \{0,1\}^{<\omega_1} \to \{0,1\}$ there exists a $g \in \{0,1\}^{\omega_1}$ such that for every $f \in \{0,1\}^{\omega_1}$, the set $\{\alpha < \omega_1 : F(f{\restriction}\alpha) = g(\alpha)\}$ is stationary.

(27.17) is a consequence of $\diamondsuit$ and fails under $\mathrm{MA}_{\aleph_1}$.

Trees in L

Let κ be an infinite cardinal. Generalizing Definition 9.12, we have:

Definition 27.6. A κ^+-*Suslin tree* is a tree of height κ^+ such that every branch and every antichain have cardinality at most κ.

The following result generalizes Theorem 15.26:

Theorem 27.7 (Jensen). *If $V = L$ then for every infinite cardinal κ there exists a κ^+-Suslin tree.*

When κ is regular, the proof is a straightforward generalization of the construction of a Suslin tree using $\diamondsuit$: instead we use $\diamondsuit(E_\kappa^{\kappa^+})$. We construct a tree by induction on levels. At limit levels α of cofinality $< \kappa$ we extend all branches in T_α; since $\kappa^{<\kappa} = \kappa$, the αth level has size κ. If $\operatorname{cf} \alpha = \kappa$ then we use Diamond to destroy potential antichains of size κ^+. Note that since all branches have been extended at lower cofinalities, every $x \in T_\alpha$ has an α-branch in T_α going through x. The proof that the resulting tree is a κ^+-Suslin tree is exactly as in Theorem 15.26.

When κ is singular, this approach does not work as there are κ^+-branches in T_α when $\operatorname{cf} \alpha = \operatorname{cf} \kappa$. By not extending all of them we cannot guarantee that at a later stage β, T_β has β-branches at all. Jensen's proof succeeds by involving not only $\diamondsuit$, but the $\square_\kappa$ principle as well. The proof shows that if $\square_\kappa$ holds and if $\diamondsuit_{\kappa^+}(E)$ for all E, then a κ^+-Suslin tree exists. For a proof, see Devlin [1984], Theorem IV.2.4.

Let us recall (Definition 9.24) that a tree of height ω_1 is a *Kurepa tree* if it has countable levels and at least $\aleph_2$ uncountable branches.

Theorem 27.8 (Solovay). *If $V = L$ then there exists a Kurepa tree.*

Proof. Assume $V = L$. We shall construct a family of subsets of ω_1 that satisfy (9.12).

For each $\alpha < \omega_1$, there is a smallest elementary submodel M of $(L_{\omega_1}, \in)$ such that $\alpha \in M$. Moreover (see Exercise 13.17), $M = L_\gamma$ for some $\gamma < \omega_1$, and we denote γ by $f(\alpha)$:

$$f(\alpha) = \text{the least } \gamma \text{ such that } \alpha \in L_\gamma \prec (L_{\omega_1}, \in). \tag{27.18}$$

Let $\mathcal{F}$ be the following family of subsets of ω_1:

$$\mathcal{F} = \{X \subset \omega_1 : X \cap \alpha \in L_{f(\alpha)} \text{ for every } \alpha < \omega_1\}. \tag{27.19}$$

It is immediately clear that $\{X \cap \alpha : X \in \mathcal{F}\}$ is countable for each $\alpha < \omega_1$; and hence if we show that $|\mathcal{F}| = \aleph_2$, $\mathcal{F}$ will satisfy (9.12).

Assume that $|\mathcal{F}| \le \aleph_1$. Then $\mathcal{F}$ has an enumeration

$$C = \langle X_\xi : \xi < \omega_1 \rangle \tag{27.20}$$

and any such enumeration is in L_{ω_2}. If we let C be the $<_L$-least such C in L_{ω_2}, then since the function f is a definable element of L_{ω_2} (by the definition (27.18)) and the X_ξ satisfy (27.19) in $(L_{\omega_2}, \in)$, it follows that C is a definable element of $(L_{\omega_2}, \in)$.

