34. Supercompact Cardinals and the Real Line

In this chapter we present results showing the effect of very large cardinals (such as supercompact) on the structure of sets of real numbers. In earlier chapters we showed that if $\aleph_1$ is inaccessible in every $L[x]$ (where $x \in \boldsymbol{R}$) then all $\boldsymbol{\Sigma}^1_2$ sets of reals are Lebesgue measurable, have the Baire property, and the perfect set property. If $x^\sharp$ exists for all $x \in \boldsymbol{R}$ then every $\boldsymbol{\Pi}^1_1$ game is determined. Thus already the existence of moderately large cardinals (such as measurable) has an effect on regularity of projective sets (but recall that—by Silver's Theorem 32.20—measurability is still weak to influence $\boldsymbol{\Sigma}^1_3$ sets, as measurable cardinals are consistent with a Σ^1_3 well-ordering of $\boldsymbol{R}$). It follows from the results presented below that if a supercompact cardinal exists, then all sets of reals in $L(\boldsymbol{R})$ have the regularity properties mentioned above.

Woodin Cardinals

As we mentioned in the last chapter (Theorem 33.27), the consistency strength of Determinacy is below a supercompact cardinal; the appropriate large cardinal concept (a Woodin cardinal) was isolated in the course of investigations leading to the proof of AD. Let us elaborate on the definition (Definition 20.31) of Woodin cardinals: Let κ and $\lambda \geq \kappa$ be cardinals, and let A be an arbitrary set. We say that κ is λ-*strong for* A if there exists an elementary embedding $j : V \to M$ with critical point κ such that

(34.1) (i) $j(\kappa) > \lambda$,
(ii) $V_\lambda \subset M$,
(iii) $A \cap V_\lambda = j(A) \cap V_\lambda$.

Hence κ is λ-strong if it is λ-strong for $\emptyset$, and by definition, δ is a Woodin cardinal if for every $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ that are λ-strong for A for all $\lambda < \delta$. We now present a different definition of Woodin cardinals and show that it is equivalent to Definition 20.31.

Definition 34.1. A cardinal δ is a *Woodin cardinal* if for every function $f : \delta \to \delta$ there exists a $\kappa < \delta$ with $f``\kappa \subset \kappa$, and an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(f)(\alpha)} \subset M$.

Every supercompact cardinal is Woodin, and is a limit of Woodin cardinals (Exercise 34.1). An immediate consequence of Definition 34.1 is that every Woodin cardinal is a Mahlo cardinal, and in fact has a stationary set of measurable cardinals. The following lemma proves the equivalence of Definitions 20.31 and 34.1:

Lemma 34.2. *The following are equivalent:*

(i) *For every $A \subset V_\delta$ there exists a $\kappa < \delta$ that is λ-strong for A for all $\lambda < \delta$.*
(ii) *For every $A \subset V_\delta$ the set of all $\kappa < \delta$ that are λ-strong for A for all $\lambda < \delta$ is stationary.*
(iii) *For every $f : \delta \to \delta$ there exists a $\kappa < \delta$ with $f``\kappa \subset \kappa$, and an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(f)(\alpha)} \subset M$.*
(iv) *For every $f : \delta \to \delta$ there exists a $\kappa < \delta$ with $f``\kappa \subset \kappa$, and an extender $E \in V_\kappa$ with critical point κ such that $j_E(f)(\kappa) = f(\kappa)$ and $V_{f(\kappa)} \subset \mathrm{Ult}_E$.*

Proof. It suffices to show that (i) implies (iv) and that (iii) implies (ii).

Assume that (i) holds, and let $f : \delta \to \delta$. By (i) there exists a $\kappa < \delta$ that is λ-strong for A for all $\lambda < \delta$. Let $\lambda < \delta$ be sufficiently large, and let E be an extender with critical point κ such that $V_{f(\kappa)} \subset \mathrm{Ult}_E$ and $f \cap V_\lambda = j_E(f) \cap V_\lambda$; such an extender exists in V_δ. Clearly, $f``\kappa \subset \kappa$, and since λ is sufficiently large, we have $j_E(f)(\kappa) = f(\kappa)$. Therefore (iv) holds.

