35. Inner Models for Large Cardinals

This chapter is an introduction to the highly technical theory of inner models for large cardinals. We present the fundamental concepts and ideas of the theory and state, mostly without a proof (or giving an outline of a proof) some significant results.

There are two major themes in the theory of inner models. One is that with a given large cardinal property one can associate a minimal inner model of ZFC for that property. An example is the model $L[U]$ for a measurable cardinal. The other is the construction of *core models* for large cardinals. These generalize the Dodd-Jensen core model K that we describe in some detail. K is an inner model of ZFC that satisfies GCH and either the Covering Theorem holds for K (see below) or $L[U]$ exists.

Definition 35.1. Let M be an inner model of ZFC. We say that the *Covering Theorem* holds for M if for every uncountable set X of ordinals there exists some $Y \in M$ such that $|Y| = |X|$.

If the Covering Theorem holds for some inner model M that satisfies GCH then (exactly as in Corollaries 18.31–18.33) the Singular Cardinal Hypothesis holds, every singular cardinal is singular in M, and $(\kappa^+)^M = \kappa^+$ for every singular cardinal κ.

A theory of core models has been developed for large cardinals up to a Woodin cardinal. While the Covering Theorem does not hold beyond K, a generalized core model possesses the following feature: If there exists no inner model for a large cardinal with a given property then the core model M for such a property is "close to" the universe V; typically, $(\kappa^+)^M = \kappa^+$ for every singular strong limit cardinal. This feature makes core models a tool for gauging the consistency strength of set-theoretical conjectures.

As an example, Dodd-Jensen's Covering Theorem for K gives a lower bound for the consistency of the failure of SCH: If SCH fails then the Covering Theorem for K fails and therefore there exists an inner model for a measurable cardinal.

The Core Model

The origin of the core model theory was the construction, by Dodd and Jensen, of the core model K. The Dodd-Jensen core model ("the core model up to a measurable cardinal") is an inner model K that contains much of the large cardinal structure without the existence of measurable cardinals. Its main features are:

(1) K has a definable well-ordering, satisfies GCH and combinatorial principles such as $\square$.
(2) There exists a nontrivial elementary embedding $j : K \to K$ if and only if $L[U]$ exists.
(3) If $L[U]$ does not exist then the Covering Theorem holds for K.

If $L[U]$ exists then K has a simple definition:

Definition 35.2. Assume that $L[U]$ exists. The *core model* is the inner model

$$K = \bigcap_{\alpha \in Ord} \mathrm{Ult}_U^{(\alpha)}(L[U]).$$

It is easy to verify that K is an inner model. Only the lower parts of the iterated ultrapowers matter; see Exercise 35.1.

Central to the theory of inner models is the "internal" definition of K. The main idea underlying the theory, including the generalizations of K, is that the core model is approximated by transitive models (sets), so called *mice*. Mice are the building blocks of K, as much as the models L_α are for L.

The preferred hierarchy for the fine structure of L is the Jensen hierarchy J_α. For K, we use its relativization J_α^A for the language $\{\in, A\}$ where A is a unary predicate: We modify Definition 27.2 (of rudimentary functions) by adding the function $F(x) = x \cap A$ to obtain functions *rudimentary in* A, and let, for any set A,

(35.1) $\mathrm{rud}_A(M) =$ the closure of $M \cup \{M\}$ under functions rudimentary in A.

Definition 35.3. $J_0^A = \emptyset$, $J_{\alpha+1}^A = \mathrm{rud}_A(J_\alpha^A)$, $J_\alpha^A = \bigcup_{\beta<\alpha} J_\beta$ if α is a limit ordinal.

It follows that $L[A] = \bigcup_{\alpha \in Ord} J_\alpha^A$. See Exercise 35.2 for some properties of the relativized Jensen hierarchy. Each J_α^A is a transitive set and we abuse the notation by using J_α^A to denote also the model $(J_\alpha^A, \in, A \cap J_\alpha^A)$.

