33. Determinacy

With each subset A of ω^ω we associate the following game G_A, played by two players I and II. First I chooses a natural number a_0, then II chooses a natural number b_0, then I chooses a_1, then II chooses b_1, and so on. The game ends after ω steps; if the resulting sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is in A, then I wins, otherwise II wins.

A *strategy* (for I or II) is a rule that tells the player what move to make depending on the previous moves of both players. A strategy is a *winning strategy* if the player who follows it always wins. The game G_A is *determined* if one of the players has a winning strategy.

The Axiom of Determinacy (AD) states that for every $A \subset \omega^\omega$, the game G_A is determined.

Determinacy and Choice

First some definitions: Let $A \subset \omega^\omega$ be given and let G_A denote the corresponding game. A *play* is a sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in \omega^\omega$; for each n, a_n is the nth *move of player* I and b_n is the nth *move of player* II. A *strategy* for I is a function σ with values in ω whose domain consists of finite sequences $s \in Seq$ of even length. Player I plays $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ *by the strategy* σ if $a_0 = \sigma(\emptyset)$, $a_1 = \sigma(\langle a_0, b_0\rangle)$, $a_2 = \sigma(\langle a_0, b_0, a_1, b_1\rangle)$, and so on; it is clear that if I plays by σ, then the play is determined by σ and the sequence $b = \langle b_n : n \in \omega\rangle$. We denote the play by $\sigma * b$. A strategy σ is a *winning strategy* for I if

$$\{\sigma * b : b \in \mathcal{N}\} \subset A,$$

in other words, if all plays that I plays by σ are in A. Similarly, a strategy for II is a function τ with values in ω, defined on finite sequences $s \in Seq$ of odd length. If $a \in \mathcal{N}$ and if τ is a strategy for II, then $a * \tau$ denotes the play in which I plays a and II plays by τ. A strategy τ for II is a winning strategy if

$$\{a * \tau : a \in \mathcal{N}\} \subset \mathcal{N} - A.$$

We sometimes consider games G_A whose moves are not natural numbers but elements of an arbitrary set S. A play is then a sequence $p \in S^\omega$, and the result of the game depends on whether $p \in A$ or $p \notin A$ (here A is a subset of S^ω). The other relevant notions are defined accordingly.

Since the number of strategies is $2^{\aleph_0}$, an easy diagonal argument shows that the Axiom of Choice is incompatible with the Axiom of Determinacy:

Lemma 33.1. *Assuming the Axiom of Choice, there exists $A \subset \omega^\omega$ such that the game G_A is not determined.*

Proof. Let $\{\sigma_\alpha : \alpha < 2^{\aleph_0}\}$ and $\{\tau_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate all strategies for I and all strategies for II. We construct sets $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$, subsets of $\mathcal{N}$, as follows: Given $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$, let us choose some y_α such that $y_\alpha = \sigma_\alpha * b$ for some b and $y_\alpha \notin \{x_\xi : \xi < \alpha\}$ (such y_α exist because the set $\{\sigma_\alpha * b : b \in \mathcal{N}\}$ has size $2^{\aleph_0}$); similarly, let us choose x_α such that $x_\alpha = a * \tau_\alpha$ for some a and $x_\alpha \notin \{y_\xi : \xi \le \alpha\}$. It is clear that the sets X and Y are disjoint, that for each α there is b such that $\sigma_\alpha * b \notin X$, and there is a such that $a * \tau_\alpha \in X$. Thus neither I nor II has a winning strategy in the game G_X, and hence G_X is not determined. $\square$

In contrast with this lemma, the Axiom of Determinacy implies a weak form of the Axiom of Choice:

Lemma 33.2. *The Axiom of Determinacy implies that every countable family of nonempty sets of real numbers has a choice function.*

Proof. We prove that if $\mathcal{X} = \{X_n : n \in \omega\}$ is a family of nonempty subsets of $\mathcal{N}$, then there exists f on $\mathcal{X}$ such that $f(X_n) \in X_n$ for all n. Let us consider the following game: If I plays $\langle a_0, a_1, a_2, \ldots\rangle$ and and II plays $\langle b_0, b_1, b_2, \ldots\rangle$, then II wins if and only if $b \in X_{a_0}$. It is clear that I does not have a winning strategy: Once I plays a_0, the player II finds some $b \in X_{a_0}$, plays $b = \langle b_0, b_1, b_2, \ldots\rangle$ and wins. Hence II has a winning strategy τ, and we can define f on $\mathcal{X}$ as follows: $f(X_n) = \tau * \langle n, 0, 0, 0, \ldots\rangle$. $\square$

As we show below, Determinacy has desirable consequences for sets of reals: AD implies that every set of reals is Lebesgue measurable, has the Baire property and the perfect set property. Thus it is natural to postulate that Determinacy holds to the extent it does not contradict the Axiom of Choice. The appropriate postulate turns out to be that AD holds in the model $L(\boldsymbol{R})$, and therefore all sets of reals definable from a real parameter are determined. This implies, in particular, that the game G_A is determined for every projective set—*Projective Determinacy* (PD). It has been established that both $\mathrm{AD}^{L(\boldsymbol{R})}$ and PD are large cardinal axioms; we shall elaborate on this later in this chapter.

Throughout the rest of the present chapter we work in ZF + the Principle of Dependent Choices.

Some Consequences of AD

We shall now prove that under the assumption of Determinacy, sets of real numbers are well behaved.

Theorem 33.3. *Assume the Axiom of Determinacy. Then*:

(i) *Every set of reals is Lebesgue measurable.*
(ii) *Every set of reals has the property of Baire.*
(iii) *Every uncountable set of reals contains a perfect subset.*

Proof. (i) It suffices to prove the following lemma:

Lemma 33.4. *Assuming* AD, *let S be a set of reals such that every measurable $Z \subset S$ is null. Then S is null.*

It is easy to see that Lemma 33.4 implies that every set X is Lebesgue measurable: Let $A \supset X$ be a measurable set with the property that every measurable $Z \subset A - X$ is null. Then $A - X$ is null and hence X is measurable.

Proof. Thus let S be a set of reals with the property

$$\text{if } Z \subset S \text{ is Lebesgue measurable, then } Z \text{ is null;} \tag{33.1}$$

we shall use AD to show that S is null. It is clear that we can restrict ourselves to subsets of the unit interval; thus assume that $S \subset [0, 1]$. In order to show that S is null, it suffices to show that the outer measure $\mu^*(S)$ is less than any $\varepsilon > 0$. Thus let ε be a fixed positive real number.

33.5. The Covering Game. Given S and ε, let us set up a game as follows: If $\langle a_0, a_1, a_2, \ldots\rangle$ is a sequence of 0's and 1's, let a be the real number

$$a = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}}. \tag{33.2}$$

For each $n \in \omega$, let G^n_k, $k = 0, 1, 2, \ldots$, be an enumeration of the set K_n of all sets G such that

(33.3) (i) G is a union of finitely many intervals with rational endpoints;
(ii) $\mu(G) \le \varepsilon/2^{2(n+1)}$.

The rules of the game are that player I tries to play a real number $a \in S$, and player II tries to cover the real a by the union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all n. More precisely, a play $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is won by player I if

(i) $a_n = 0$ or 1, for all n;
(ii) $a \in S$; and
(iii) $a \notin \bigcup_{n=0}^{\infty} G^n_{b_n}$.