Now assume that (iii) holds; let $A \subset V_\delta$ and let $C \subset \delta$ be a closed unbounded set. To prove (ii) we need a $\kappa \in C$ that is λ-strong for A for all $\lambda < \delta$. For each $\alpha < \delta$ let $f(\alpha)$ be a limit ordinal $\beta \in C$ such that if there exists a $\lambda < \delta$ such that α is not λ-strong for A, then such a λ exists below β. By (iii) there exists some $\kappa < \delta$ with $f``\kappa \subset \kappa$, and an elementary $j : V \to M$ with critical point $\kappa$ such that $V_{j(f)(\kappa)} \subset M$. Since $f``\kappa \subset \kappa$, $C \cap \kappa$ is closed unbounded in κ and hence $\kappa \in j(C)$. By elementarity it suffices to show that

$$M \vDash \kappa \text{ is } \lambda\text{-strong for } j(A) \text{ for all } \lambda < j(\delta). \tag{34.2}$$

Assume that (34.2) fails, and let λ be the least λ such that in M, κ is not λ-strong for $j(A)$. By definition of f we have $\lambda < j(f)(\kappa)$. Let E be the (κ, λ)-extender derived from j. It is routine to verify that $V_\lambda \subset \mathrm{Ult}_E$ and since $\mathrm{Ult}_E = \{(jf)(a) : f \in V,\ a \in V_\lambda\} \subset M$, it follows that $E \in M$. In M, E is a (κ, λ)-extender, and $V_\lambda^M = V_\lambda \subset \mathrm{Ult}_E(M)$.

We complete the proof by showing that

$$j(A) \cap V_\lambda^M = j_E^M(j(A)) \cap V_\lambda^M. \tag{34.3}$$

Each of the following equalities is easily verified: $j(A) \cap V_{j(\kappa)}^M = j(A \cap V_\kappa) = j_E(A \cap V_\kappa) = j_E^M(A \cap V_\kappa) = j_E^M(j(A) \cap V_\kappa) = j_E^M(j(A)) \cap j_E^M(V_\kappa)$, and (34.3) holds because $\lambda < j_E^M(\kappa)$ and $V_\lambda^M = V_\lambda \subset \mathrm{Ult}_E^M$. Therefore (ii) holds. $\square$

Property (iv) in Lemma 34.2 is Π^1_1 over V_δ and so the least Woodin cardinal is Π^1_1-describable and therefore not weakly compact.

Semiproper Forcing

A property of forcing somewhat weaker than properness, called *semiproperness*, have been of considerable use in the theory of large cardinals. We shall investigate it in some detail in Chapter 37; at this point we introduce semiproper forcing, prove basic properties and use it in an application involving $L(\boldsymbol{R})$ and supercompact cardinals.

Modifying condition (ii) in Lemma 31.6 and the characterization of properness in Theorem 31.7 we obtain the following: Let P be a notion of forcing and let λ be sufficiently large. Let M be a countable elementary submodel of $(H_\lambda, \in, <)$. A condition q is (M, P)-*semigeneric* if for every name $\dot\alpha \in M$ such that $\Vdash \dot\alpha$ is a countable ordinal,

$$q \Vdash \exists \beta \in M\ \dot\alpha = \beta. \tag{34.4}$$

Definition 34.3. A notion of forcing P is *semiproper* if for every sufficiently large λ there is a closed unbounded set in $[H_\lambda]^\omega$ of countable elementary submodels such that

$$\forall p \in M\ \exists q \le p\ q \text{ is } (M, P)\text{-semigeneric.}$$

Thus semiproperness is a weaker property than properness: Definition 34.3 is obtained by replacing arbitrary ordinal names in Lemma 31.6 by names for countable ordinals. While the condition in Theorem 31.7 is equivalent to preservation of stationary sets in $[\lambda]^\omega$, only the second part of the proof of Theorem 31.7 remains valid for semiproper forcing, and we get:

Theorem 34.4. *If P is semiproper then every stationary set $S \subset \omega_1$ remains stationary in V^P.* □

If P is semiproper and $\dot Q$ is semiproper in V^P, then $P * \dot Q$ is semiproper. Semiproperness is generally not preserved under countable support iteration; the proof of Theorem 31.15 does not generalize to iterations of semiproper forcing. (The reason is that a semiproper forcing may change the cofinality of a regular uncountable cardinal to ω: It is not necessarily the case that a countable set of ordinals in $V[G]$ is included in a set that is countable in V.) When the iteration adds no new countable sets, however, the proof of Theorem 31.15 does go through for semiproper forcing, and we have:

Lemma 34.5. *If P is a countable support iteration of semiproper forcing notions and if P is ω-distributive, then P is semiproper.* □

In Chapter 37 we shall deal with iterations of semiproper forcings.

The Model $L(\boldsymbol{R})$

We shall now show that if there exists a supercompact cardinal then every set of reals in $L(\boldsymbol{R})$ is Lebesgue measurable, has the Baire property and the perfect set property. In fact, the reals in $L(\boldsymbol{R})$ behave exactly as the reals in Solovay's model in Theorem 26.14(i). The regularity of sets of reals in $L(\boldsymbol{R})$ follows of course from Theorem 33.26, and we shall outline later in this chapter the methods that lead to the proof of that theorem. We shall prove the following:

Theorem 34.6 (Woodin). *Let κ be a superstrong cardinal and let $V[G]$ be the generic extension of V by the Lévy collapse* $\text{Col}(\aleph_0, {<}\kappa)$. *Then there exists an elementary embedding*

$$j : L(\boldsymbol{R}) \to (L(\boldsymbol{R}))^{V[G]}.$$

(For superstrong cardinals, see Exercise 34.2.)

