37. Martin's Maximum

This chapter is devoted to a generalization of the Proper Forcing Axiom. The stronger axiom is obtained by replacing "proper notion of forcing" in Definition 31.20 by "stationary set preserving notion of forcing." A notion of forcing P is *stationary set preserving* if every stationary set $S \subset \omega_1$ remains stationary in V^P.

Definition 37.1 (Martin's Maximum (MM)). If $(P, <)$ is a stationary set preserving notion of forcing and if $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

As every proper notion of forcing is stationary set preserving, MM is a strengthening of PFA which in turn is a strengthening of MA. The axiom MM has been dubbed "Martin's Maximum" as it is ostensibly the strongest possible generalization of Martin's Axiom: If P is not stationary set preserving then the corresponding axiom for P is false (Exercise 37.1).

Below we establish the consistency of Martin's Maximum, and present some applications.

RCS iteration of semiproper forcing

The proof of the consistency of MM is modeled after the consistency proof of either MA or PFA: By iterated forcing one obtains a generic extension in which every stationary set preserving P satisfies the statement of MM.

The straightforward approach, iterating stationary set preserving forcings, does not work: If $g : \omega_1 \to \omega_1$ dominates the canonical functions f_η, $\eta < \omega_2$, (mod I_{NS}) then there is a stationary preserving forcing notion P_g that produces a function $f < g$ mod I_{NS} (and still above the f_η) (Exercise 37.2). An ω-iteration of such forcing collapses ω_1.

It turns out that semiproper forcing can be iterated, yielding the consistency of the following principle:

Definition 37.2 (Semiproper Forcing Axiom (SPFA)). If $(P, <)$ is a semiproper notion of forcing and if $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P then there exists a $\mathcal{D}$-generic filter on P.

Clearly, MM implies SPFA and SPFA implies PFA. In the next section we show that SPFA is in fact equivalent to MM and so for the consistency of MM it is enough to construct a model of SPFA.

While under special circumstances semiproperness may by preserved under countable support iteration (see Lemma 34.5), in general this is not the case. The reason is that a semiproper forcing notion may change the cofinality of ordinals from uncountable to countable. (An example of such forcing is the Prikry forcing, see Exercise 37.7.)

The iteration applicable to semiproper forcings is the *revised countable support* (RCS) iteration. Informally, a support of a condition is not just a countable set, but even a *name* for a countable set.

Definition 37.3. Let $\alpha \geq 1$. A forcing notion P_α is an RCS (revised countable support) iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ if it is an iteration (cf. Definition 16.29) consisting of all α-sequences p that satisfy

(37.1) for every $q \leq p$ there exist a $\gamma < \alpha$ and an $r \leq_\gamma q{\restriction}\gamma$ such that $r \Vdash_\gamma \operatorname{cf} \alpha = \omega$ or $\forall \beta \geq \gamma\ p{\restriction}[\gamma, \beta) \Vdash_{P_{\gamma,\beta}} p(\beta) = 1$.

In (37.1), q ranges over elements of the inverse limit of the $\dot{Q}_\beta$, cf. (16.12), and $P_{\gamma,\beta}$ is the restriction of the inverse limit to the interval $[\gamma, \beta) = \{\xi : \gamma \leq \xi < \beta\}$.

The main result on RCS iterations is that they preserve semiproperness:

Theorem 37.4 (Shelah). *If P_α is an* RCS *iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ such that every $\dot{Q}_\beta$ is a semiproper forcing notion in $V^{P_\alpha \restriction \beta}$ then P_α is semiproper.*

Theorem 37.4 can be proved along the lines of the proof of Theorem 31.15. We shall outline the proof of a special case of Theorem 37.4 (Proposition 37.8 below) which suffices for the consistency proof of SPFA. (To be precise, Shelah proved Theorem 37.4 for a more complicated definition of RCS iteration; the current Definition 37.3 is based on simplifications by Schlindwein and Donder).