Definition 35.4. A *mouse* is a transitive model $M = J_\alpha^U$ such that

(i) U is a normal κ-complete iterable M-ultrafilter on some $\kappa < \alpha$,
(ii) all iterated ultrapowers of J_α^U by U are well-founded,
(iii) $M = H_1^M(\gamma \cup p)$ (the Σ_1 Skolem hull) for some $\gamma < \kappa$ and some finite $p \subset \alpha$.

More specifically, M is a mouse at κ.

Some remarks about the definition: Iterability is the condition (19.17) that makes possible iterating the ultrapower. In Chapter 19 we assumed that M is a model of ZF^- which is not the case for mice (the requirement (iii) precludes it; see Exercise 35.3). Łoś's Theorem is not true in general in ultrapowers of mice, only for Σ_0 formulas (actually for Σ_1 formulas as we discuss below). One uses the fine structure to overcome this difficulty. Finally, for (ii) it is sufficient that the ω_1st iterate is well-founded.

Definition 35.5. $K = L\{M : M \text{ is a mouse}\}$.

Below we outline a proof of the following theorem:

Theorem 35.6 (Dodd-Jensen).

(i) *K is an inner model of* ZFC *and has a Σ_2 well-ordering.*
(ii) *K satisfies* GCH.
(iii) *$\boldsymbol{R}^K$ has a Σ^1_3 well-ordering.*
(iv) *$K^K = K$, and $K^{V[G]} = K$ for every generic extension.*
(v) *If $L[U]$ exists then $K = \bigcap_{\alpha\in Ord} \mathrm{Ult}^{(\alpha)}_U L[U]$.*
(vi) *In K, $L[U]$ does not exist.*
(vii) *If $0^\sharp$ does not exist then $K = L$. If $0^\sharp$ exists then $0^\sharp \in K$. More generally, for every $x \in K$, if $x^\sharp$ exists then $x^\sharp \in K$.* □

We now outline the basic theory of mice and techniques used in the core model theory. First we state a special case of (vii):

Lemma 35.7. *A mouse exists if and only if $0^\sharp$ exists.*

Proof. If a mouse exists at κ, then the iterates $\kappa^{(\alpha)}$ are indiscernibles for L.

Conversely, let $0^\sharp$ exist and let i_α be the Silver indiscernibles. For each α, let $j_\alpha : L \to L$ be the unique elementary embedding with critical point i_α such that $j_\alpha(i_\alpha) = i_{\alpha+1}$; Let U_α be the corresponding L-ultrafilter. Using indiscernibility, one shows that $j_{U_\alpha} = j_\alpha$. Each U_α is iterable and all iterates $\mathrm{Ult}^{(\beta)}_{U_\alpha}(L)$ are well-founded.

Now consider $\kappa = i_0$ and $U = U_0$, and let $M = J^U_{\kappa+1}$. One proves that $U \subset J^U_{\kappa+1} \subset L$, and $\mathrm{Ult}^{(\alpha)}_U(J^U_{\kappa+1}) = J^{U_\alpha}_{i_\alpha+1}$. Finally, one verifies that $J^U_{\kappa+1} = H^M_1(\emptyset)$, and hence M is a mouse. □

Instrumental in the core model theory is the *comparison* of mice, a Σ_2 well-ordering of the class of all mice obtained by comparing the transfinite iterates of mice.

For every regular uncountable cardinal λ, let $\mathcal{C}_\lambda$ denote the closed unbounded filter on λ. Let $M = J^U_\alpha$ be a mouse at κ, and let λ be a regular cardinal greater than κ^+. Then (as in Chapter 19), the λth iterate of M has the form

$$\mathrm{Ult}_U^\lambda(M) = J_\beta^{\mathcal{C}_\lambda}. \tag{35.2}$$

Clearly, $J_\beta^{\mathcal{C}_\lambda}$ is constructible from M, but M is also constructible from $J_\beta^{\mathcal{C}_\lambda}$: M is isomorphic to the Σ_1 Skolem hull of $\gamma \cup i_{0,\lambda}(p)$ in $J_\beta^{\mathcal{C}_\lambda}$.

For a given mouse M, γ and p are fixed to be least possible such that $M = H_1^M(\gamma \cup p)$, in the following sense: γ is the least such γ, and then p is the least p in the lexicographic ordering of finite descending sequences of ordinals.