Now, we construct an elementary chain of submodels of $(L_{\omega_2}, \in)$:

$$N_0 \prec N_1 \prec \ldots \prec N_\nu \prec \ldots \prec (L_{\omega_2}, \in) \qquad (\nu < \omega_1)$$

as follows: N_0 is the smallest elementary submodel of L_{ω_2}; $N_{\nu+1}$ is the smallest $N \prec L_{\omega_2}$ such that $N_\nu \subset N$ and $N_\nu \in N$; if η is a limit ordinal, then $N_\eta = \bigcup_{\nu<\eta} N_\nu$. Note that each N_ν is countable, and $\omega_1 \cap N_\nu = \alpha_\nu$, for some $\alpha_\nu < \omega_1$ (see Exercise 13.18). Moreover,

$$\langle \alpha_\nu : \nu < \omega_1 \rangle \tag{27.21}$$

is a continuous increasing sequence of countable ordinals.

Now, we let $X = \{\alpha_\nu : \alpha_\nu \notin X_\nu\}$. Obviously, $X \ne X_\xi$ for all $\xi < \omega_1$, and we shall show that X satisfies the condition in (27.19), which will contradict the assumption that (27.20) is an enumeration of *all* elements of $\mathcal{F}$.

We want to show that $X \cap \alpha \in L_{f(\alpha)}$ for all $\alpha < \omega_1$. By induction on α, if α is not a limit point of the sequence (27.21), then let α_ν be the largest $\alpha_\nu < \alpha$. Then either $X \cap \alpha = X \cap \alpha_\nu$ or $X \cap \alpha = (X \cap \alpha_\nu) \cup \{\alpha_\nu\}$; in either case, since $X \cap \alpha_\nu \in L_{f(\alpha_\nu)} \subset L_{f(\alpha)}$ (by the induction hypothesis), we have $X \cap \alpha \in L_{f(\alpha)}$. Thus it suffices to show that $X \cap \alpha_\eta \in L_{f(\alpha_\eta)}$ whenever η is a limit ordinal.

We shall show that

$$\begin{aligned} &\text{(i)}\ \langle \alpha_\nu : \nu < \eta \rangle \in L_{f(\alpha_\eta)};\\ &\text{(ii)}\ \langle X_\xi \cap \alpha_\eta : \xi < \alpha_\eta \rangle \in L_{f(\alpha_\eta)}. \end{aligned} \tag{27.22}$$

Since $L_{f(\alpha_\eta)}$ is a model of ZF^-, the set $X \cap \alpha_\eta$ has the following definition in $L_{f(\alpha_\eta)}$:

$$X \cap \alpha = \{\alpha_\nu : \nu < \eta \text{ and } \alpha_\nu \notin X_\nu \cap \alpha_\eta\}.$$

For each $\nu < \omega_1$, let π_ν be the transitive collapse of N_ν. Each N_ν is isomorphic to some $L_{\delta(\nu)}$, and since $\omega_1 \cap N_\nu = \alpha_\nu$, we have $\pi_\nu(\omega_1) = \alpha_\nu$. Since C is a definable element of L_{ω_2}, we have $C \in N_\nu$ for all ν and one can see that $\pi_\nu(C) = \langle X_\xi \cap \alpha_\nu : \xi < \alpha_\nu \rangle$.

Note that α_η is uncountable in $L_{\delta(\eta)}$, while it is countable in $L_{f(\alpha_\eta)}$. It follows that $\delta(\eta) < f(\alpha_\eta)$, and we have $\pi_\eta(C) \in L_{\delta(\eta)} \subset L_{f(\alpha_\eta)}$, which proves (27.22)(ii).