We claim that player I does not have a winning strategy in the game. To show this, notice that if σ is a winning strategy for I, then the function f that to each $b = \langle b_0, b_1, b_2, \ldots\rangle \in \mathcal{N}$ assigns the real number $a = f(b)$ such that $\langle a_0, b_0, a_1, b_1, \ldots\rangle = \sigma * b$ is continuous and hence the set $Z = f(\mathcal{N})$ is analytic and hence measurable. Moreover $Z \subset S$, and therefore Z is null. Now a null set can be covered by a countable union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in K_n$ for all n, and therefore, if II plays $\langle b_0, b_1, b_2, \ldots\rangle$ where $G^n_{b_n} = H_n$ and I plays by σ, then II wins. Thus σ cannot be a winning strategy for I.

Assuming AD, the covering game is determined, and therefore player II has a winning strategy. Let τ be such a strategy. For each finite sequence $s = \langle a_0, \ldots, a_n\rangle$ of 0's and 1's, let $G_s \in K_n$ be the set $G^n_{b_n}$, where $\langle b_0, \ldots, b_n\rangle$ are the moves that II plays by τ in response to $a_0, \ldots, a_n$. Since τ is a winning strategy, every $a \in S$ is in the set $\bigcup\{G_s : s \subset a\}$ and hence

$$S \subset \bigcup\{G_s : s \in Seq(\{0,1\})\} = \bigcup_{n=1}^{\infty} \bigcup_{s\in\{0,1\}^n} G_s. \tag{33.4}$$

Now for every $n \geq 1$, if $s \in \{0,1\}^n$, then $\mu(G_s) \leq \varepsilon/2^{2n}$ and hence

$$\mu\big(\textstyle\bigcup_{s\in\{0,1\}^n}\big) \leq \frac{\varepsilon}{2^{2n}} \cdot 2^n = \frac{\varepsilon}{2^n}.$$

It follows that $\mu(\bigcup_{n=1}^{\infty} \bigcup_{s\in\{0,1\}^n} G_s) \leq \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$ and thus $\mu^*(S) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, S is null. This completes the proof. □

(ii) Next we consider the property of Baire:

33.6. The Banach-Mazur Game. Let X be a subset of the Baire space $\mathcal{N}$, and let us consider the following game: Player I plays a finite sequence $s_0 \in Seq$; then II plays a proper extension $t_0 \supset s_0$; then I plays $s_1 \supset t_0$, etc.:

$$s_0 \subset t_0 \subset s_1 \subset t_1 \subset \ldots. \tag{33.5}$$

The sequence (33.5) converges to some $x \in \mathcal{N}$. If $x \in A$, then I wins, and otherwise II wins.

First we verify that this game can be reformulated as a game G_A of the kind introduced at the beginning of the section (i.e., when the moves are natural numbers). Let u_k, $k \in \boldsymbol{N}$, be an enumeration of the set Seq. If $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is a sequence of numbers, then consider the sequence

$$u_{a_0},\ u_{b_0},\ u_{a_1},\ u_{b_1},\ \ldots \tag{33.6}$$

and let A be the set of all $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in \mathcal{N}$ such that: Either there is n such that

$$u_{a_0} \subset u_{b_0} \subset \ldots u_{a_n} \not\subset u_{b_n}$$

or the sequence (33.6) is increasing and converges to some $x \in X$. It is clear I wins the Banach-Mazur game if and only if I wins the game G_A.

Thus if AD holds, the game is determined, for every $X \subset \mathcal{N}$. We will use this to show that every $X \subset \mathcal{N}$ has the Baire property.

Lemma 33.7. *Player II has a winning strategy in the Banach-Mazur game if and only if X is meager.*

Proof. Let Y be the complement of X in $\mathcal{N}$. For each $s \in Seq$, $O(s)$ denotes the basic open set $\{x \in \mathcal{N} : s \subset x\}$.

(a) If X is a meager set, then there exist open dense sets G_n, $n \in \boldsymbol{N}$, such that $Y \supset \bigcap_{n=0}^{\infty} G_n$. It is easy to find a winning strategy τ for II: If I plays s_0, let $t_0 = \tau(\langle s_0 \rangle)$ be some $t_0 \supset s_0$ such that $U_{t_0} \subset G_0$; such t_0 exists because G_0 is dense. Then if I plays $s_1 \supset t_0$, let $t_1 = \tau(\langle s_0, t_0, s_1 \rangle)$ be some $t_1 \supset s_1$ such that $U_{t_1} \subset G_1$, and so on. It is clear that every such play $s_0 \subset t_0 \subset s_1 \subset \ldots$ converges to $x \in \bigcap_{n=0}^{\infty} G_n$, and hence τ is a winning strategy for II.

(b) Conversely, assume that II has a winning strategy τ. A *correct position* is a finite sequence $\langle s_0, t_0, \ldots, s_n, t_n \rangle$ such that $s_0 \subset t_0 \subset \ldots \subset t_n$ and $t_0 = \tau(\langle s_0 \rangle)$, $t_1 = \tau(\langle s_0, t_0, s_1 \rangle)$, etc. We shall first prove the following claim: Let $x \in \mathcal{N}$ and assume that for every correct position $p = \langle s_0, \ldots, t_n \rangle$ with $t_n \subset x$ there exists $s \supset t_n$ such that $\tau(p^\frown s) \subset x$. Then $x \in Y$.

To prove the claim, let x satisfy the condition. To begin, there exists s_0 such that $\tau(\langle s_0 \rangle) \subset x$; let $t_0 = \tau(\langle s_0 \rangle)$. Then there exists $s_1 \supset t_0$ such that $t_1 = \tau(\langle s_0, t_0, s_1 \rangle) \subset x$; then there is $s_2 \supset t_1$ such that $\tau(\langle s_0, t_0, s_1, t_1, s_2 \rangle) \subset x$; and so on. The sequence $s_0 \subset t_0 \subset s_1 \subset t_1 \subset \ldots$ converges to x and is a play in which II plays by τ. Hence $x \in Y$.

For every correct position $p = \langle s_0, \ldots, t_n \rangle$, let

$$F_p = \{x \in \mathcal{N} : t_n \subset x \text{ and } (\forall s \supset t_n)\, \tau(p^\frown s) \not\subset x\}.$$

By the claim, for every $x \in X$ there is a correct position p such that $x \in F_p$; in other words,

$$X \subset \bigcup \{F_p : p \text{ is a correct position}\}.$$

It is easy to see that for each p, $O(t_n) - F_p$ is on open dense set in $O(t_n)$; hence F_p is a closed nowhere dense set. The number of correct positions is countable and hence X is meager. □

Corollary 33.8. *Let $X \subset \mathcal{N}$. Player I has a winning strategy in the Banach-Mazur game if and only if for some $s \in Seq$, $O(s) - X$ is meager.*

Proof. Note that I has a winning strategy if and only if there exists $s \in Seq$ (the first move of I) such that player II has a winning strategy in the following game: I plays $t_0 \supset s$, II plays $s_0 \supset t_0$, I plays $t_1 \supset s_0$, etc.; and I wins if $t_0 \subset s_0 \subset t_1 \subset \ldots$ converges to $x \in U_s - X$. By Lemma 33.7, II has a winning strategy in this game if and only if $O(s) - X$ is meager. □

Now part (ii) of Theorem 33.3 follows. If $X \subset \mathcal{N}$, then since the Banach-Mazur game is determined, either X is meager or for some $s \in Seq$, $O(s) - X$ is meager. Thus let $X \subset \mathcal{N}$ be arbitrary. If X is meager, then X has the Baire property. If X is nonmeager, then let $G = \bigcup\{O(s) : O(s) - X$ is meager$\}$. Clearly, $G - X$ is meager, and $X - G$ must be meager too because otherwise there would exist some s such that $O(s) - (X - G)$ is meager, which contradicts the definition of G. It follows that X has the Baire property.