Definition 35.8. Let $M = J_\alpha^U = H_1^M(\gamma \cup p)$ and $M' = J_{\alpha'}^{U'} = H_1^{M'}(\gamma' \cup p')$ be mice, and let λ be (any) sufficiently large regular cardinal. Let $i_{0,\lambda} : M \to J_\beta^{\mathcal{C}_\lambda}$ and $i'_{0,\lambda} : M' \to J_{\beta'}^{\mathcal{C}_\lambda}$ be the iterated ultrapowers, with $q = i_{0,\lambda}(p)$ and $q' = i'_{0,\lambda}(p')$. We define $M < M'$ as follows:

(i) either $\beta < \beta'$,
(ii) or $\beta = \beta'$ and $\gamma < \gamma'$,
(iii) or $\beta = \beta'$ and $\gamma = \gamma'$, and $q < q'$ (in the descending lexicographic ordering).

Lemma 35.9. *$<$ is a well-ordering of mice, and if $M \le M'$ then $M \in L[M']$.*

Proof. If $\beta < \beta'$ then $J_\beta^{\mathcal{C}_\lambda} \in J_{\beta'}^{\mathcal{C}_\lambda}$, and $M \in L[J_\beta^{\mathcal{C}_\lambda}]$. □

An analysis of the complexity of $<$ reveals that it is a Σ_2 relation (and that $<$ on $\boldsymbol{R}^K$ is Σ_3^1). We recall that the constructible hierarchy is Σ_1; the added complexity in K is caused by the condition that every iterated ultrapower of a mouse is well-founded.

Being a mouse is absolute for transitive models of ZF, and so is the well-ordering of mice. Thus if M is an inner model then $K^M = L\{N : N$ is a mouse and $N \in M\}$, and $K^K = K$. Since the well-ordering of mice is definable in K, K is a model of ZFC.

If $V[G]$ is a generic extension of V (by a set forcing) then for all sufficiently large regular cardinals λ, the closed unbounded filter $\mathcal{C}_\lambda$ on λ in $V[G]$ is generated by the closed unbounded filter in V. Hence $J_\beta^{\mathcal{C}_\lambda}$ is the same in $V[G]$ as in V, and so every mouse in $V[G]$ is in V. Hence $K^{V[G]} = K$.

If $L[U]$ exists then every mouse is in $L[\mathcal{C}_\lambda]$ for some λ; but $L[\mathcal{C}_\lambda] = \mathrm{Ult}_U^{(\lambda)} L[U]$. Hence $K \subset \bigcap_{\alpha \in Ord} \mathrm{Ult}^{(\alpha)} L[U]$. If x is a set of ordinals in $\bigcap_\alpha \mathrm{Ult}^{(\alpha)}$ then for some $\lambda > \sup x$, $x \in L[U^{(\lambda)}]$. Hence there exists a mouse $M \prec_{\Sigma_1} J_{\lambda^+}^{U^{(\lambda)}}$ such that $x \in M$. Therefore $K = \bigcap_{\alpha \in Ord} \mathrm{Ult}^{(\alpha)} L[U]$. The latter model has no submodel with a measurable cardinal and so neither does K.

One important feature of K is the following, which we state without a proof:

Lemma 35.10. *If mice exist then $K = \bigcup\{M : M$ is a mouse$\}$.* □

Proofs in the core model theory such as the proof of Lemma 35.10 involve iterations of mice. One of the difficulties is that since mice do not satisfy ZF$^-$, the resulting embeddings are not fully elementary. It is easy to verify that $i_{0,1} : M \to \text{Ult}_U M$ is Σ_0-elementary, and an additional argument shows that $i_{0,1}$ is Σ_1-elementary; similarly for $i_{\alpha,\beta}$. While the finite iterates of a mouse are mice, arbitrary iterates are not. See Exercises 35.5–35.7.

The proof of GCH in K resembles somewhat Silver's proof of GCH in $L[U]$. Instrumental in the proof (and proofs of combinatorial principles in K) are condensation arguments, similar to Lemmas 18.38 and 27.5. These proofs use heavily the fine structure of K (including projecta, standard codes and parameters), reducing arguments about Σ_n to Σ_1.