To prove (27.22)(i), let us construct, inside $L_{f(\alpha_\eta)}$ (which is a model of ZF^-), an elementary chain N'_ν, $\nu < \eta$ of submodels of $(L_{\delta(\eta)}, \in)$: N'_0 is the smallest elementary submodel of $L_{\delta(\eta)}$; $N'_{\nu+1}$ is the smallest $N \prec L_{\delta(\eta)}$ such that $N'_\nu \cup \{N'_\nu\} \subset N$, etc. It is not difficult to show, by induction on $\nu < \eta$, that for each ν, N'_ν is isomorphic to N_ν. Then the transitive collapse of N'_ν is $L_{\delta(\nu)}$, and so $\langle L_{\delta(\nu)} : \nu < \eta \rangle \in L_{f(\alpha_\eta)}$. It follows that $\langle \alpha_\nu : \nu < \eta \rangle \in L_{f(\alpha_\eta)}$, proving (27.22)(i). $\square$

One consequence of the foregoing proof is that a Kurepa tree exists unless $\aleph_2$ is inaccessible in L (Exercise 27.5). This is complemented by the following consistency result:

Theorem 27.9 (Silver [1971c]). *If there exists an inaccessible cardinal then there is a generic extension in which there are no Kurepa trees.*

Proof. Let λ be an inaccessible cardinal. Let $(P, <)$ be the Lévy collapse of λ to $\aleph_2$: forcing conditions are countable functions p on subsets of $\lambda \times \omega_1$ such that $p(\alpha, \xi) < \alpha$ for every $(\alpha, \xi) \in \operatorname{dom}(p)$ and p is stronger than q if $p \supset q$.

$(P, <)$ is $\aleph_0$-closed, and so V and $V[G]$ have the same countable sequences in V. Also, $\aleph_1^{V[G]} = \aleph_1$, and $\aleph_2^{V[G]} = \lambda$.

Lemma 27.10. *If P is an $\aleph_0$-closed notion of forcing and T is an ω_1-tree in the ground model such that every level of T is countable, then T has no new branches in $V[G]$.*

Proof. Assume that T has a branch $b \in V[G]$ that is not in V; since $V[G]$ has no new countable sets, b has length ω_1. There is a name $\dot{b}$ for b and a condition $p_0 \in G$ such that $p_0 \Vdash \dot{b} \neq \check{a}$ for all $a \in V$. We construct, by induction, conditions $p_s < p_0$ and nodes $x_s \in T$ for all finite sequences s of 0's and 1's. Having constructed p_s, we can find two incomparable nodes $x_{s\frown 0}$ and $x_{s\frown 1}$ both $> x_s$, and two conditions $p_{s\frown 0}$ and $p_{s\frown 1}$, both stronger than p_s such that $p_{s\frown 0} \Vdash x_{s\frown 0} \in \dot{b}$ and $p_{s\frown 1} \Vdash x_{s\frown 1} \in \dot{b}$. Moreover, we can find such $x_{s\frown 0}$ and $x_{s\frown 1}$ at the same level of T. Let $\alpha < \omega_1$ be such that all x_s lie below level α in T. For each $f : \omega \to \{0, 1\}$, let p_f be a condition stronger than all $p_{f \restriction n}$, $n \in \omega$. Since $p_0 \Vdash \dot{b}$ is uncountable, there exist $q < p_f$ and x_f at the αth level of T such that $q \Vdash x_f \in \dot{b}$. Now it is clear that $x_f \neq x_g$ whenever f and g are distinct 0–1 functions on ω. Thus the αth level of T has at least $2^{\aleph_0}$ elements, contrary to our assumption. $\square$

It follows immediately from the lemma that in $V[G]$, no tree $T \in V$ whose levels are countable can be a Kurepa tree: Since every branch of T in $V[G]$ is in V, T has at most $(2^{\aleph_1})^V$ branches, but $(2^{\aleph_1})^V < \lambda = \aleph_2^{V[G]}$, and so T has (in $V[G]$) fewer than $\aleph_2$ branches.

A similar argument can be used for any tree in $V[G]$, with a slight modification. For each $\alpha < \lambda$, let P_α denote the set of all conditions whose domain is a subset of $\alpha \times \omega_1$; similarly, let $P^\alpha = \{p \in P : \operatorname{dom}(p) \subset (\kappa - \alpha) \times \omega_1\}$. Clearly, P is (isomorphic to) the product $P_\alpha \times P^\alpha$. Let $X \in V[G]$ be a subset of ω_1, and let $\dot{X}$ be a name of X; since P satisfies the λ-chain condition, there exists for each $\xi < \omega_1$ a set of conditions $W_\xi \subset P$ of size less than λ such that $\|\xi \in \dot{X}\| = \sum\{p : p \in W_\xi\}$. There exists an $\alpha < \lambda$ such that $W_\xi \subset P_\alpha$, for all $\xi < \omega_1$. It follows that $X \in V[G \cap P_\alpha]$.