(iii) We will use AD to prove that every uncountable set in the Cantor space $\boldsymbol{C} = \{0,1\}^\omega$ has a perfect subset. We consider the following game:

33.9. The Perfect Set Game. Let X be a subset of $\{0,1\}^\omega$. The game is defined as follows: Player I plays a sequence $s_0 \in Seq(\{0,1\})$ of 0's and 1's (possibly the empty sequence), then player II plays $n_0 \in \{0,1\}$, then I plays $s_1 \in Seq(\{0,1\})$, and so on. Let $x = s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \ldots$. Player I wins if $x \in X$, and II wins if $x \notin X$.

The game can be reformulated as a game G_A, for some $A \subset \omega^\omega$.

Lemma 33.10. *Let $X \subset \boldsymbol{C}$. If II has a winning strategy in the perfect set game, then X is countable.*

Proof. Let τ be a winning strategy for II. A *correct position* is a finite sequence $\langle s_0, n_0, \ldots, s_k, n_k\rangle$ such that $n_0 = \tau(\langle s_0\rangle)$, $n_1 = \tau(\langle s_0, n_0, s_1\rangle)$, etc. By the same argument as in Lemma 33.7, we get the following claim: Let $x \in \{0,1\}^\omega$ and assume that for every correct position $p = \langle s_0, \ldots, n_k\rangle$ if $s_0{}^\frown n_0{}^\frown \ldots {}^\frown n_k \subset x$, then there exists an $s \in Seq(\{0,1\})$ such that $s_0{}^\frown n_0{}^\frown \ldots {}^\frown n_k{}^\frown s^\frown \tau(p^\frown s) \subset x$. Then $x \notin X$.

It follows that $X \subset \bigcup\{F_p : p \text{ is a correct position}\}$, where

$$F_p = \{x \in \boldsymbol{C} : s_0{}^\frown \ldots {}^\frown n_k \subset x \text{ and } \forall s\, (s^\frown \ldots {}^\frown n_k{}^\frown s^\frown \tau(p^\frown s) \not\subset x)\}.$$

The lemma will follow if we show that each F_p has exactly one element $x \in \boldsymbol{C}$. This element x is uniquely determined as follows (because each $x(m)$ is either 0 or 1); first, for some $l \in \boldsymbol{N}$, $\langle x(0), \ldots, x(l-1)\rangle = s_0{}^\frown n_0{}^\frown \ldots {}^\frown n_k$; then $x(l) = 1 - \tau(p^\frown \emptyset)$, $x(l+1) = 1 - \tau(p^\frown \langle x(l)\rangle)$, $x(l+2) = 1 - \tau(p^\frown \langle x(l), x(l+1)\rangle)$, and so on. □

Now part (iii) of Theorem 33.3 follows. If $X \subset \boldsymbol{C}$ is uncountable, then II does not have a winning strategy; and since the game is determined, I has a winning strategy σ. For each $x = \langle n_0, n_1, \ldots\rangle \in \boldsymbol{C}$, let $F(x) \in \boldsymbol{C}$ denote the 0–1 sequence

$$s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \ldots$$

where $s_0 = \sigma(\emptyset)$, $s_1 = \sigma(\langle s_0{}^\frown n_0\rangle)$, $s_2 = \sigma(\langle s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1\rangle)$, etc. The function f is continuous and one-to-one, and hence $f(\boldsymbol{C})$ is a perfect set. But $X \supset f(\boldsymbol{C})$ and hence X has a perfect subset. □

We proved earlier that if $\aleph_1 = \aleph_1^{L[a]}$ for some $a \subset \omega$, then there is an uncountable set without a perfect subset. Thus we have:

Corollary 33.11. *If* AD *holds, then $\aleph_1$ is inaccessible in $L[a]$, for every $a \subset \omega$.* □

AD and Large Cardinals

To illustrate the relationship between the Axiom of Determinacy and the theory of large cardinals, we show that AD implies that $\aleph_1$ and $\aleph_2$ are measurable cardinals.

Theorem 33.12 (Solovay). *The Axiom of Determinacy implies that*:

(i) $\aleph_1$ *is a measurable cardinal, and moreover, the closed unbounded filter on* $\aleph_1$ *is an ultrafilter.*
(ii) $\aleph_2$ *is a measurable cardinal.*

Proof. (i) We first show that AD implies that ω_1 is measurable. We already know that ω_1 is inaccessible in every $L[a]$, $a \subset \omega$.

Let us consider the following partial ordering of the Baire space:

$$x \preccurlyeq y \quad \text{if and only if} \quad x \in L[y] \tag{33.7}$$

and the corresponding equivalence relation

$$x \equiv y \quad \text{if and only if} \quad x \preccurlyeq y \text{ and } y \preccurlyeq x. \tag{33.8}$$

We say that $A \subset \mathcal{N}$ is $\equiv$-*closed* if ($x \in A$ and $y \equiv x$) implies $y \in A$. Note that the collection $\mathcal{B}$ of all $\equiv$-closed sets in $\mathcal{N}$ is a complete Boolean algebra.

If $x_0 \in \mathcal{N}$, then we let

$$\operatorname{cone}(x_0) = \{x \in \mathcal{N} : x_0 \preccurlyeq x\} = \{x : x_0 \in L[x]\} \tag{33.9}$$

and call $\operatorname{cone}(x_0)$ a *cone*. Clearly, every cone is $\equiv$-closed. Let

$$\mathcal{F} = \{A \in \mathcal{B} : A \text{ contains a cone}\}.$$

We claim that $\mathcal{F}$ is a σ-complete filter on $\mathcal{B}$. Let $A_0, A_1, \ldots, A_n, \ldots$ be elements of $\mathcal{F}$. For each n, we choose $x_n \in \mathcal{N}$ such that $A_n \supset \operatorname{cone}(x_n)$. Let $x \in \mathcal{N}$ be defined as follows: $x(\langle n, m\rangle) = x_n(m)$ for all $n, m \in \boldsymbol{N}$ (where $\langle\ \rangle$ is a pairing function). It is clear that for each n, $x_n \in L[x]$ and hence $\operatorname{cone}(x) \subset \operatorname{cone}(x_n) \subset A_n$. Thus $\bigcap_{n=0}^{\infty} A_n$ is in $\mathcal{F}$.

Lemma 33.13. AD *implies that for every* $\equiv$*-closed* $A \subset \mathcal{N}$, *either* A *or its complement contains a cone. Hence* $\mathcal{F}$ *is a* σ*-complete ultrafilter on* $\mathcal{B}$.

Proof. We show that if I has a winning strategy in the game G_A, then A contains a cone (and similarly, if II has a winning strategy, then $\mathcal{N} - A \in \mathcal{F}$). Let σ be a winning strategy for I. It suffices to show that A contains the cone $\{x \in \mathcal{N} : \sigma \in L[x]\}$.

Let $x \in \mathcal{N}$ be such that $\sigma \in L[x]$. Then $a = \sigma * x$ is in A because σ is a winning strategy. Clearly, $x \in L[a]$, and because $\sigma \in L[x]$, we also have $a \in L[x]$ and hence $x \equiv a$. Since A is $\equiv$-closed, we have $x \in A$. $\square$

Now we can use AD to find a nonprincipal σ-complete ultrafilter U on ω_1. For each $x \in \mathcal{N}$, let $f(x) = \aleph_1^{L[x]}$; $f(x)$ is a countable ordinal. Note that if $x \equiv y$, then $f(x) = f(y)$, and hence for every $X \subset \omega_1$, the set $f_{-1}(X) \subset \mathcal{N}$ is $\equiv$-closed. Let

$$U = \{X \subset \omega_1 : f_{-1}(X) \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is a σ-complete ultrafilter on $\mathcal{B}$, U is a σ-complete ultrafilter on ω_1. It remains to show that U is nonprincipal. But for every $\alpha < \omega_1$, if $x \in \mathcal{N}$ is such that $\aleph_1^{L[x]} = \alpha$, then there is $y \succcurlyeq x$ such that $\aleph_1^{L[y]} > \alpha$; and hence $f_{-1}(\{\alpha\}) \notin \mathcal{F}$.