Corollary 34.7. *If there exists a superstrong cardinal then every set of reals in $L(\boldsymbol{R})$ is Lebesgue measurable, has the Baire property, and has the perfect set property. In particular, there is no projective well-ordering of $\boldsymbol{R}$.* □

The main result used in the proof of Theorem 34.6 is the following result on saturated ideals:

Theorem 34.8. *If κ is a superstrong cardinal then there exists an ω-distributive κ-c.c. notion of forcing P such that in V^P, $\kappa = \aleph_2$ and there exists a normal $\aleph_2$-saturated ideal on ω_1.*

Let us show how Theorem 34.8 implies Theorem 34.6:

Proof of Theorem 34.6. Let P be the notion of forcing from Theorem 34.8, and let M be the generic extension of V by P. In M, let I be a normal $\aleph_2$-saturated ideal on ω_1, and let Q be the notion of forcing $P(\omega_1)/I$. Q yields an M-generic M-ultrafilter G on ω_1; let $N = \text{Ult}_G(M)$ be the generic ultrapower. If $j : M \to N$ is the generic elementary embedding then (by the results proved in Chapter 22), ω_1 is the critical point, $j(\omega_1) = \omega_2^M = \kappa$, and $(P(\omega_1))^N = (P(\omega_1))^{M[G]}$. Hence $\boldsymbol{R}^N = \boldsymbol{R}^{M[G]}$, and since $\boldsymbol{R}^M = \boldsymbol{R}$, j yields an elementary embedding

$$j : L(\boldsymbol{R}) \to (L(\boldsymbol{R}))^{V^{P*Q}}. \tag{34.5}$$

Let $B = B(P * Q)$. Since P satisfies the κ-chain condition and I is κ-saturated in V^P, B satisfies the κ-chain condition. Since P collapses all cardinals below κ to ω_1, and Q collapses ω_1 (because $\aleph_1^N = j(\aleph_1) = \aleph_2^M$),

B makes $\kappa = \aleph_1$. Every complete subalgebra of B generated by fewer than κ elements has size less than κ (by weak compactness of κ), and hence we have

(34.6) (i) B is κ-c.c.,
(ii) $B = \bigcup_{\alpha<\kappa} B_\alpha$ where $|B_\alpha| < \kappa$, and $B_\alpha <_{\text{reg}} B_\beta$ for all $\alpha < \beta < \kappa$,
(iii) every $\gamma < \kappa$ is countable in V^{B_α} for some $\alpha < \kappa$.

It follows from (34.6) that B is isomorphic to the Lévy collapse $\text{Col}(\omega, {<}\kappa)$ (see Exercise 34.5), and Theorem 34.6 now follows from (34.5). □

Proof of Theorem 34.8. The notion of forcing P is a countable support iteration of length κ, where κ is a superstrong cardinal. The goal is to build a model V^P in which for some stationary set $A \subset \omega_1$, the nonstationary ideal estricted to A, i.e., $I_{\text{NS}}{\restriction}A$, is $\aleph_2$-saturated. We shall first describe the iterates.

Let us fix a set $A \subset \omega_1$ such that both A and $\omega_1 - A$ are stationary. Let $\{A_i : i \in W\}$ be a maximal almost disjoint collection of stationary subsets of A (in this context, *almost disjoint* means that $A_i \cap A_k$ is nonstationary). If $|W| > \aleph_1$, consider the following notion of forcing Q_W: First let Q be the forcing that collapses $|W|$ to $\aleph_1$ with countable conditions. In V^Q, let $\dot{S} = \sum_{i\in W} A_i$, and let $P_{\dot{S}}$ be the forcing (from Theorem 23.8) that shoots a closed unbounded set through $(\omega_1 - A) \cup \dot{S}$. Let $Q_W = Q * P_{\dot{S}}$. Equivalently, let Q_W be the set of all pairs (q,p) such that

(34.7) (i) $q : \gamma + 1 \to W$ for some $\gamma < \omega_1$, and
(ii) $p \subset \omega_1$ is a closed countable set such that $\alpha \in p \cap A$ implies $\alpha \in \bigcup_{\xi<\alpha} A_{q(\xi)}$.

A condition (q', p') is stronger than (q,p) if $q' \supset q$ and p' is an end-extension of p.

The forcing Q_W preserves stationary subsets of ω_1 (Exercise 34.6) but is not necessarily semiproper. If W is not maximal then Q_W makes it maximal, and preserves all stationary subsets of $\omega_1 - A$ and of all A_i (Exercise 34.7). Note also that if A is a nonstationary set then the forcing Q_W as defined in (34.7) has a dense subset that is countably closed.