The fine structure of K makes it possible to generalize combinatorial properties such as $\diamondsuit$ and $\square$ from L to K.

There is an alternative way of developing the theory of K and proving the main theorems. This method, due to Magidor, uses the closed unbounded filter directly. Instead of using mice, K can be defined using Definition 35.12 below (this definition is equivalent to the Dodd-Jensen definition).

Definition 35.11. The closed unbounded filter $\mathcal{C}_\kappa$ on κ *survives at* β if for every n and $f : [\kappa]^n \to \{0,1\}$ with $f \in L_{\beta+1}[\mathcal{C}_\kappa]$ there is a set $C \in \mathcal{C}_\kappa$ homogeneous for f.

Definition 35.12. A set x belongs to K if and only if for some $\kappa > \text{rank}(x)$ and some β, $x \in L_\beta[\mathcal{C}_\kappa]$ and $\mathcal{C}_\kappa$ survives at β.

If $\mathcal{C}_\kappa$ survives at β then it survives at all $\beta' < \beta$. If it survives at all β then $L[\mathcal{C}_\kappa]$ is the inner model for one measurable cardinal. $\mathcal{C}_\kappa$ survives vacuously at every $\beta < \kappa$, and survives at κ if and only if $0^\sharp$ exists (Exercise 35.8).

If $L[U]$ exists then for every sufficiently large regular κ, $L[\mathcal{C}_\kappa] = L[U^{(\kappa)}]$, and so $\bigcup\{L_\beta[\mathcal{C}_\kappa] \cap V_\kappa : \mathcal{C}_\kappa \text{ survives at } \beta\} = \bigcup\{L[U^{(\kappa)}] \cap V_\kappa : \kappa > \omega \text{ regular}\} = K$.

The Covering Theorem for *K*

The two main results on the core model are that unless $L[U]$ exists, K is rigid and the Covering Theorem holds for K:

Theorem 35.13 (Dodd-Jensen). *The following are equivalent*:

(i) *$L[U]$ exists.*
(ii) *There exists a nontrivial elementary embedding $j : K \to K$.* □

Theorem 35.14 (Dodd-Jensen's Covering Theorem for K). *If $L[U]$ does not exist, then for every uncountable set X of ordinals there exists a set $Y \supset X$ in K such that $|Y| = |X|$.* □

If $L[U]$ exists then the ultrapower by U yields an elementary embedding $j : K \to K$. The proof of the converse shows somewhat more: If $j : K \to M$ is elementary, then necessarily $M = K$ (and $L[U]$ exists). The proofs of both theorems use the fine structure of K, but a great deal of the fine structure can be eliminated when using Magidor's approach.

Since K is a model of ZFC, Corollaries 18.31, 18.32 and 18.33 all remain true when L is replaced by K:

Corollary 35.15. *If $L[U]$ does not exist then every singular cardinal is singular in K, $(\kappa^+)^K = \kappa^+$ for every singular κ, and the Singular Cardinal Hypothesis holds.* □

The Covering Theorem for $L[U]$

By Prikry's Theorem 21.10 there is a generic extension of $L[U]$ in which the measurable cardinal κ of $L[U]$ remains a cardinal while $\operatorname{cf} \kappa = \omega$. It follows that the Covering Theorem for $L[U]$ fails. However, it turns out that the existence of a Prikry sequence is the only obstacle to the Covering Theorem:

Theorem 35.16 (Dodd-Jensen's Covering Theorem for $L[U]$). *Assume that there is an inner model with a measurable cardinal, let κ be the least such cardinal and let U be a measure on κ in $L[U]$. Then*

(i) *either $0^\dagger$ exists, or*
(ii) *the Covering Theorem holds for $L[U]$, or*
(iii) *there exists an ω-sequence $S \subset \kappa$ Prikry generic over $L[U]$, such that the Covering Theorem holds for $L[U][S]$.* □

Note that by Theorem 21.14, $L[U][S] = L[S]$.