Thus AD implies that ω_1 is a measurable cardinal.

Lemma 33.14. *Assume* AD. *Then for every $S \subset \omega_1$, the set $\{x \in \mathrm{WO} : \|x\| \in S\}$ is $\mathbf{\Pi}^1_1$. Consequently, there is some $a \subset \omega$ such that $S \in L[a]$.*

Proof. If $x \in \mathcal{N}$, then for each $n \in \boldsymbol{N}$ we let $x_n \in \mathcal{N}$ be such that $x_n(m) = x(\langle n, m\rangle)$ for all $m \in \boldsymbol{N}$. We consider the following game:

33.15. The Solovay Game. Let $S \subset \omega_1$. Player I plays $a = \langle a(0), a(1), \ldots\rangle$, and II plays $b = \langle b(0), b(1), \ldots\rangle$. If $a \notin \mathrm{WO}$, then II wins; if $a \in \mathrm{WO}$, then II wins if

$$\{\alpha \in S : \alpha \le \|a\|\} \subset \{\|b_n\| : n \in \omega\} \subset S.$$

We claim that I does not have a winning strategy in the Solovay game. Let σ be a winning strategy for I; for each $b \in \mathcal{N}$, let $f(b)$ be the $a \in \mathcal{N}$ such that $\langle a(0), b(0), a(1), b(1), \ldots\rangle = \sigma * b$. The set $f(\mathcal{N})$ is a $\mathbf{\Sigma}^1_1$ subset of WO, and by the Boundedness Lemma, there is an $\alpha < \omega_1$ such that $\|f(b)\| < \alpha$ for all $b \in \mathcal{N}$. Hence let $b \in \mathcal{N}$ be such that $\{\|b_n\| : n \in \omega\} = S \cap \alpha$. Then $\sigma * b$ is a play won by player II, and hence σ cannot be a winning strategy for I.

Now the lemma follows: Let $S \subset \omega_1$. By AD, player II has a winning strategy τ in the Solovay game. For each a, let $g(a)$ be the $b \in \mathcal{N}$ such that $\langle a(0), b(0), \ldots\rangle = a * \tau$. It follows that for each $a \in \mathrm{WO}$,

$$\|a\| \in S \quad \text{if and only if} \quad \exists n\, \|a\| = \|(g(a))_n\|$$

and consequently the set $\{x \in \mathrm{WO} : \|x\| \in S\}$ is $\mathbf{\Pi}^1_1$. By Lemma 25.22, $S \in L[a]$ for some $a \subset \omega$. □

We can now complete the proof of (i). If $X \subset \omega_1$, then $X \in L[a]$ for some $a \subset \omega$. Since $\aleph_1$ is a measurable cardinal, $a^\sharp$ exists, and it follows that either X or $\omega_1 - X$ contains a closed unbounded subset. Thus the closed unbounded filter on ω_1 is an ultrafilter.

By the Countable Axiom of Choice, the closed unbounded filter is σ-complete, and we therefore conclude (as we work in ZF + the Principle of Dependent Choices) that AD implies that the closed unbounded filter on ω_1 is the unique σ-complete normal ultrafilter on ω_1.

(ii) We shall now show that, assuming AD, $\aleph_2$ is a measurable cardinal. For each $x \in \mathcal{N}$, let $f(x)$ denote the successor cardinal of the (real) cardinal $\aleph_1$ in $L[x]$:

$$f(x) = ((\aleph_1)^+)^{L[x]}.$$

If $x \subset \omega$, then because $x^\sharp$ exists, $f(x)$ is an ordinal less than $\aleph_2$. Moreover, if $x \equiv y$, then $f(x) = f(y)$, and hence $f_{-1}(X)$ is $\equiv$-closed for each $X \subset \omega_2$. Thus let us define an ultrafilter U on ω_2 as follows:

$$U = \{X \subset \omega_2 : f_{-1}(X) \in \mathcal{F}\}.$$

We wish to show that U is an $\aleph_2$-complete nonprincipal ultrafilter on ω_2. Since $\mathcal{F}$ is σ-complete, U is σ-complete. It is also easy to see that U is nonprincipal: If $\alpha < \omega_2$ and $f(x) = \alpha$, then there exists an $S \subset \omega_1$ such that α is not a cardinal in $L[S]$; then by Lemma 33.14 there is a $y \in \mathcal{N}$ such that $x \in L[y]$ and $f(y) > \alpha$. Hence $f_{-1}(\{\alpha\})$ does not contain a cone.

It remains to show that U is $\aleph_2$-complete. Since U is σ-complete, it suffices to show that if

$$X_0 \supset X_1 \supset \ldots \supset X_\alpha \supset \ldots \qquad (\alpha < \omega_1) \tag{33.10}$$

is a descending sequence of subsets of ω_2 such that each $f_{-1}(X)$ contains a cone then $f_{-1}(\bigcap_{\alpha<\omega_1} X_\alpha)$ contains a cone.

Let us consider such a sequence (33.10), and let $X = \bigcap_{\alpha<\omega_1} X_\alpha$. We shall use the following game: Player I plays $a = \langle a(0), a(1), \ldots\rangle$, and II plays $b = \langle b(0), b(1), \ldots\rangle$. If $a \notin \text{WO}$, then I loses; if $a \in \text{WO}$ and $\|a\| = \alpha$, then II wins if $\text{cone}(b) \subset f_{-1}(X_\alpha)$.

We claim that I does not have a winning strategy in this game: If σ is a winning strategy for I, then the set of all $a \in \mathcal{N}$ that I plays by σ against all possible $b \in \mathcal{N}$, is a $\mathbf{\Sigma}^1_1$ subset of WO and hence there is α such that $\|a\| < \alpha$ for all these a's. Now II can beat I simply by playing some $b \in \mathcal{N}$ such that $\text{cone}(b) \subset f_{-1}(X_\alpha)$.

Thus II has a winning strategy τ, and we intend to show that $f_{-1}(X)$ contains the cone $\{x \in \mathcal{N} : \tau \in L[x]\}$. Let $\alpha < \omega_1$ and let $x \in \mathcal{N}$ be such that $\tau \in L[x]$; we want to show that $f(x) \in X_\alpha$.

Let P_α be the notion of forcing that collapses α onto ω: The conditions are finite sequences of ordinals less than α. Since $\aleph_1$ is inaccessible in $L[x]$, $L[x]$ has only countably many subsets of P_α, and therefore there exists an $L[x]$-generic filter G on P_α. Let $a \in \text{WO}$ be such that $\|a\| = \alpha$ and let $L[a] = L[G]$ and let $y \in \mathcal{N}$ be such that $L[y] = L[x][G] = L[x][a]$.

Since G is generic on P_α over $L[x]$, all cardinals in $L[x]$ greater than α are preserved in $L[x][G]$. In particular, $(\aleph_1^+)^{L[x]}$ is preserved and hence $f(y) = f(x)$.

Now if I plays $a = \langle a(0), a(1), \ldots\rangle$ and if II plays against a by his winning strategy τ, II produces $b = \langle b(0), b(1), \ldots\rangle$ such that $\text{cone}(b) \subset f_{-1}(X_\alpha)$. But since $b \in L[\tau, a]$ and $\tau \in L[x]$, we have $b \in L[x, a] = L[y]$ and therefore

$y \in \mathrm{cone}(b)$. It follows that $f(y) \in X_\alpha$; and because $f(x) = f(y)$, we have $f(x) \in X_\alpha$, as we wanted to prove.