A two-step iteration of semiproper forcings is semiproper, cf. Exercise 37.8. The proof of Theorem 37.4 proceeds by induction, showing

(37.2) for all $\gamma < \beta \leq \alpha$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper;

here $B_\beta = B(P_\beta)$, and $B_\beta : B_\gamma$ is the complete Boolean algebra in V^{P_γ} such that $B_\gamma * (B_\beta : B_\gamma) = B_\beta$ (see Exercise 16.4). One property of RCS that is used in the proof is that $B_\beta : B_\gamma$ is (in V^{P_γ}) an RCS iteration (Exercise 37.9).

The following three lemmas, special cases of Theorem 37.4, can be proved in a similar way as Theorem 31.15:

Lemma 37.5. *Let P_ω be the inverse limit iteration of semiproper forcings $\{\dot{Q}_n : n < \omega\}$. Then P_ω is semiproper.* □

Lemma 37.6. *Let P_{ω_1} be a countable support iteration such that for all $\gamma < \beta < \alpha$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper. Then P_{ω_1} is semiproper.* □

Lemma 37.7. *Let λ be a regular uncountable cardinal. Assume that*

(i) *P_λ is a direct limit,*
(ii) *for every $\alpha < \lambda$ of cofinality ω, P_α is the inverse limit,*
(iii) *for all $\gamma < \beta < \lambda$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper,*
(iv) *P_λ satisfies the λ-chain condition.*

Then P_λ is semiproper. □

We shall now prove a version of Theorem 37.4 that will be used in the consistency proof of MM:

Proposition 37.8. *If P_α is an* RCS *iteration of semiproper forcings $\{\dot{Q}_\beta : \beta < \alpha\}$ such that for every $\beta < \alpha$, $\Vdash_{\beta+1} |P_\beta| \leq \aleph_1$, then P_α is semiproper.*

Proof. We proceed by induction, proving (37.2). As successor stages present no problem, let α be a limit ordinal. By the induction hypothesis, for every $\gamma < \beta < \alpha$, $\Vdash_\gamma B_\beta : B_\gamma$ is semiproper; we shall prove that P_α is semiproper, and (37.2) for α then follows by Exercise 37.8.

Case I. Let $p \in P$ and $\gamma < \alpha$ be such that $p{\restriction}\gamma \Vdash \operatorname{cf}\alpha = \omega$. We will show that $P{\restriction}p$ is semiproper. In this case, $p{\restriction}\gamma$ forces that $B_\alpha : B_\gamma$ is the inverse limit, and in fact, an inverse limit iteration of length ω of semiproper forcings, hence semiproper by Lemma 37.5. It follows that $P{\restriction}p$ is semiproper.

Case II. Let $p \in P$ be such that $\forall\gamma < \alpha\ p{\restriction}\gamma \Vdash \operatorname{cf}\alpha > \omega$, and let $\gamma < \alpha$ be such that $p{\restriction}\gamma \Vdash \operatorname{cf}\alpha = \omega_1$. Again, we will show that $P{\restriction}p$ is semiproper. In this case, $p{\restriction}\gamma$ forces that $B_\alpha : B_\gamma$ is a direct limit iteration of length ω_1 of semiproper forcings, and hence semiproper by Lemma 37.6. Therefore $P{\restriction}p$ is semiproper.

Case III. Let $p \in P$ be such that $\forall\gamma < \alpha\ p{\restriction}\gamma \Vdash \operatorname{cf}\alpha > \aleph_1$; we will show that in this case too, $P{\restriction}p$ is semiproper. This will complete the proof that P is semiproper.

Without loss of generality, assume that $p = 1$, and since P_α is in this case a direct limit of the P_β, it is a direct limit of the P_{β_i}, $i < \operatorname{cf}\alpha$ (where $\alpha = \lim_{i<\operatorname{cf}\alpha} \beta_i$), so we can assume that α is a regular cardinal. For every $\gamma < \alpha$, since $\Vdash_{\gamma+1} |P_\gamma| \leq \aleph_1 < \operatorname{cf}\alpha$, we have $|P_\gamma| < \alpha$. Also since $\Vdash_{\gamma+1} \operatorname{cf}\alpha > \aleph_1$, there is a stationary set of $\beta < \alpha$ (those for which $\forall\gamma \Vdash_\gamma \operatorname{cf}\beta \geq \aleph_1$) at which P_β is a direct limit. By Theorem 16.30, P_α satisfies the α-chain condition. Therefore P is semiproper by Lemma 37.7. □

Consistency of MM

Theorem 37.9 (Foreman, Magidor and Shelah). *If there exists a supercompact cardinal then there is a generic model that satisfies* MM.