The Core Model for Sequences of Measures

The theory of K has been generalized by W. Mitchell who constructed a core model K^m for sequences of measures (the "core model up to $o(\kappa) = \kappa^{++}$"). In analogy with K,

(i) K^m has a definable well-ordering, satisfies GCH and □.
(ii) There exists a nontrivial $j : K^m \to K^m$ if and only if there is an inner model for a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.
(iii) If there is no model for $o(\kappa) = \kappa^{++}$ then a "weak" covering theorem holds for K^m.

Mitchell's core model is the union of mice where a mouse is an appropriate generalization of the Dodd-Jensen mouse. The main result on K^m is as follows:

Theorem 35.17 (Mitchell).

(i) *K^m is a model of* ZFC + GCH.
(ii) *K^m has a Σ_2 well-ordering and $\square$ holds; $\boldsymbol{R} \cap K^m$ has a Σ^1_3 well-ordering.*

If there exists no inner model for $o(\kappa) = \kappa^{++}$, then:

(iii) *If U is a normal iterable K^m-ultrafilter with $\mathrm{Ult}_U(K^m)$ well-founded then $U \in K^m$.*
(iv) *If $j : K^m \to M$ is a nontrivial elementary embedding then j is an iterated ultrapower using measures in K^m. (Hence there is no nontrivial $j : K^m \to K^m$.)*
(v) *If κ is a singular strong limit cardinal then $(\kappa^+)^{K^m} = \kappa^+$.* □

Clause (v) is often called "the Weak Covering Theorem."

Theorem 35.17 is a useful tool for obtaining lower bounds for the consistency strength. As an example, we present the following application:

Corollary 35.18 (Mitchell). *Assume that κ is a measurable cardinal and $2^\kappa > \kappa^+$. Then there is an inner model with a measurable λ of order λ^{++}.*

Proof. If there is no such model then (iii) and (iv) hold. Let D be a normal measure on κ and $j_D : V \to M = \mathrm{Ult}_D(V)$; let $j = j_D{\restriction}K^m : K^m \to N$. By (iv), j is an iterated ultrapower, $j = i_{0,\vartheta} : K^m \to \mathrm{Ult}^{(\vartheta)} = N$, by measures in K^m. Let N_ν, $\nu \leq \vartheta$, be the iterates; $N_0 = K^m$ and $N_\vartheta = N$. If $\nu < \vartheta$ is a limit ordinal then there exist $\xi_\nu < \nu$ and $U_\nu \in N_{\xi_\nu}$ such that $N_{\nu+1} = \mathrm{Ult}_{i_{\xi_\nu,\nu}(U_\nu)}(N_\nu)$. Since $o(\kappa) < \kappa^{++} \leq 2^\kappa \leq \vartheta$, there is a stationary set $S \subset \kappa^{++}$ of ordinals of cofinality ω such that $\xi_\nu = \xi$ and $U_\nu = U$ are constant for $\nu \in S$. Let $\nu \in S$ be a limit point of S, let $\langle \nu_n : n < \omega \rangle$ be cofinal in $S \cap \nu$, and let κ_n be the critical point of $i_{\nu_n,\nu}$, for each n. The sequence $\langle \kappa_n : n < \omega \rangle$ generates the measure $i_{\xi,\nu}(U)$ and belongs to M, hence $i_{\xi,\nu}(U) \in M$. By (iii), $i_{\xi,\nu}(U) \in (K^m)^M = N_\vartheta$ but this is impossible since $i_{\xi,\nu}(U) \notin N_{\nu+1}$. □

This, combined with a theorem of Gitik [1989] shows that the existence of a measurable cardinal κ such that $2^\kappa > \kappa^+$ is equiconsistent with the existence of a measurable cardinal κ of Mitchell order κ^{++}.

Another result of Gitik (cf. [1989] and [1991]) shows that the consistency strength of the failure of SCH is exactly a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.

Up to a Strong Cardinal

The current core model theory employs sequences of extenders rather than sequences of measures. This not only enables one to generalize the theory to large cardinals beyond measurable but also has some technical advantages even in the case of K^m.