This completes the proof of Theorem 33.12. □

It turns out that under Determinacy there exist many measurable cardinals. Of particular interest have been the *projective* ordinals $\boldsymbol{\delta}^1_n$. By definition

$$\boldsymbol{\delta}^1_n = \sup\{\xi : \xi \text{ is the length of a } \boldsymbol{\Delta}^1_n \text{ prewellordering of } \mathcal{N}\}.$$

By the results in Chapter 25, $\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 \le \omega_2$. It has been established (under AD) that all the $\boldsymbol{\delta}^1_n$ are measurable cardinals, along with other properties, such as $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$. The size of each $\boldsymbol{\delta}^1_{2n+1}$ has now been calculated exactly; in particular, $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ and $\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$. The analysis of the $\boldsymbol{\delta}^1_n$'s depends heavily on calculations of length of ultrapowers by measures on projective ordinals.

An important ordinal (isolated by Moschovakis) is

$$\Theta = \sup\{\xi : \xi \text{ is the length of a prewellordering of } \mathcal{N}\}.$$

AD implies that $\Theta = \aleph_\Theta$, and if in addition $V = L(\boldsymbol{R})$ then Θ is a regular cardinal (Solovay). Θ is the limit of measurable cardinals (Kechris and Woodin), and for every $\lambda < \Theta$, there exists a normal ultrafilter on $[\lambda]^\omega$ (Solovay). As for the consistency strength of AD, we have:

Theorem 33.16 (Woodin). *Assume* AD *and* $V = L(\boldsymbol{R})$. *Then there exists an inner model with infinitely many Woodin cardinals.* □

Theorem 33.16 is optimal, as the existence of infinitely many Woodin cardinals is equiconsistent with AD; see Theorem 33.26. (We remark that the proof of Theorem 33.16 uses the following result: If AD and $V = L(\boldsymbol{R})$, then Θ is a Woodin cardinal in the model *HOD*.)

Projective Determinacy

In this section we address the question how strong is the determinacy assumption when restricted to games that have a simple enough definition. In particular, we turn our attention to the game G_A where $A \subset \mathcal{N}$ is a projective set.

When A is open (or closed) then G_A is determined:

Lemma 33.17. *If $A \subset \mathcal{N}$ is an open set, then G_A is determined.*

Proof. Player I plays $\langle a_0, a_1, \ldots\rangle$, player II plays $\langle b_0, b_1, \ldots\rangle$, and I wins if $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in A$. Let us assume that player I does not have a winning strategy, and let us show that II has a winning strategy. The strategy for II is as follows: When I plays a_0, then because I does not have a winning strategy, there exists b_0 such the position $\langle a_0, b_0\rangle$ is not yet lost for II. That is, I does not have a winning strategy in the game $G_A^{\langle a_0,b_0\rangle}$ that starts at $\langle a_0, b_0\rangle$, in which I plays $\langle a_1, a_2, \ldots\rangle$ and II plays $\langle b_1, b_2, \ldots\rangle$, and in which I wins when $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in A$.

Let II plays such b_0. When I plays a_1, then again, because II is not yet lost at $\langle a_0, b_0\rangle$, there exists b_1 such that II is not yet lost at $\langle a_0, b_0, a_1, b_1\rangle$. Let II play such b_1. And so on. We claim that this strategy for II is a winning strategy.

Let $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$ be a play which II plays by the above strategy; We want to show that $x \notin A$. If $x \in A$, then because A is open, there is $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle \subset x$ such that $O(s) \subset A$. But then it is clear that II is lost at s; a contradiction. □

The same argument (interchanging the players) would show that every closed game is determined. Or, we can show that every closed game is determined as follows: If A is closed, then I has a winning strategy in G_A if and only if there is $a_0 \in \boldsymbol{N}$ such that II does not have a winning strategy in the open game $G_A^{a_0}$ in which II make a first move b_0, then I plays a_1, etc., and II wins if $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is in the open set $\mathcal{N} - A$. Since $G_A^{a_0}$ is determined for all $a_0 \in \boldsymbol{N}$, G_A is determined.

One of the major results in descriptive set theory is Martin's proof that for every Borel set A the game G_A is determined:

Theorem 33.18 (Martin [1975]). *All Borel games are determined.* □

We shall not give a proof. It can be found either in Martin's paper [1975], or in the survey article [1980] by Kechris and Martin; furthermore, Martin gives a simplification of his proof in [1985].

Analytic Determinacy, i.e., determinacy of all analytic games, is already a large cardinal assumption:

Theorem 33.19. *Let $a \in \mathcal{N}$. Every $\Sigma^1_1(a)$ game is determined if and only if $a^\sharp$ exists.*

Thus Analytic Determinacy is equivalent to the statement

$$a^\sharp \text{ exists for all } a \in \mathcal{N}. \tag{33.11}$$

The proof of Analytic Determinacy from (33.11) is due to Martin [1969/70]. The necessity of (33.11) is a result of Harrington [1978]. We omit Harrington's proof and prove a corollary of Martin's result. We note however that the proof of the corollary can be converted into a proof of the "if" part of Theorem 33.19 without much difficulty.

Corollary 33.20. *If there exists a measurable cardinal, then all analytic games are determined.*

Proof. Let κ be a measurable cardinal and let $A \subset \mathcal{N}$ be an analytic set. We want to show that the game G_A is determined.

Let us use the tree representation of analytic sets. There is a tree $T \subset Seq_2$ such that for all $x \in \mathcal{N}$,

$$x \in A \leftrightarrow T(x) \text{ is ill-founded.}$$

Let $\preccurlyeq$ be the linear ordering of the set Seq that extends the partial ordering $\supset$: If $s, t \in Seq$, then $s \preccurlyeq t$ either if $s \supset t$, or if s and t are incompatible, and $s(n) < t(n)$ where n is the least n such that $s(n) \neq t(n)$. Thus

$$x \in A \leftrightarrow T(x) \text{ is not well-ordered by } \preccurlyeq.$$

We also recall that $T(x) = \{t : (x{\restriction}n, t) \in T \text{ for some } n\}$ and that the first n levels of $T(x)$ depend only on $x{\restriction}n$. For $s \in Seq$, we let $T_s = \{t : (u, t) \in T$ for some $u \subset s\}$; then $T_{x\restriction n}$ is exactly the first n levels of the tree $T(x)$. We need some further notation. Let $t_0, t_1, \ldots, t_n, \ldots$ be an enumeration of the set Seq. If $s \in Seq$ is a sequence of length $2n$, let K_s be the finite set $\{t_0, \ldots, t_{n-1}\} \cap T_s$ and let $k_s = |K_s|$.

We shall now define an auxiliary game G^*: Player I plays natural numbers $a_0, a_1, a_2, \ldots$, and player II plays pairs $(b_0, h_0), (b_1, h_1), (b_2, h_2), \ldots$ where $b_0, b_1, b_2, \ldots$ are natural numbers, and for each n, h_n is an order-preserving mapping from $(K_s, \preccurlyeq)$ into κ where $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle$ such that $h_0 \subset h_1 \subset h_2 \subset \ldots \subset h_n \subset \ldots$. If player II is able to follow these rules throughout the game, then he wins. Otherwise, I wins.

It is clear that the game G^* is determined: If I does not have a winning strategy, then he cannot prevent II from following the rules and thus II has a winning strategy, namely his each move is to reach a position in which I does not have a winning strategy. (The argument is the same as in the proof of determinacy of open games; in fact, G^* is an open game in a suitable topology.)