The effect of Q_W is that in the generic extension, $\sum_{i\in W} A_i = A \pmod{I_{\text{NS}}}$ and $|W| = \aleph_1$. (In the intermediate extension by Q there could exist a new stationary subset of A almost disjoint from each A_i, but it is destroyed by $P_{\dot{S}}$, and in V^{Q_W}, $\sum_{i\in W} A_i \cup (\omega_1 - A)$ contains a closed unbounded set.)

Now we define a countable support iteration P_α; and then we let $P = P_\kappa$. Using some book-keeping device (standard in forcing iterations), at stage α we consider (in V^{P_α}) a maximal almost disjoint collection $\{A_i : i \in W\}$ of stationary subsets of A such that $|W| > \aleph_1$. If Q_W is semiproper, we let $\dot{Q}_\alpha = Q_W$; otherwise we let $\dot{Q}_\alpha$ be the collapse with countable conditions of $2^{\aleph_2}$ to $\aleph_1$.

Thus P_α is a countable support iteration of semiproper forcing notions. The role of the set A is to guarantee that P_α is ω-distributive. To show that, consider the generic extension $V[G]$ obtained by shooting a closed unbounded set through $\kappa - A$. In $V[G]$, each $\dot{Q}_\alpha$ has a countably close dense subset, and so P_α is a countable support iteration of countably closed forcing notions. Hence P_α is ω-distributive in $V[G]$, and therefore in V.

Thus by Lemma 34.5, each P_α is semiproper. Since κ is inaccessible and P_κ is the direct limit of small forcing notions, P_κ satisfies the κ-chain condition. Since at cofinally many stages $\dot{Q}_\alpha$ collapses $2^{\aleph_2}$ (of V^{P_α}) to $\aleph_1$, κ becomes $\aleph_2$ in the model V^P. The model V^P has no new countable sets of ordinals, and every stationary subset of ω_1 remains stationary. Moreover, if $S \in V^P$ is a subset of ω_1 and is stationary in some V^{P_α}, then it remains stationary: This is because V^P is a semiproper forcing extension of V^{P_α}—the iteration from α to κ is a countable support iteration of semiproper forcings in V^{P_α}, and is ω-distributive (in V^{P_α}).

We shall now prove that in V^P, the ideal $I_{\mathrm{NS}}{\restriction}A$ is $\aleph_2$-saturated. Let G be a generic filter on P_κ and assume that in $V[G]$ there exists a maximal almost disjoint family of stationary subsets of A, such that $|W| > \aleph_1$ (hence $|W| = \aleph_2^{V[G]} = \kappa$). Let W be such a family, and assume further that W is chosen by our book-keeping to be the family considered at stage κ of the iteration.

Let $j : V \to M$ be an elementary embedding with critical point κ such that $V_{j(\kappa)} \subset M$. For all $\alpha < j(\kappa)$, $(P_\alpha)^M = P_\alpha$; $(P_{j(\kappa)})^M = j(P_\kappa)$ is the direct limit of the P_α while $P_{j(\kappa)}$ is the direct or inverse limit, depending on cf $j(\kappa)$ in V. Let H be such that $G * H$ is a generic filter on $P_{j(\kappa)}$. Let $\tilde{H} = \bigcup_{\kappa \le \alpha < j(\kappa)} H{\restriction}\alpha$; $G * \tilde{H}$ is an M-generic filter on $j(P_\kappa)$, and $j : V \to M$ extends (in $V[G * H]$) to an elementary embedding $j : V[G] \to M[G * \tilde{H}]$.

One more remark before we proceed. If $X \subset \omega_1$ is a stationary set in $M[G][\tilde{H}]$ then it is stationary in $V[G][H]$. This is because $X \in M[G][H{\restriction}\alpha]$ for some $\alpha < j(\kappa)$, hence X is stationary in $V[G][H{\restriction}\alpha]$, and $V[G][H]$ is a semiproper forcing extension of $V[G][H{\restriction}\alpha]$.

Lemma 34.9. *The forcing notion Q_W is semiproper in $V[G]$.*

This will complete the proof: If Q_W is semiproper then $Q_\kappa = Q_W$. It follows that $A = \sum W$, after forcing with Q_κ, hence in $M[G][\tilde{H}]$. This is a contradiction, since $j(W)$ is an almost disjoint family of stationary subsets of A, and $W \subset j(W)$ and $W \ne j(W)$, since $|W| = \kappa$ in $V[G]$.