Following the proof of Theorem 31.21, we construct a model that satisfies SPFA. Instead of proper forcings, we iterated semiproper forcings, and use the RCS iteration. At each stage α of the iteration, in addition to using the notion of forcing presented by the Laver function, we also collapse (with countable conditions) the cardinal $|P_\alpha|$ to $\aleph_1$. By Proposition 37.8, such iteration is semiproper. An argument similar to the one in the proof of Theorem 31.21 shows that the iteration up to a supercompact cardinal yields a model in which SPFA holds.

The consistency of MM then follows from this result:

Theorem 37.10 (Shelah). SPFA *implies that every stationary set preserving notion of forcing is semiproper. Therefore* SPFA *implies* MM.

Proof. Let X be a set of countable elementary submodels of $H_\lambda = (H_\lambda, \in, <)$. We denote $X^\perp$ the set

$$X^\perp = \{M \in [H_\lambda]^\omega : M \prec H_\lambda \text{ and } N \notin X \text{ for every countable } N \text{ that satisfies } M \prec N \prec H_\lambda \text{ and } N \cap \omega_1 = M \cap \omega_1\}. \tag{37.3}$$

As in Chapter 31, we call an *elementary chain* (of length $\vartheta \le \omega_1$) a sequence $\langle M_\alpha : \alpha < \vartheta\rangle$ of countable elementary submodels of $(H_\lambda, \in, <)$ such that $M_\alpha \subset M_\beta$ and $M_\alpha \in M_\beta$ if $\alpha < \beta$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ if α is a limit ordinal. (Note that $\alpha \subset M_\alpha$ for every α.)

Lemma 37.11. *Assume* SPFA, *and let $\omega_1 \le \kappa < \lambda$ with λ regular and sufficiently large. Let $Y \subset [H_\kappa]^\omega$ be stationary, and let $X = \{M \in [H_\lambda]^\omega : M \cap H_\kappa \in Y\}$ be the lifting of Y to H_λ. There exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of submodels of $(H_\lambda, \in, <)$ such that $M_\alpha \in X \cup X^\perp$ for every α.*

Proof. Let P be the notion of forcing that shoots an elementary chain through $X \cup X^\perp$: Conditions are elementary chains $\langle M_\alpha : \alpha \le \gamma\rangle$ in $X \cup X^\perp$ where γ is a countable ordinal; a stronger condition is an extension. We shall prove that P is semiproper; then, using SPFA applied to the dense sets $D_\xi = \{\langle M_\alpha : \alpha \le \gamma\rangle : \gamma \ge \xi\}$ (cf. Exercise 37.10), we obtain an elementary chain of length ω_1 in $X \cup X^\perp$.

To show that P is semiproper, let $\mu > \lambda$ be sufficiently large, let $M \prec (H_\mu, \in, <)$ be countable, with $P \in M$, and let $p \in P \cap M$. It suffices (cf. Exercise 37.6) to find a $q \le p$ that is (M, P)-semigeneric.

Claim 37.12. *There exists a countable N, $M \prec N \prec H_\mu$ such that $N \cap \omega_1 = M \cap \omega_1$ and $N \cap H_\lambda \in X \cup X^\perp$.*

Proof. If $M \cap H_\lambda \in X^\perp$ let $N = M$. Otherwise, there exists a countable $N' \prec H_\lambda$ such that $M \cap H_\lambda \subset N'$, $N' \cap \omega_1 = M \cap \omega_1$ and $N' \in X$. Let N be the Skolem hull of $M \cup (N' \cap H_\kappa)$ in $(H_\mu, \in, <)$. We claim that $N \cap H_\kappa = N' \cap H_\kappa$; hence $N \cap \omega_1 = M \cap \omega_1$ and $N \cap H_\kappa \in Y$, hence $N \cap H_\lambda \in X$.