If II wins a play in the game G^*, then he has constructed an order-preserving mapping $h = \bigcup_{n=0}^{\infty} h_n$ of $(T(x), \preccurlyeq)$ into κ, where $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$; hence $\preccurlyeq$ well-orders $T(x)$ and so $x \notin A$. Thus we can view the game G^* as a variant of G_A, but more difficult for player II: II tries to make sure that $x \notin A$, and in addition, he tries to construct an embedding of $(T(x), \preccurlyeq)$ in κ. Hence it is fairly obvious that if II has a winning strategy in the game G^*, then II has a winning strategy in G_A: If τ^* is a winning strategy for II in G^*, let τ be as follows. When I plays a_0, let $\tau(\langle a_0\rangle) = b_0$ where $(b_0, h_0) = \tau^*(\langle a_0\rangle)$; then when I plays a_1, let $\tau(\langle a_0, b_0, a_1\rangle) = b_1$ where $(b_1, h_1) = \tau^*(\langle a_0, (b_0, h_0), a_1\rangle)$; etc.

Since G^* is determined, it suffices to prove the following lemma in order to show that G_A is determined:

Lemma 33.21. *If I has a winning strategy in G^*, then I has a winning strategy in G_A.*

Proof. Let σ^* be a winning strategy for I in G^*. After $2n+2$ moves, the players have produced a sequence $s = \langle a_0, b_0, \ldots, a_n, b_n\rangle$, and II has constructed order-preserving functions $h_0 \subset \ldots \subset h_n$; the strategy σ^* then tells player I what to play next. Let E be the range of h_n; E is a finite subset of κ, and in fact its size is k_s. We observe that there is one and only one way II could have constructed $h_0, \ldots, h_n$ so that $\operatorname{ran}(h_n) = E$; the reason is that h_n is the unique order-preserving one-to-one function between $(K_s, \preccurlyeq)$ and E. Thus σ^* depends (as long as II plays correctly) only on $s \in Seq$ and the finite set $E \subset \kappa$.

For each $s \in Seq$ of even length, let F_s be the following function from $[\kappa]^{k_s}$ into ω:

$$F_s(E) = \sigma^*(s, E). \tag{33.12}$$

Each F_s is a partition of $[\kappa]^{k_s}$ into ω pieces; and because κ is measurable, there exists a set $H \subset \kappa$ of size κ homogeneous for each F_s. Let us denote by $\sigma(s)$ the unique value of $F_s(E)$ for $E \in [H]^{k_s}$.

We shall complete the proof by showing that σ is a winning strategy for I in the game G_A. Let $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$ be a play in which I plays by σ. We shall show that $x \in A$.

Assume that on the contrary, $x \notin A$. Then $(T(x), \preccurlyeq)$ is well-ordered and has order-type $< \omega_1$. Since H is uncountable, there exists an embedding h of $(T(x), \preccurlyeq)$ into H. Let us consider the following play of the game G^*: I plays a_0. Then II plays (b_0, h_0) where h_0 is the restriction of h to $K_{\langle a_0, b_0\rangle}$. Then I plays a_1 and II plays (b_1, h_1) where h_1 is the restriction of h to $K_{\langle a_0, b_0, a_1, b_1\rangle}$. And so on.

We show that in this play, player I plays by the strategy σ^*. Clearly, $a_0 = \sigma(\emptyset) = \sigma^*(\emptyset, \emptyset)$. Then $a_1 = \sigma(\langle a_0, b_0\rangle)$, and by the definition of σ it is clear that $\sigma(\langle a_0, b_0\rangle) = \sigma^*(\langle a_0, b_0\rangle, h(K_{\langle a_0, b_0\rangle}))$ and therefore a_1 is a move according to σ^*. And so on: All the moves $a_0, a_1, \ldots, a_n, \ldots$ are by σ^*.

This is a contradiction because σ^* is a winning strategy for I in G^*, but the play we described is won by player II. It follows that $x \in A$ and hence σ is a winning strategy for I in the game G_A. $\square$

This completes the proof of $\mathbf{\Sigma}^1_1$ Determinacy assuming a measurable cardinal. This assumption can be weakened to the assumption that $a^\sharp$ exists for all $a \subset \omega$. The above proof is then modified as follows: We play the auxiliary game as before; κ is an uncountable cardinal. The definition of the auxiliary game is absolute for the model $L[T]$, and it follows that either I or II has a winning strategy for G^*, which si in $L[T]$. In particular, in Lemma 33.21, we may take $\sigma^* \in L[T]$. Then the collection $\{F_s : s \in Seq\}$, where F_s is defined by (33.12), is in $L[T]$, and an indiscernibility argument shows that there is an uncountable set $H \subset \kappa$ of indiscernibles for $L[T]$ such that each F_s has the same value for all $E \in [H]^{k_s}$. The rest of the proof is the same.

Determinacy of all projective games is considerably stronger than Analytic Determinacy: $\mathbf{\Delta}^1_2$ Determinacy yields an inner model with a Woodin cardinal, and for every n, $\mathbf{\Delta}^1_{n+1}$ Determinacy yields an inner model with n Woodin cardinals.

The proof of Theorem 33.3 shows that Projective Determinacy implies that every projective set of reals is Lebesgue measurable, has the Baire property, and if uncountable, contains a perfect subset. The most important consequence of PD for the structure of projective sets of reals is the existence of scales. The following is a generalization of (25.28) and (25.34):

Definition 33.22. A Π^1_n-*norm* on a Π^1_n set A is a norm φ on A with the property that there exist a Π^1_n relation $P(x,y)$ and a Σ^1_n relation $Q(x,y)$ such that for every $y \in A$ and all x,

$$x \in A \text{ and } \varphi(x) \le \varphi(y) \leftrightarrow P(x,y) \leftrightarrow Q(x,y).$$

A Σ^1_n-*norm* on a Σ^1_n set A is defined similarly, as is a Π^1_n-*scale* on a Π^1_n set (or a Σ^1_n-*scale* on a Σ^1_n set).

We say that the class Π^1_n has the *prewellordering property* (the *scale property*) if every Π^1_n set has a Π^1_n-norm (a Π^1_n-scale). Π^1_n has the *uniformization property* if every Π^1_n relation on $\mathcal{N} \times \mathcal{N}$ is uniformized by a Π^1_n function. Similarly for Σ^1_n.

Theorem 33.23 (Moschovakis [1971]). *Assume Projective Determinacy. Then the following classes have the scale property (for every $a \in \mathcal{N}$):*

$$\Pi^1_1(a),\ \Sigma^1_2(a),\ \Pi^1_3(a),\ \Sigma^1_4(a),\ \ldots,\ \Pi^1_{2n+1}(a),\ \Sigma^1_{2n+2}(a),\ \ldots.$$

Corollary 33.24. *Assume* PD. *The classes $\Pi^1_{2n+1}(a)$ and $\Sigma^1_{2n+2}(a)$ have the prewellordering property and the uniformization property and satisfy the reduction principle; the classes $\Sigma^1_{2n+1}(a)$ and $\Pi^1_{2n+2}(a)$ satisfy the separation principle.*

The scale property generalizes the prewellordering property, and implies uniformization (using the proof of Kondô's Theorem 25.26; cf. Exercise 33.4). The prewellordering property implies the reduction principle (as in Exercise 25.7; see Exercise 33.5), which in turn implies the separation principle for the dual class (cf. Exercise 25.9).

Moreover, since reduction holds for Π^1_{2n+1} and Σ^1_{2n+2}, separation fails for these classes (see Exercise 25.11 and Exercise 33.6). Hence reduction, prewellordering and scale properties fail for the dual classes Σ^1_1, Π^1_2, Σ^1_3, ...

Instead of proving Theorem 33.23 we shall prove the weaker statement: Assuming PD, every Π^1_{2n+1} and every Σ^1_{2n+2} have the prewellordering property. The full result is proved by a similar, somewhat more complicated, method.