Proof. Assume that Q_W is not semiproper. Let $N = (H_{\kappa^+})^{V[G]}$; there is a $p \in Q_W$ such that the set

$$S = \{M \in [N]^\omega : p \in M \text{ and no } q \le p \text{ is } (M, Q_W)\text{-semigeneric}\} \tag{34.8}$$

is stationary. Since Q_W is not semiproper, the forcing $\dot{Q}_\kappa$ is the collapse (with countable conditions) of 2^κ to $\aleph_1$. Let G_κ be a generic filter on $\dot{Q}_\kappa$; since $\dot{Q}_\kappa$ is ω-closed, S remains stationary in $V[G][G_\kappa]$. Since $S \in M[G]$, S is in $M[G][G_\kappa]$ a stationary subset of $[N]^\omega$ and N has cardinality $\aleph_1$. Let π

be (in $M[G][G_\kappa]$) a one-to-one correspondence between N and ω_1, and let $\tilde{S} = \omega_1 \cap \pi``S$. $\tilde{S}$ is, in $M[G][G_\kappa]$, a stationary subset of ω_1.

Now work in $V[G][H]$ and consider the forcing notion $Q_{j(W)} = j(Q_W)$ and the condition $p \in Q_W$ from (34.8). By Exercise 34.8, $j(p) = p$ forces that $j(S)$ is nonstationary. In the generic extension, $j(S)$ is a nonstationary subset of $[j(N)]^\omega$, and hence $j``S$ is a nonstationary subset of $[j``N]^\omega$ and therefore S is a nonstationary subset of $[N]^\omega$. It follows that (in V[G][H])

$$p \Vdash_{Q_{j(W)}} \tilde{S} \text{ is a nonstationary subset of } \omega_1. \tag{34.9}$$

The set $\tilde{S}$ is stationary in $M[G][G_\kappa]$ and therefore in $M[G][\tilde{H}]$ (which is a semiproper forcing extension of $M[G][G_\kappa]$). The family $j(W)$ is, in $M[G][\tilde{H}]$, a maximal almost disjoint family of stationary subsets of A and therefore intersects either $\omega_1 - A$ or some $E \in W$ in a stationary set; for instance let $E \in j(W)$ be such that $\tilde{S} \cap E$ is stationary. Thus $\tilde{S} \cap E$ is stationary in $V[G][H]$, and (by Exercise 34.7), remains stationary after forcing (over $V[G][H]$) with $Q_{j(W)}$. This contradicts (34.9). □

Stationary Tower Forcing

We shall describe a forcing notion, due to Hugh Woodin, that is used, among other applications, to generalize Theorem 34.6 and prove Theorem 33.33.

Definition 34.10 (Stationary Tower Forcing). Let κ be an inaccessible cardinal. The forcing notion $Q = Q_{<\kappa}$ consists of conditions (V_α, S) where $\alpha < \kappa$ and S is a stationary subset of $[V_\alpha]^\omega$. A condition (V_β, T) is stronger than (V_α, S) if $\alpha \le \beta$ and $T \restriction V_\alpha \subset S$.

Equivalently, $(V_\beta, T) \le (V_\alpha, S)$ if $\alpha \le \beta$ and $T \subset S^{V_\beta}$ where S^{V_β} is the lifting of S to $[V_\beta]^\omega$; see Theorem 8.27. The forcing $Q_{<\kappa}$ is not separative: Two conditions (V_α, S) and (V_β, T) are equivalent if and only if for some (all) $\gamma \ge \alpha, \beta$, $S^{V_\gamma} \simeq T^{V_\gamma}$ mod the nonstationary ideal on $[V_\gamma]^\omega$.

If (V_α, S) is a condition, V_α is determined by S ($V_\alpha = \bigcup S$), so we can abuse the notation by calling S a condition in $Q_{<\kappa}$; we say that V_α is the support of S.

If G is a generic filter then for each $\alpha < \kappa$, $G \cap [V_\alpha]^\omega$ is a normal ultrafilter extending the closed unbounded filter. In $V[G]$, we define a *generic ultrapower* $\mathrm{Ult}_G(V)$ as follows: Consider formulas $f \in V$ defined on some V_α, $\alpha < \kappa$, and let, for f on V_α and g on V_β,

(34.10) $f =_G g$ if for some $S \in G$ with support $\ge \alpha, \beta$, $f(x \cap V_\alpha) = g(x \cap V_\beta)$ for all $x \in S$;

$f \in_G g$ is defined similarly. Below we prove that if κ is a Woodin cardinal then $\mathrm{Ult}_G(V)$ is well-founded.

The following definition was inspired by the earlier sections of this chapter, in particular Exercise 34.9: Let $M \subset N$ be countable models; we say that N *end-extends* M if for all $u \in M$, $u \cap N = u \cap M$.

Definition 34.11. Let A be a dense set of conditions in $Q_{<\kappa}$. A is *semiproper* if for all sufficiently large λ there is a closed unbounded set in $[H_\lambda]^\omega$ of countable elementary submodels M such that for some countable $N \prec H_\lambda$,

(34.11) (i) $M \subset N$ and N end-extends $M \cap V_\kappa$,
(ii) $\exists S \in A \cap N$ with support V_α such that $N \cap V_\alpha \in S$.