The equality $N \cap H_\kappa = N' \cap H_\kappa$ holds because $N \cap H_\kappa \subset N'$: notice that for every Skolem function h for H_μ, $h \cap H_\kappa \in M \cap H_\lambda \subset N'$. □

Continuing the proof of Lemma 37.11, let N be as in Claim 37.12. We can find a decreasing sequence of conditions $p_n \in N$ with $p_0 = p$ such that $p_n = \langle M_\alpha : \alpha \le \gamma_n \rangle$, such that every name for a countable ordinal in N is decided by some p_n (as an ordinal in N) and $\bigcup_{n=0}^\infty \bigcup_{\alpha \le \gamma_n} M_\alpha = N \cap H_\lambda$. Let $\gamma = \bigcup_{n=0}^\infty \gamma_n$ and $M_\gamma = N \cap H_\lambda$. Since $N \cap H_\lambda \in X \cup X^\perp$, $q = \langle M_\alpha : \alpha \le \gamma \rangle$ is a condition, and is (N, P)-semigeneric. Since $M \subset N$ and $M \cap \omega_1 = N \cap \omega_1$, q is (M, P)-semigeneric. □

Now we finish the proof of Theorem 37.10. Assuming SPFA, let Q be a stationary set preserving notion of forcing that is not semiproper. Let κ be sufficiently large (so that all Q-names for countable ordinals are in H_κ). Since Q is not semiproper, there exists some $p \in Q$ such that the set $Y = \{M \prec H_\kappa : \text{there exists no } (M, Q)\text{-semigeneric } q \le p\}$ is stationary. Let $\lambda > \kappa$ be regular and let X be the lifting of Y to H_λ; since κ is sufficiently large, $X = \{M \prec H_\lambda : \text{there is no } (M, Q)\text{-semigeneric } q \le p\}$. We may assume that $p = 1$ is the trivial condition.

By Lemma 37.11 there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ in $X \cup X^\perp$. We claim that the set $S = \{\alpha < \omega_1 : M_\alpha \in X\}$ is nonstationary. Assume that S is stationary and let G be a generic filter on Q. Since Q is stationary set preserving, S is stationary in $V[G]$. Let $\dot\delta_\xi$, $\xi < \omega_1$, enumerate all the names in $\bigcup_{\alpha < \omega_1} M_\alpha$ for countable ordinals. In $V[G]$, let

$$C = \{\alpha < \omega_1 : M_\alpha \cap \omega_1 = \alpha \text{ and } (\forall \xi < \alpha)[\dot\delta_\xi \in M_\alpha \text{ and } \dot\delta_\xi^G < \alpha]\}.$$

The set C is closed unbounded, and if $\alpha \in C$ then there exists some $q \in G$ such that for every $\dot\delta_\xi \in M_\alpha$, $q \Vdash (\exists \beta \in M_\alpha)\ \dot\delta_\xi = \beta$; therefore q is (M_α, Q)-semigeneric. Therefore S is nonstationary in $V[G]$, and hence in V.

Thus there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ in $X^\perp$. Let $\mu > \lambda$ be sufficiently large; we shall finish the proof by showing that for every countable $M \prec (H_\mu, \in, <, Q, \langle M_\alpha : \alpha < \omega_1 \rangle)$, for every $p \in M$ there exists an (M, Q)-semigeneric $q \le p$.

Let M be such; if $\delta = M \cap \omega_1$, then $M \cap H_\lambda \supset M_\delta$ and $\delta = M_\delta \cap \omega_1$, and since $M_\delta \in X^\perp$ we have $M \cap H_\lambda \notin X$ and we are done. □

Applications of MM

The first application deals with cardinal arithmetic. Since MM implies PFA, it follows (by Theorem 31.23) that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. It turns out that from MM one can prove much more, including the Singular Cardinal Hypothesis:

Theorem 37.13 (Foreman, Magidor and Shelah). MM *implies that for every regular* $\kappa \geq \aleph_2$, $\kappa^{\aleph_1} = \kappa$.

Corollary 37.14. MM *implies* $2^{\aleph_0} = \aleph_2$. □

Corollary 37.15. MM *implies* SCH.