Now let $T \in V[G]$ be an ω_1-tree with countable levels. There exists an $\alpha < \lambda$ such that $T \in V[G \cap P_\alpha]$. By the Product Lemma, $G \cap P^\alpha$ is P^α-generic over $V[G \cap P_\alpha]$ and $V[G] = V[G \cap P_\alpha][G \cap P^\alpha]$. Since $V[G \cap P_\alpha]$ and V have the same countable sequences in V, it follows that P^α is $\aleph_0$-closed not only in V, but also $V[G \cap P_\alpha] \vDash P^\alpha$ is $\aleph_0$-closed. Thus Lemma 27.10 applies and every branch of T in $V[G]$ is in $V[G \cap P_\alpha]$. However, $(2^{\aleph_1})^{V[G \cap P_\alpha]} < \lambda = \aleph_2^{V[G]}$, and so T is not a Kurepa tree in $V[G]$. This completes the proof. □

Canonical Functions on ω_1

For ordinal functions on ω_1, let $f < g$ if $\{\xi < \omega_1 : f(\xi) < g(\xi)\}$ contains a closed unbounded set. The rank of f in $<$ is the Galvin-Hajnal norm $\|f\|$; cf. Definition 24.4. By induction on α, the αth *canonical function* f_α is defined (if it exists) as the $<$-least ordinal function greater than each f_β, $\beta < \alpha$. If f_α exists then it is unique up to the equivalence $=_{I_{\text{NS}}}$. Lemma 24.5 shows that for every $\alpha < \omega_2$ the αth canonical function exists; see also Exercise 27.6.

It is possible that the constant function ω_1 is the ω_2nd canonical function (see Exercise 27.7) but this is known to have large cardinal consequences; in particular, in L there is a function $f : \omega_1 \to \omega_1$ such that $\|f\| = \omega_2$ (Exercise 27.8).

If canonical functions f_α exist for all α, then the ideal I_{NS} is precipitous (Exercise 27.10) and hence there is an inner model with a measurable cardinal. Conversely, a combination of the method from Jech and Mitchell [1983] with the proof of Theorem 23.10 yields the consistency, relative to a measurable cardinal, of canonical functions for all α.

The following result shows that in L, the ω_2nd canonical function does not exist.

Theorem 27.11 (Hajnal). *If $V = L$ then there is no ω_2nd canonical function on ω_1.*

Proof. Assume $V = L$, and assume that there is an ω_2nd canonical function. This statement can be expressed in L_{ω_2}:

$$(\exists f : \omega_1 \to \omega_1)\, \forall \eta\, (f_\eta < f) \text{ and}$$
$$(\forall \text{ stationary } S)(\forall g <_S f)(\exists \text{ stationary } T \subset S)\, \exists \eta\, g{\restriction}T = f_\eta{\restriction}T.$$

Let γ be the least ordinal such that $(L_\gamma, \in)$ is elementarily equivalent to $(L_{\omega_2}, \in)$. Let f be the ω_2nd canonical function in $(L_\gamma, \in)$ and let $\delta = \omega_1^{L_\gamma}$. We shall find a $\xi < \delta$ such that $(L_\xi, \in) \equiv L_\gamma$, reaching a contradiction.

Consider the generic ultrapower of L_γ by the nonstationary ideal $(I_{\text{NS}})^{L_\gamma}$ on $\delta = \omega_1^{L_\gamma}$ (using functions in L_γ). As f is the $\omega_2^{L_\gamma}$nd canonical function, the ultraproduct $\prod_{\xi<\delta} f(\xi)/G$ has order-type γ, and moreover, the ultraproduct $\mathrm{Ult}_G = \prod_{\xi<\delta} L_{f(\xi)}/G$ is isomorphic to L_γ. Thus if a sentence σ is true in $(L_\gamma, \in)$ then it is forced to be true in Ult_G by every stationary $S \subset \delta$ in L_γ, and so if we let