The definition has equivalent variants:

Lemma 34.12. *Each of the following two properties is equivalent to semiproperness of A:*

(i) *There is a closed unbounded set of countable $M \prec V_{\kappa+1}$ such that some countable $N \prec V_{\kappa+1}$ satisfies* (34.11).
(ii) *For all sufficiently large λ, for every countable $M \prec H_\lambda$ such that $A \in M$ there is a countable $N \prec H_\lambda$ that satisfies* (34.11).

Proof. For the nontrivial implication (i) $\Rightarrow$ (ii) see Exercise 34.10. □

The following is the key lemma. If δ is a Woodin cardinal and A is a dense subset of $Q_{<\delta}$ then for a closed unbounded set of $\kappa < \delta$, $A \cap Q_{<\kappa}$ is dense in $Q_{<\kappa}$, and for a stationary set of κ, κ is λ-strong for A for all $\lambda < \delta$.

Lemma 34.13. *Let $\kappa < \delta$ be such that $A \cap Q_{<\kappa}$ is dense in $Q_{<\kappa}$ and that κ is λ-strong for A for all $\lambda < \delta$. Then $A \cap Q_{<\kappa}$ is semiproper in $Q_{<\kappa}$.*

Proof. Toward a contradiction, assume that the set

$$S = \{M \prec V_{\kappa+1} : \text{there is no } N \prec V_{\kappa+1} \text{ such that (34.11) holds}\}$$

is stationary. Let $\lambda > \kappa + 1$ ($\lambda < \delta$) be such that $(V_\lambda, \in) \prec (V_\delta, \in)$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$, $V_\lambda \in M$ and $j(A) \cap V_\lambda = A \cap V_\lambda$. We have $S \in M$, $S \in j(Q_{<\kappa})$, and $M \vDash j(A)$ is dense in $j(Q_{<\kappa})$, and so there exists a $T < S$ such that $T \in j(A) \cap V_\lambda = A \cap V_\lambda$. Note that $T < S$ means that for every $z \in T$, $z \cap V_{\kappa+1} \in S$.

Let V_α be the support of T. We shall find a countable $x \prec V_{\kappa+1}$, a countable $y \prec j(V_\alpha)$, and a countable $z \prec V_\alpha$ such that $y \cap V_\alpha = z \in T$, $z \cap V_{\kappa+1} = x$, and $T \in y$. Then y end-extends $j(x \cap V_\kappa) = x \cap V_\kappa$, $T \in j(A) \cap y$, and $y \cap V_\alpha \in T$. This implies (by (34.11)) that $j(x) \notin j(S)$, but $z \in T < S$ implies that $x \in S$, a contradiction.

To find x, y, and z, let $F : V_\alpha^{<\omega} \to V_\alpha$ be the function $F(e \cup \{f\}) = j(f)(T, e)$ (if defined and $\in V_\alpha$; e is a finite subset of V_α and $f \in V_\alpha$ is a function), and let $z \in T$ be closed under F. Let $y = \{(jf)(T, e) : f \in z \text{ and } e \in z^{<\omega}\}$ and $x = z \cap V_{\kappa+1}$. We have $y \prec j(V_\alpha)$, $y \cap V_\alpha = z$ and $T \in y$, as desired. □

Lemma 34.13 is used to prove the following theorem on the stationary tower forcing:

Theorem 34.14 (Woodin, [1988]). *Let δ be a Woodin cardinal and let $Q_{<\delta}$ be the stationary tower forcing. Let G be a generic filter on $Q_{<\delta}$, and let $j : V \to \text{Ult}_G$ be the canonical elementary embedding into the generic ultrapower. Then*

(i) Ult_G *is well-founded.*
(ii) $j(\omega_1) = \delta$.
(iii) *In $V[G]$, the model Ult_G is closed under $<\delta$-sequences.*

We sketch the proof of (i) and refer the reader to Woodin [1988] for the details of the complete proof. (Woodin's paper states the theorem for a supercompact cardinal but the proof can be easily adapted. See also Woodin [1999], Theorem 2.36.)

Proof. (i) If A is a dense set and N is a countable model, we say that N *captures* A if (34.11)(ii) holds. First we claim that if $A \subset Q_{<\kappa}$ is semiproper then for every condition $p \in Q_{<\kappa}$ there is a stronger condition q such that every $N \in q$ captures A. This is proved by showing that the set

$$q = \{N \prec V_{\kappa+1} : N \cap V_\alpha \in p \text{ and } N \text{ captures } A\}$$

(where V_α is the support of p) is stationary. To show this, let $F : V_{\kappa+1}^{<\omega} \to V_{\kappa+1}$ and let $M \prec H_\lambda$ for some λ be such that $A \in M$, $F \in M$, and $M \cap V_\alpha \in p$. Let $N \supset M$ be such that N end-extends $M \cap V_\kappa$ and captures A. Then $N \cap V_{\kappa+1} \in q$ and is closed under F.