Proof. For every cardinal λ of cofinality ω, if $2^{\aleph_0} < \lambda$ then $\lambda^{\aleph_0} \leq (\lambda^+)^{\aleph_1} = \lambda^+$, and SCH follows by Silver's Theorem 8.13. □

Proof of Theorem 37.13. Let A_α, $\alpha < \kappa$, be disjoint stationary subsets of $E^\kappa_\omega = \{\xi < \kappa : \operatorname{cf} \xi = \omega\}$. We shall prove the following claim that implies the theorem: For every increasing $f : \omega_1 \to \kappa$ there exists an ordinal $\gamma_f < \kappa$ of cofinality ω_1 such that

$$(37.4) \qquad \forall \alpha < \kappa \ \alpha \in \operatorname{ran}(f) \text{ if and only if } A_\alpha \cap \gamma_f \text{ is stationary.}$$

(It follows that $f \neq g$ implies $\gamma_f \neq \gamma_g$.)

Thus let $f : \omega_1 \to \kappa$ be an increasing function, and let S_α, $\alpha < \omega_1$, be disjoint stationary subsets of ω_1 such that $\bigcup_\alpha S_\alpha = \omega_1$ and that for every stationary S there exists an α such that $S \cap S_\alpha$ is stationary. We shall use MM to find a continuous increasing function $F : \omega_1 \to \kappa$ such that for every $\delta < \omega_1$, if $\delta \in S_\alpha$ then $F(\delta) \in A_{f(\alpha)}$.

Then if we let $\gamma_f = \sup_{\delta<\omega_1} F(\delta)$, $\gamma_f \cap \bigcup_{\alpha<\omega_1} A_{f(\alpha)}$ contains a closed unbounded set $\{F(\delta) : \delta < \omega_1\}$, and (37.4) holds.

Let P be the following notion of forcing: A condition is a continuous increasing function $p = \langle p(\delta) : \delta \leq \gamma \rangle$ where $\gamma < \omega_1$ such that for every $\delta \leq \gamma$, if $\delta \in S_\alpha$ then $p(\delta) \in A_{f(\alpha)}$. A stronger condition is an extension. We will show that P is stationary set preserving and that for every $\alpha < \omega_1$ the set $D_\alpha = \{p \in P : \alpha \in \operatorname{dom}(p)\}$ is dense. Then MM applied to the sets D_α produces the desired function F.

We prove the second claim first, by induction on α. Let α be a limit ordinal, and assume that all D_β, $\beta < \alpha$, are dense; let $p \in P$. Let γ be such that $\alpha \in S_\gamma$. Let λ be sufficiently large, and let $M \prec H_\lambda$ be a countable model with $P, p, \alpha \in M$ such that $\eta = \sup(M \cap \kappa) \in A_\gamma$ (M exists because A_γ is stationary). Let $\langle \alpha_n \rangle_n$ be an increasing sequence with limit α, and let $\langle \eta_n \rangle_n$ be an increasing sequence with limit η. We construct a sequence of conditions $p = p_0 \subset p_1 \subset \ldots \subset p_n \subset \ldots$, each $p_n \in M$, as follows: Given $p_n \in M$, $D_{\alpha_n} \in M$ is dense and so there exists a $p_{n+1} \in M$ such that $p_{n+1} \supset p_n$, $\alpha_n \in \operatorname{dom}(p_{n+1})$ and $p_{n+1}(\alpha_n + 1) \geq \eta_n$. The function $q = \bigcup_{n=0}^\infty p_n \cup \{(\alpha, \eta)\}$ is a condition, proving that D_α is dense.

Now we complete the proof by showing that P is stationary set preserving. Let S be a stationary subset of ω_1, let $p \in P$ and let $\dot{C}$ be a name for a closed unbounded set. We shall find a $q \leq p$ and some $\delta \in S$ such that $q \Vdash \delta \in \dot{C}$.