In Chapter 25 we proved that every Π^1_1 set has a Π^1_1-norm and that every Σ^1_2 set has a Σ^1_2-norm. The latter statement is easily derived from the former (Exercise 25.6). The same proof shows that if Π^1_{2n+1} has the prewellordering property then so does Σ^1_{2n+2} (Exercise 33.7). Thus it suffices to prove the following:

Lemma 33.25. *Assume that every $\mathbf{\Delta}^1_{2n}$ game is determined, and that every Σ^1_{2n} set has a Σ^1_{2n}-norm. Then every Π^1_{2n+1} set has a Π^1_{2n+1}-norm.*

Proof. Assume that the hypotheses hold and let B be a Π^1_{2n+1} set

$$x \in B \leftrightarrow \forall u\,(x,u) \in A$$

where A is Σ^1_{2n}. Let ψ be a Σ^1_{2n}-norm on A. For $x, y \in \mathcal{N}$ consider the game $G(x,y)$ where I plays $a(0)$, $a(1)$, ..., $a(k)$, ... and II plays $b(0)$, $b(1)$, ..., $b(k)$, ... and II wins if $(y,b) \notin A$, or $(x,a) \in A$ and $\psi(x,a) \le \psi(y,b)$. The game $G(x,y)$ is determined: If $y \notin B$ then II can win by playing b such that $(y,b) \notin A$; if $y \in B$ then

$$\text{II wins } G(x,y) \leftrightarrow P(x,a,y,b) \leftrightarrow Q(x,a,y,b)$$

and so the payoff set is $\mathbf{\Delta}^1_{2n}$ and hence determined.

For $x, y \in B$, define

$$x \preccurlyeq y \leftrightarrow \text{II has a winning strategy in } G(x,y).$$

We will show that $\preccurlyeq$ is a prewellordering of B and the corresponding norm is a Π^1_{2n+1}-norm.

Clearly, $x \preccurlyeq x$ for every $x \in B$ (II wins by copying I's moves).

To check that $\preccurlyeq$ is transitive, let $x \preccurlyeq y$ and $y \preccurlyeq z$. Thus II has winning strategies both in $G(x,y)$ and $G(y,z)$. We describe a winning strategy for II in $G(x,z)$: Let $k \ge 0$. When I plays $a(k)$ in $G(x,z)$, consider this the kth move in $G(x,y)$ and apply the strategy in $G(x,y)$ to respond $b(k)$. Consider $b(k)$ to be the kth move of I in $G(y,z)$ and apply the strategy in $G(y,z)$ to respond $c(k)$. This $c(k)$ is then the kth move of II in $G(x,z)$. It is clear that II wins.

Now assume that $x, y \in B$ and $x \not\preccurlyeq y$. Then I has a winning strategy in $G(x,y)$ (because II does not); we describe a winning strategy for II in $G(y,x)$ so that $y \preccurlyeq x$: Let $k \ge 0$. When I plays $a(k)$ in $G(y,x)$, let $b(k)$ be the move by I's winning strategy in $G(x,y)$ (responding to II's $a(k-1)$). Let II play $b(k)$ in $G(y,x)$. As I wins in $G(x,y)$, we have $\psi(x,a) > \psi(y,b)$, and so II wins.

To verify that $\preccurlyeq$ is well-founded, we assume to the contrary that $x_0 \succ x_1 \succ \ldots \succ x_n \succ \ldots$ is a descending chain, that I has a winning strategy in each of the games $G(x_i, x_{i+1})$. Let $a_0(0)$, $a_1(0)$, ..., $a_i(0)$, ... be the first moves of I by the winning strategies in the games $G(x_i, x_{i+1})$, and for each $k \ge 1$, let $a_0(k)$, $a_1(k)$, ..., $a_i(k)$, ... be I's moves responding to $a_1(k-1)$, $a_2(k-1)$, ..., $a_{i+1}(k-1)$, ... II's moves in these games. Since I wins all these games, we have $\psi(x_0,a_0) > \psi(x_1,a_1) > \ldots > \psi(x_i,a_i) > \ldots$, a contradiction.

Finally, for every $y \in B$,

$$x \in B \text{ and } x \preccurlyeq y \leftrightarrow \exists \tau\, \forall a\,(x,a) \le_\psi (y, a*\tau) \leftrightarrow \forall \sigma\, \exists b\,(x, \sigma * b) \le_\psi (y,b)$$

(where σ and τ denote strategies for I and II) and since ψ is a Σ^1_{2n}-norm on A, it follows that the norm associated with $\preccurlyeq$ is a Π^1_{2n+1}-norm on B. $\square$

Consistency of AD

The following theorem confirms what has been expected since the early 1970's: Determinacy is a large cardinal axiom:

Theorem 33.26 (Martin-Steel-Woodin). *If there exist infinitely many Woodin cardinals and a measurable cardinal above them, then the Axiom of Determinacy holds in* $L(\boldsymbol{R})$. □

In the rest of this chapter we shall outline some ideas on which this result is based. But first we state two related results:

Theorem 33.27 (Woodin). *The following are equiconsistent*:

(i) ZFC + "*There exist infinitely many Woodin cardinals.*"
(ii) ZF + AD. □

Theorem 33.28 (Martin-Steel). *Let* $n \in \boldsymbol{N}$. *If there exist* n *Woodin cardinals with a measurable cardinal above them then every* $\boldsymbol{\Pi}^1_{n+1}$ *game is determined.* □

The crucial concept in these proofs is that of a *homogeneous tree.*

Following the terminology and notation of Chapter 25, and specifically Definition 25.8, let K be a set and let T be a tree on $\omega \times K$ (or more generally, on $\omega^r \times K$). For $s \in Seq$ let

$$T_s = \{t : (s,t) \in T\}. \tag{33.13}$$

In the present context, a *measure* is a σ-complete ultrafilter, not necessarily nonprincipal.

Definition 33.29. A tree T on $\omega \times K$ is *homogeneous* if there are measures μ_s, $s \in Seq$, such that μ_s is a measure on T_s and:

(i) If t extends s then $\pi_{s,t}(\mu_t) = \mu_s$ where $\pi_{s,t}$ is the natural projection map from T_t to T_s.
(ii) If $x \in p[T]$ then the direct limit of the ultrapowers by $\{\mu_{x\restriction n} : n \in \omega\}$ is well-founded.

(See Exercise 33.8 for an explicit formulation of (ii).)

A tree T is κ-*homogeneous* (where κ is a regular uncountable cardinal) if the measures μ_s are all κ-complete. A set $A \subset \mathcal{N}$ is (κ-)*homogeneously Suslin* if $A = p[T]$ for some (κ-)homogeneous tree T.

Homogeneous trees are an abstraction of Martin's proof of $\boldsymbol{\Pi}^1_1$ Determinacy from a measurable cardinal. First, an analysis of Martin's proof shows the following:

Lemma 33.30. *If* $A \subset \mathcal{N}$ *is* $\boldsymbol{\Pi}^1_1$ *and* κ *is a measurable cardinal then* A *is* κ*-homogeneously Suslin.*

Proof. Exercise 33.10. □

Martin's proof essentially uses this (Exercise 33.11):

Lemma 33.31. *If $A \subset \mathcal{N}$ is homogeneously Suslin then A is determined.* □

A related concept is a *weakly homogeneous tree*:

Definition 33.32. A tree T on $\omega \times K$ is *weakly homogeneous* if there are measures $\mu_{s,t}$, where $s, t \in Seq$ and length(s) = length(t), such that $\mu_{s,t}$ is a measure on T_s and

(i) If $\bar{s} \supset s$ and $\bar{t} \supset t$ then $\pi_{s,\bar{s}}(\mu_{\bar{s},\bar{t}}) = \mu_{s,t}$.
(ii) If $x \in p[T]$ then there exists a $y \in \mathcal{N}$ such that the direct limit of the ultrapowers by $\{\mu_{x\restriction n, y\restriction n} : n \in \omega\}$ is well-founded.