One proves similarly that if A_n, $n < \omega$, are semiproper then for every p there exists a $q < p$ such that every $N \in q$ captures every A_n.

Now let $\langle \dot{f}_n : n < \omega\rangle$ be a sequence of names of functions in the generic ultrapower, names for a descending sequence of ordinals. For each n there is a dense set A_n such that for each $S \in A_n$ there is an ordinal function f_n^S on S such that $S \Vdash \dot{f}_n = f_n^S$. Let $\kappa < \delta$ be such that each $A_n \cap Q_{<\kappa}$ is semiproper, and let $p \in G$ be such that every $N \in p$ captures each A_n.

Now we define, for each $n < \omega$, a function f_n on p as follows: If $N \in p$, let $f_n(N) = f_n^S(N)$ where $S \in A_n$ (with support V_α) is such that $N \cap V_\alpha \in S$. The functions f_n are defined for almost all (mod I_{NS}) $N \in p$, and $f_{n+1}(N) < f_n(N)$ for all n, producing a descending sequence of ordinals.

(iii) is proved similarly; one can show that if A_α, $\alpha < \gamma$, with $\gamma < \delta$ are semiproper then for every p there exists a $q < p$ such that every $N \in q$ captures A_α for all $\alpha \in N$.

(ii) follows by showing that δ remains a regular cardinal in $V[G]$ and that every $\alpha < \delta$ is collapsed to ω. While the proof of regularity of δ is similar to the proof of (iii), the proof that α becomes countable is a consequence of the following fact that is easy to verify: If $S \in Q_{<\delta}$ has support V_α then

$$S \Vdash j``V_\alpha \in j(S).$$

□

Weakly Homogeneous Trees

Let δ be a Woodin cardinal. By Theorem 34.14 there exists a generic elementary embedding $j : V \to M$ such that $\boldsymbol{R}^M = \boldsymbol{R}^{V[G]}$ and $j(\omega_1) = \delta$; G is a generic filter on $Q = Q_{<\delta}$. Consider the following forcing notion P in $V[G]$: A forcing condition p is a V-generic filter on the Lévy collapse $\mathrm{Col}(\omega, <\lambda)$ for some $\lambda < \delta$; p is stronger than q if $p \supset q$. The forcing P does not add reals and if $H \subset P$ is $V[G]$-generic then H is a V-generic filter on $\mathrm{Col}(\omega, <\delta)$. Under additional assumptions on δ, such as that δ is also a limit of Woodin cardinals, every countably generated subalgebra of $Q_{<\delta}$ has cardinality less than δ, and $\boldsymbol{R}^{V[G]} = \boldsymbol{R}^{V[H]}$. Hence there exists an elementary embedding $j : L(\boldsymbol{R}) \to L(\boldsymbol{R})^{\mathrm{Col}(\omega,<\delta)}$ and consequently, the sets of reals in $L(\boldsymbol{R})$ have the regularity properties stated in Corollary 34.7.

The above argument yields a stronger result:

Corollary 34.15. *If δ is a Woodin cardinal and a limit of Woodin cardinals, if P is a forcing notion such that $|P| < \delta$, and if G is a generic filter on P, then the model $L(\boldsymbol{R})^{V[G]}$ is elementarily equivalent to $L(\boldsymbol{R})$.*

Proof. As δ remains a Woodin cardinal in $V[G]$, we can find a V-generic filter H on $\mathrm{Col}(\omega, <\delta)$ such that $V[G] \subset V[H]$ and $V[H]$ is a $\mathrm{Col}(\omega, <\delta)$-generic extension of $V[G]$, and elementary embeddings $j : L(\boldsymbol{R}) \to L(\boldsymbol{R})^{V[H]}$ and $k : L(\boldsymbol{R})^{V[G]} \to L(\boldsymbol{R})^{V[H]}$. □

This property of Woodin cardinals (that the theory of $L(\boldsymbol{R})$ is unchanged by small forcing) has been exploited by Woodin to prove the following theorem. In [1988] these results are stated under the assumption that a supercompact cardinal exists, but Woodin subsequently proved the theorem under the assumption stated below. The assumption is close to optimal as ω Woodin cardinals do not suffice; compare also with Theorem 35.20. The proof of (ii) uses the result in (i), and is a restatement of Theorem 33.33, establishing Determinacy in $L(\boldsymbol{R})$.