Let α be such that $S \cap S_\alpha$ is stationary. Let λ be sufficiently large and let $M \prec H_\lambda$ be a countable model with $P, p, \dot{C} \in M$ such that $\eta = \sup(M \cap \kappa) \in$

A_α and $\delta = M \cap \omega_1 \in S \cap S_\alpha$ (see Exercise 37.11). Let $\langle \alpha_n \rangle_n$ be an increasing sequence with limit δ and let $\langle \eta_n \rangle_n$ be an increasing sequence with limit η. As before, we find a sequence of conditions $p = p_0 \subset \ldots \subset p_n \subset \ldots$ in M such that $\alpha_n \in \operatorname{dom}(p_{n+1})$, $p_{n+1}(\alpha_n + 1) \geq \eta_n$, and such that for some $\beta_n \geq \alpha_n$ in M, $p_{n+1} \Vdash \beta_n \in \dot{C}$. The function $q = \bigcup_{n=0}^{\infty} p_n \cup \{(\delta, \eta)\}$ is a condition and since $\delta = \lim_n \beta_n$, we have $q \Vdash \delta \in \dot{C}$. □

Another important application of MM is the saturation of the nonstationary ideal on $\aleph_1$:

Theorem 37.16 (Foreman, Magidor and Shelah). MM *implies that the nonstationary ideal on* $\aleph_1$ *is* $\aleph_2$*-saturated.*

Proof. Assume MM and let $\{A_i : i \in W\}$ be a maximal almost disjoint collection of stationary subsets of ω_1. We shall find a set $Z \subset W$ of size $\leq \aleph_1$ such that $\sum_{i \in Z} A_i$ contains a closed unbounded set. That will prove that I_{NS} is $\aleph_2$-saturated.

Let P be the set of all pairs (q, p) such that

(37.5) (i) $q : \gamma + 1 \to W$ for some $\gamma < \omega_1$, and
(ii) $p \subset \omega_1$ is a closed countable set such that $\alpha \in p$ implies $\alpha \in \bigcup_{\xi < \alpha} A_{q(\xi)}$.

A condition (q', p') is stronger than (q, p) if $q' \supset q$ and p' is an end-extension of p. (See also (34.7).)

P can be viewed as a two-step iteration $Q * P_{\dot{S}}$ where Q collapses $|W|$ to $\aleph_1$ with countable conditions and $P_{\dot{S}}$ shoots a closed unbounded set through $\dot{S} = \sum_{i \in W} A_i$. P is stationary set preserving: If $A \subset \omega_1$ is stationary then for some $i \in W$, $A \cap A_i$ is stationary and remains stationary in V^Q. Hence $A \cap A_i \cap \dot{S}$ is stationary and remains stationary in V^P. Hence A is stationary in V^P. See Exercises 34.6 and 23.6.

For each $\alpha < \omega_1$, let $D_\alpha = \{(q, p) \in P : \alpha \leq \max(p)\}$. Each D_α is dense and so by MM there is a filter G on P that meets all the D_α. Let

$$F = \bigcup \{q : (q, p) \in G \text{ for some } p\}, \quad C = \bigcup \{p : (q, p) \in G \text{ for some } q\}.$$

The set C is closed unbounded, and is equal to the set $\{\alpha : (\exists \xi < \alpha)\ \alpha \in A_{F(\xi)}\} = \sum_{i \in \operatorname{ran}(F)} A_i$. □

Reflection Principles

An important consequence of MM are reflection principles. These combinatorial principles imply some major consequences of MM.

Definition 37.17 (Reflection Principle (RP)). For every regular $\lambda \geq \aleph_2$ the following holds:

RP(λ) If S is a stationary set in $[\lambda]^\omega$ then for every $X \subset \lambda$ of cardinality $\aleph_1$ there exists a $Y \subset \lambda$ of cardinality $\aleph_1$ such that $X \subset Y$ and that $S \cap [Y]^\omega$ is stationary in $[Y]^\omega$.

RP follows from Martin's Maximum, see Theorems 37.21 and 37.23 below. One consequence of RP is that every stationary set preserving notion of forcing is semiproper (Exercise 37.13). This in turn implies that I_{NS} on ω_1 is precipitous (Foreman, Magidor and Shelah [1988], Theorem 26) and is therefore a large cardinal property.

Theorem 37.18 (Todorčević). RP(ω_2) *implies that* $2^{\aleph_0} \leq \aleph_2$.