A tree T is *κ-weakly homogeneous* if the $\mu_{s,t}$ are κ-complete. A set A is (κ-)*weakly homogeneously Suslin* if $A = p[T]$ for some (κ-)weakly homogeneous tree T.

It is not difficult to show that a set $A \subset \mathcal{N}$ is κ-weakly homogeneously Suslin if and only if it is a projection of a homogeneously Suslin set $B \subset \mathcal{N}\times\mathcal{N}$ (Exercises 33.12 and 33.13).

Theorem 33.26 follows, via Lemma 33.31, from the following two deep results:

Theorem 33.33 (Woodin [1988]). *If there exist infinitely many Woodin cardinals with a measurable cardinal above, then every subset of $\mathcal{N}$ in $L(\boldsymbol{R})$ is δ^+-weakly homogeneously Suslin, for some Woodin cardinal δ.* □

Theorem 33.34 (Martin and Steel [1988]). *If $A \subset \mathcal{N}$ is δ^+-weakly homogeneously Suslin, where δ is a Woodin cardinal, then $\mathcal{N} - A$ is homogeneously Suslin.* □

We shall return to Theorem 33.33 in a later chapter. As for Theorem 33.34, assume that $A = p[T]$ where T is weakly homogeneous. Then one constructs a tree $\tilde{T}$ such that $\mathcal{N} - A = p[\tilde{T}]$ in a manner similar to the tree representation for $\boldsymbol{\Pi}^1_2$ sets in Theorem 32.14. The heart of the argument in Martin-Steel's proof is to show that $\tilde{T}$ is a homogeneous tree.

Exercises

33.1. (i) The function $f(b) = \sigma * b$ is a one-to-one continuous function.
(ii) The set $\{\sigma * b : b \in \mathcal{N}\}$ contains a perfect subset.

33.2. I has a winning strategy in the perfect set game if and only if X has a perfect subset. II has a winning strategy if and only X is at most countable.

33.3. Let $n > 0$. If G_A is determined for every $\mathbf{\Sigma}^1_n$ set A, then G_A is determined for every $\mathbf{\Pi}^1_n$ set, and vice versa.

33.4. If every Π^1_{2n+1} set has a Π^1_{2n+1}-scale then every Π^1_{2n+1} relation is uniformized by a Π^1_{2n+1} function.

33.5. If every Π^1_{2n+1} set has a Π^1_{2n+1}-norm then Π^1_{2n+1} satisfies the reduction principle.

33.6. If Π^1_{2n+1} satisfies the reduction principle then it does not satisfy the separation principle.

33.7. If every Π^1_{2n+1} set has a Π^1_{2n+1}-norm (has a Π^1_{2n+1}-scale) then every Σ^1_{2n+2} set has a Σ^1_{2n+2}-norm (has a Σ^1_{2n+2}-scale).

33.8. Property (ii) in Definition 33.29 is equivalent to this: If $x \in p[T]$ and A_1, $A_2, \ldots$ are such that $\mu_{x\restriction n}(A_n) = 1$, then there exists an $f \in K^\omega$ such that $(x, f) \in [T]$ and $f\restriction n \in A_n$ for all n.

33.9. Every closed set is homogeneously Suslin.

[T is on $\omega \times \omega$ and each μ_s is principal.]

33.10. Let κ be a measurable cardinal. If A is $\mathbf{\Pi}^1_1$ then there is a κ-homogeneous tree T on $\omega \times \kappa$ such that $A = p[T]$.

[As A is $\mathbf{\Pi}^1_1$ there are linear orders $<_s$, $s \in Seq$, such that $<_s$ orders $\{0, \ldots, n-1\}$ where $n = \text{length}(s)$, $<_t$ extends $<_s$ if $s \subset t$, and such that $A = \{x : <_x \text{ is a well-ordering}\}$ where $<_x$ is the limit of the $<_{x\restriction n}$. Let T be the tree on $\omega \times \kappa$ such that $[T] = \{(x, f) : f \text{ is order-preserving from } (\omega, <_x) \text{ into } (\kappa, <)\}$. Let U be a normal measure on κ and let for s of length n, let μ_s on T_s be induced by U_n (on $[\kappa]^n$).]

33.11. If $A = p[T]$ and T is a homogeneous tree then the game G_A is determined.

[Use an auxiliary game G^* as in the proof of Corollary 33.20.]

33.12. If $B \subset \mathcal{N}^2$ is weakly homogeneously Suslin then so is the projection of B.

33.13. If T is a weakly homogeneous tree on $\omega \times K$ then there exists a homogeneous tree U on $(\omega \times \omega) \times K$ such that $p[T]$ is the projection of $p[U]$.

33.14. Let T be a homogeneous tree on $(\omega \times \omega) \times K$, and let $T' = \{(s, (t, u)) : ((s, t), u) \in T\}$. Then T' is a weakly homogeneous tree on $\omega \times (\omega \times K)$.

Historical Notes

Infinite games were first considered in the 1930. Mazur described an infinite game and conjectured its connection to Baire category, which was then proved by Banach.

In [1953] Gale and Stewart investigated infinite games in general and proved that the Axiom of Choice implies that there exist undetermined games and that open games are determined.

In [1962] Mycielski and Steinhaus proposed an axiom and called it the Axiom of Determinateness (AD). In [1963/64, 1966] Mycielski gave a comprehensive account of consequences of AD and related open problems.

Theorem 33.3(i) is due to Mycielski and Świerczkowski [1964]; the present proof (and the covering game) is due to Harrington. Theorem 33.3(iii) is due to Morton Davis [1964].

Following Solovay's discovery that AD implies that $\aleph_1$ is a measurable cardinal, attention has been turned to the relation between Determinacy and large cardinals. There have been numerous results in this direction, and a vast of literature exists on the subject. The reader can find an excellent account of current research on AD in Kanamori's book [1994]; a comprehensive treatment of the subject is expected to appear in the near future (Woodin et al. [∞]).

Theorem 33.12 is due to Solovay; the present proof of measurability of $\aleph_1$ (Lemma 33.13) is due to Martin [1968].

Projective ordinals $\boldsymbol{\delta}^1_n$ as well as the cardinal Θ were introduced by Moschovakis [1970] and studied extensively by Kechris [1974, 1978]. The calculation of the size of the $\boldsymbol{\delta}^1_5$ was accomplished by Steve Jackson, cf. [1988, 1999]. For the results on Θ, see e.g. Kechris [1985].

Theorem 33.18 (Borel Determinacy) is due to Martin [1975]; see also Martin [1985] and Kechris and Martin [1980].

Theorem 33.19: In [1969/70], Martin proved that analytic games are determined if $a^\sharp$ exists for all $a \in \mathcal{N}$; the converse was proved by Harrington in [1978].

Moschovakis' Theorem 33.23, cf. [1971], is the culmination of applications of Projective Determinacy to classical descriptive set theory: among others, see Blackwell [1967], Addison and Moschovakis [1968] and Martin [1968]. For a comprehensive survey, see Kechris and Moschovakis [1978].

Consistency of AD follows from the results of Martin, Steel and Woodin, cf. Martin and Steel [1988, 1989] and Woodin [1988].

Homogeneous trees are implicit in Martin and Solovay [1969] and in Martin [1969/70]. They were explicitly isolated by Kechris [1981]. Weakly homogeneous trees figured prominently in Woodin [1988].