Strong cardinals (see Chapter 20) were introduced by Dodd and Jensen who also provided their characterization in terms of extenders. They also constructed an inner model of the form $L[\mathcal{E}]$ where $\mathcal{E}$ is a (transfinite) sequence of extenders such that

(i) $L[\mathcal{E}]$ is a model of ZFC,
(ii) in $L[\mathcal{E}]$, $\mathcal{E}$ witnesses that there exists a strong cardinal,
(iii) $L[\mathcal{E}]$ satisfies GCH, $\Box$, and has a Σ^1_3 well-ordering of the reals.

They also introduced a real, the *sharp for a strong cardinal*, that exists if and only if there exists a nontrivial elementary embedding $j : L[\mathcal{E}] \to L[\mathcal{E}]$.

The theory of core models up to a strong cardinal uses mice of the form $J^{\mathcal{E}}_\alpha$ where $\mathcal{E}$ is a sequence of $J^{\mathcal{E}}_\alpha$-extenders. The crucial fact that makes the generalization of the Dodd-Jensen theory possible is that one uses sequences of *non-overlapping* extenders. (Two extenders overlap if one is a (κ, λ)-extender, the other a (κ', λ')-extender and $\kappa \le \kappa' < \lambda$.) This fact allows the comparison of mice by iteration, and while the generalization is far from routine, one obtains a result similar to those for K and K^m.

Theorem 35.19. *There exists an inner model* K^{strong} *such that*:

(i) K^{strong} *is a model of* ZFC + GCH.
(ii) K^{strong} *has a* Σ_2 *well-ordering and* $\Box$ *holds*; $\boldsymbol{R} \cap K^{\text{strong}}$ *has a* Σ^1_3 *well-ordering.*

If there exists no inner model for a strong cardinal then:

(iii) *If* $j : K^{\text{strong}} \to M$ *is a nontrivial elementary embedding then* j *is an iterated ultrapower by extenders in* K^{strong}. (*Hence there is no nontrivial* $j : K^{\text{strong}} \to K^{\text{strong}}$.)
(iv) *If* κ *is a singular strong limit cardinal then* $(\kappa^+)^{K^{\text{strong}}} = \kappa^+$. $\Box$

Inner Models for Woodin Cardinals

Inner models for very large cardinals employ a new method of comparison of mice. Due to the presence of overlapping extenders, a "linear" iteration of mice does not work and a new technique has been developed—the theory of *iteration trees*. Iteration trees were introduced by Martin and Steel, who used the technique to construct inner models for Woodin cardinals

Theorem 35.20 (Martin-Steel). *If there are* n *Woodin cardinals then there is an inner model that has* n *Woodin cardinals, and its reals have a* Σ^1_{n+2} *well-ordering.* $\Box$

The Σ^1_{n+2} result is best possible: If there are n Woodin cardinals with a measurable cardinal above them then Π^1_{n+1} determinacy holds (Theorem 33.26) and so $\boldsymbol{R}$ does not have a Σ^1_{n+2} well-ordering.

The fine structure for iteration trees was developed further by Mitchell and Steel who constructed an inner model for a Woodin cardinal that satisfies GCH. Then Steel constructed a core model up to a Woodin cardinal, under an additional assumption of a measurable cardinal above.

Let Ω be a measurable cardinal. *Steel's core model* K^{steel} is an inner model of V_Ω and if V_Ω has no inner model with a Woodin cardinal then K^{steel} is both rigid and satisfies the Weak Covering Theorem:

Theorem 35.21. *Let Ω be a measurable cardinal.*

(i) $(K^{\text{steel}})^{V_\Omega[G]} = K^{\text{steel}}$, *for every generic extension of V^Ω (by forcing in V_Ω).*

If V_Ω has no inner model with a Woodin cardinal then:

(ii) *There is no nontrivial elementary embedding $j : K^{\text{steel}} \to K^{\text{steel}}$.*
(iii) *For every singular cardinal $\lambda < \Omega$, $(\lambda^+)^{K^{\text{steel}}} = \lambda^+$.* □

The following is an application of Steel's core model:

Theorem 35.22 (Steel). *If $\aleph_1$ carries an $\aleph_2$-saturated ideal, and if there exists a measurable cardinal, then there exists an inner model with a Woodin cardinal.* □

This is (almost) best possible, as Shelah proved that if κ is a Woodin cardinal then there is a generic extension in which $\kappa = \omega_2$ and NS_{ω_1} is ω_2-saturated.