Theorem 34.16 (Woodin, [1988]). *Assume that there exist infinitely many Woodin cardinals with a measurable cardinal above them. Let λ be the supremum of the first ω Woodin cardinals.*

(i) *For every set $A \subset \boldsymbol{R}$ in $L(\boldsymbol{R})$ there exist trees T and S such that*

$$A = p[T], \qquad \boldsymbol{R} - A = p[S]$$

and for every forcing P such that $|P| < \lambda$, if $G \subset P$ is generic then

$$V[G] \vDash p[T] \cup p[S] = \boldsymbol{R} \text{ and } p[T] \cap p[S] = \emptyset.$$

(ii) *Every set $A \subset \boldsymbol{R}$ in $L(\boldsymbol{R})$ is κ-weakly homogeneously Suslin, for all $\kappa < \lambda$.* □

Exercises

34.1. Let κ be a supercompact cardinal. Then

(i) κ is a Woodin cardinal, and
(ii) there is a normal measure on κ such that almost all $\delta < \kappa$ are Woodin.

A cardinal κ is *superstrong* if there exists an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(\kappa)} \subset M$.

34.2. If κ is supercompact then there exists a normal measure on κ such that almost all $\alpha < \kappa$ are superstrong cardinals.

34.3. If κ is superstrong then κ is a Woodin cardinal, and there exists a normal measure on κ such that almost all $\delta < \kappa$ are Woodin cardinals.

34.4. P is semiproper if and only if for every $p \in P$, player II has a winning strategy in the following game: I plays names $\dot{\alpha}_n$ for countable ordinals, II plays β_n, and II wins if $\exists q \le p\ q \Vdash \forall n\, \exists k\, (\dot{\alpha}_n = \beta_k)$.

34.5. Let B be an atomless complete Boolean algebra that satisfies (34.6). Then $B = \mathrm{Col}(\omega, {<}\kappa)$.

[Similar to Theorem 26.12.]

34.6. Q_W preserves stationary subsets of ω_1.

[If $T \subset A$ is stationary then $T \cap A_i$ is stationary for some i, and remains stationary in V^Q. Hence $T \cap A_i \cap \dot{S}$ is stationary in V^Q, and then use Exercise 23.6.]

34.7. Let W be a family of stationary subsets of $A \subset \omega_1$, and let Q_W be defined as in (34.7). If S is a stationary subset of some $A_i \in W$ or a stationary subset of $\omega_1 - A$, then S remains stationary. Also, $A = \sum W$ in V^{Q_W}.

[As in Exercises 23.6 and 34.6.]

34.8. Let N be a transitive model, $N \supset \omega_1$, let $P \in N$ and $p \in P$. Then p forces that the set

$$\{M \in [N]^\omega : M \prec N \text{ and } \exists q \le p\ q \text{ is } (M,P)\text{-semigeneric}\}$$

contains a closed unbounded set.

[Let $C = \{M : \text{if } \dot{\alpha} \in M \text{ and } \dot{\alpha}^G < \omega_1 \text{ then } \dot{\alpha}^G \in M\}$.]

Let W be a family of stationary subsets of ω_1 and let Q_W be defined as in (34.7) (i.e., $A = \omega_1$).

34.9. Q_W is semiproper if and only if for all sufficiently large λ there is a closed unbounded set of $M \prec H_\lambda$ such that exists an $N \prec H_\lambda$ with $M \subset N$ and $\omega_1 \cap M = \omega_1 \cap N$, and for some $S \in W \cap N$, $\omega_1 \cap N \in S$.

34.10. Show that (i) implies (ii) in Lemma 34.12.

[By (i) let $F \in H_\lambda$ be such that $F : (V_{\kappa+1})^{<\omega} \to V_{\kappa+1}$ and that for every $M \prec V_{\kappa+1}$ closed under F there is some $N \prec V_{\kappa+1}$ such that (34.11) holds. Now if $M \prec H_\lambda$ and $A \in M$, there exists such an F in M, and so $\tilde{M} = M \cap V_{\kappa+1}$ is closed under F. Let $\tilde{M} \subset \tilde{N} \prec V_{\kappa+1}$ be so that (34.11) holds for $\tilde{N}$. Then let $N = \{f(e) : f \in M \text{ and } e \in (\tilde{N} \cap V_\kappa)^{<\omega}\}$. Verify that $M \subset N \prec H_\lambda$ and (34.11) holds for N.]

Historical Notes

Woodin cardinals were introduced by Woodin (both Definitions 20.31 and 34.1). Strong and superstrong cardinals were considered by Mitchell [1979a], Dodd and Jensen (Dodd [1982]) and Baldwin [1986], in their study of inner models.

Semiproper forcing was introduced by Shelah and was investigated extensively by Foreman, Magidor, and Shelah in [1988]. For Theorem 34.6, see Shelah and Woodin [1990]. The proof of Theorem 34.8 was inspired by the work of Foreman, Magidor, and Shelah on Martin's Maximum.

Stationary tower forcing and its applications (Theorem 34.14, Corollary 34.15, and Theorem 34.16) are due to Woodin.