$$T_\sigma = \{\xi < \delta : L_{f(\xi)} \vDash \sigma\},$$

then (since $f \in L_\gamma$) $T_\sigma \in L_\gamma$ and

$$L_\gamma \vDash T_\sigma \text{ contains a closed unbounded set.}$$

If $\{\sigma_n : n \in \omega\}$ enumerates all sentences of ZF, then $\langle T_{\sigma_n} : n < \omega\rangle \in L_\gamma$, and

$$L_\gamma \vDash \bigcap\{T_{\sigma_n} : n \in \omega \text{ and } T_{\sigma_n} \text{ contains a closed unbounded set}\} \neq \emptyset.$$

If ξ is an element of this intersection, then $L_{f(\xi)} \equiv L_\gamma$. $\square$

The existence of an ω_2nd canonical function is not a large cardinal property, as this consistency result shows:

Theorem 27.12 (Jech-Shelah). *There is a generic extension of L in which the ω_2nd canonical function exists.*

The model is obtained by first adding (by forcing with countable conditions) an increasing sequence $\langle f_\alpha : \alpha \le \omega_2\rangle$ of ordinal functions from ω_1 into ω_1. Then one uses an iterated forcing, with countable support of length ω_2, that successively destroys all stationary subsets of ω_1 that witness that the sequence $\langle f_\alpha : \alpha \le \omega_2\rangle$ is not canonical. For details, consult Jech and Shelah [1989].

Exercises

27.1. Let $\alpha \le \beta$ be limit ordinals and assume that there exists a set $Z \subset \alpha$ that is Σ_1 over L_β but $Z \notin L_\beta$. Then there exists a $\Sigma_1(L_\beta)$ function g such that $g``\alpha = L_\beta$.

[First show that there is a Σ_1 function $g : \alpha \to \beta$ unbounded in β. Let $Z = \{\xi < \alpha : (\exists y \in L_\beta)\, \varphi(\xi, y, p)\}$ where φ is Σ_0, and let $g(\xi)$ be the least η such that $(\exists y \in L_\eta)\, \varphi$.]

27.2. If a Mahlo cardinal λ is Lévy collapsed to $\aleph_2$ (by countable conditions) then $\square_{\omega_1}$ fails in the extension.

27.3. If κ is supercompact then $\square_\lambda$ fails for all $\lambda \ge \kappa$.

27.4. If $V = L$ then $\diamondsuit_\kappa(E)$ holds for every regular uncountable κ and every stationary $E \subset \kappa$.

27.5. If $\aleph_2$ is not inaccessible in L then a Kurepa tree exists.

[There exists an $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$; modify Theorem 27.8 to construct a Kurepa tree in $L[A]$.]

27.6. If $\omega_1 \le \alpha < \omega_2$, and if g is a one-to-one function of ω_1 onto α, let $f(\xi) =$ the order-type of $g``\xi$. Show that f is the αth canonical function.

27.7. If I_{NS} is $\aleph_2$-saturated then the constant function ω_1 is the $\aleph_2$nd canonical function.

27.8. In L, find a function $f : \omega_1 \to \omega_1$ of norm ω_2.
[As in the proof of Theorem 27.8.]

27.9. If $f : \omega_1 \to Ord$ and S is stationary then $\|f\|_S = \alpha$ if S forces $j(f)(\omega_1^V) = \alpha$ in $P(\omega_1)/I_{\text{NS}}$.

27.10. If a canonical f_α exists for every α, then I_{NS} is precipitous.

Historical Notes

The fine structure theory was introduced by Jensen in [1972]. The paper gives, among others, proofs of $\Box_\kappa$ and of the existence of κ^+-Suslin trees in L. It also formulates a combinatorial principle $\diamondsuit^+$ that implies the existence of a Kurepa tree (abstracting Solovay's proof given here). Silver's model with no Kurepa trees appears in [1971c]. Theorem 27.11 is an unpublished result of András Hajnal from 1976; the model in Theorem 27.11 is from Jech and Shelah [1989].

Exercises 27.2, 27.3: Solovay.

Exercise 27.4: Jensen.