Exercises

35.1. Assume that $L[U]$ exists; then $K = \bigcup_{\alpha \in Ord}(\mathrm{Ult}^{(\alpha)}_U(L[U]) \cap V_{\kappa^{(\alpha)}})$.

35.2. (i) There is a $\Sigma_1(J^A_\alpha)$ map of ω_α onto J^A_α.
(ii) $\langle J^A_\xi : \xi < \alpha\rangle$ is $\Sigma_1(J^A_\alpha)$.
(iii) J^A_α has a $\Sigma_1(J^A_\alpha)$ well-ordering.
(iv) The relation $J^A_\alpha \vDash \varphi$ is $\Sigma_1(J^A_\alpha)$.

35.3. If M is a mouse then $\rho^1_M < \kappa$, where ρ^1_M, the Σ_1-projectum of M, is the smallest $\rho \le \alpha$ such that there exists a $\Sigma_1(M)$ function with $f``\omega\rho = J^U_\rho$.

35.4. Assume that $0^\sharp$ exists, let $a \in L[0^\sharp]$ be a real Cohen generic over L and let $M = L[a]$. Then $M \subset K$ and so $K \cap M = M$, while $K^M = L$.

Let $M = J^M_\alpha = H_1(\gamma \cup p)$ be a mouse. Let $i_{0,\xi} : M \to M_\xi = \mathrm{Ult}^{(\xi)}_U(M)$.

35.5. Łoś's Theorem holds in $\mathrm{Ult}_U(M)$ for Σ_0 formulas.

35.6. $i_{0,1}$ is a cofinal embedding of M into M_1 and therefore Σ_1-elementary. $i_{\xi,\eta} : M_\xi \to M_\eta$ is Σ_1-elementary.

35.7. $M_1 = H_1(\gamma \cup i_{0,1}(p) \cup \{\kappa\})$, $M_n = H_1(\gamma \cup i_{0,n}(p) \cup \{\kappa^{(0)}, \dots, \kappa^{(n-1)}\})$, $M_\xi = H_1(\gamma \cup i_{0,\xi}(p) \cup \{\kappa^{(\nu)} : \nu < \xi\})$.

35.8. For every regular $\kappa > \omega$, $\mathcal{C}_\kappa$ survives at κ if and only if $0^\sharp$ exists.

Historical Notes

The core model K was introduced by Dodd and Jensen in [1981, 1982a, 1982b]; see also Dodd [1982]. Theorem 35.6 is proved in [1981], Theorem 35.13 and 35.14 in [1982a], and the proof of $\diamondsuit$ and $\square$ in K is due to Welch. An overview of K, with some proofs, can be found in Mitchell [1979b] and Dodd [1983]. Magidor's approach is described in Magidor [1990] and in Kanamori's forthcoming book [∞].

The core model K^m for sequences of measures was introduced by Mitchell. Theorem 35.17 was stated in Mitchell [1984]. Its proof has never been published but a detailed sketch will appear in Mitchell [∞b] (in the forthcoming Handbook of Set Theory). Mitchell's article [∞b] and its companion [∞a] give an excellent introduction to the inner model theory, as does the more expository Mitchell [1994].

The inner model $L[\mathcal{E}]$ for a strong cardinal appeared in Dodd [1982]. The definition of K^{strong} is given explicitly in Koepke [1989] where Theorem 35.19 is stated (a proof has not been published).

Iteration trees are introduced in Martin and Steel [1994] where Theorem 35.20 is proved. Fine structure for iteration trees is developed in Mitchell and Steel [1994] obtaining an inner model with a Woodin cardinal and GCH. Steel's core model is constructed in Steel [1996], proving Theorem 35.21(i), (ii) and Theorem 35.22. Theorem 35.21(iii) is proved in Mitchell et al. [1997].

An overview of these (and of more recent results) is given in Löwe and Steel [1999].