Proof. One can show that RP(λ) implies a stronger version of RP(λ), namely that $S \cap [Y]^\omega$ is stationary for stationary many $Y \in [\lambda]^{\aleph_1}$ (Exercise 37.14). If $\omega_1 \leq \alpha < \omega_2$, let C_α be a closed unbounded subset of $[\alpha]^\omega$ of order-type ω_1, and let $D = \bigcup_{\omega_1 \leq \alpha < \omega_2} C_\alpha$. Since $|C_\alpha| = \aleph_1$ for each α, we have $|D| = \aleph_2$. By RP(ω_2), D contains a closed unbounded set: Otherwise, if $S = [\omega_2]^\omega - D$ is stationary, there exists an $\alpha \geq \omega_1$ such that $S \cap [\alpha]^\omega$ is stationary, a contradiction.

By a theorem of Baumgartner and Taylor ([1982], Theorem 3.2(a)), every closed unbounded subset of $[\omega_2]^\omega$ has size at least $2^{\aleph_0}$. Therefore $2^{\aleph_0} \leq \aleph_2$. □

RP(ω_2) is not very strong; its consistency follows from a weakly compact cardinal (a modification of Theorem 23.23).

Definition 37.19. A set $S \subset [\lambda]^\omega$ is *projective stationary* if for every stationary set $T \subset \omega_1$, the set $\{X \in S : X \cap \omega_1 \in T\}$ is stationary.

(Equivalently, for every closed unbounded $C \subset [\lambda]^\omega$, the projection $(S \cap C) \restriction \omega_1$ contains a closed unbounded set.)

Definition 37.20 (Strong Reflection Principle SRP). For every regular $\lambda \geq \aleph_2$, the following holds:

SRP(λ) If S is projective stationary in $[H_\lambda]^\omega$ then there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable models such that $M_\alpha \in S$ for all α.

Theorem 37.21. MM *implies* SRP.

Proof. Let $S \subset [H_\kappa]^\omega$ be projective stationary. Let P be the forcing notion that shoots an elementary chain through S: Conditions are elementary chains $\langle M_\alpha : \alpha \leq \gamma \rangle$ where $\gamma < \omega_1$ and $M_\alpha \in S$ for all $\alpha \leq \gamma$. We will show that P is stationary set preserving; then if G is a filter on P that meets $\{p \in P : \alpha \in \text{dom}(p)\}$ for each $\alpha < \omega_1$, $\bigcup G$ is an elementary chain in S.

Let $T \subset \omega_1$ be stationary, let $\dot{C}$ be a P-name for a closed unbounded set, and let $p \in P$. We shall find a $q \le p$ and a $\delta \in T$ such that $q \Vdash \delta \in \dot{C}$. Let λ be sufficiently large and let $M \prec (H_\lambda, \in, <, P, \dot{C}, S, T, p)$ be a countable model such that $M \cap H_\kappa \in S$ and $\delta = M \cap \omega_1 \in T$. Let $p = p_0 \ge \ldots \ge p_n \ge \ldots$ be conditions in M such that for every open dense set $D \in M$, $p_n \in D$ for some n. If $p_n = \langle M_\alpha : \alpha \le \gamma_n \rangle$ then $\delta = \lim_n \gamma_n$ and $M \cap H_\kappa = \bigcup_{n<\omega} M_{\gamma_n}$. If we let $q = \bigcup_{n<\omega} p_n \cup \{(\delta, M \cap H_\kappa)\}$, then q is a condition and $q \Vdash \delta \in \dot{C}$. □

Theorem 37.22. SRP *implies that the nonstationary ideal on* ω_1 *is* $\aleph_2$-*saturated.*

Proof. Assume SRP and let W be a maximal antichain of stationary subsets of ω_1. We will show that $|W| \le \aleph_1$. Consider the set

$$S = \{M \in [H_{\omega_2}]^\omega : M \prec H_{\omega_2},\ W \in M \text{ and } \exists A \in W \cap M\ (M \cap \omega_1 \in A)\}.$$

We claim that S is projective stationary. Let $T \subset \omega_1$ be stationary and let $A \in W$ be such that $T \cap A$ is stationary. Let C be a closed unbounded set in $[H_{\omega_2}]^\omega$. Then there exists a model $M \in C$ such that $M \cap \omega_1 \in A \cap T$; hence S is projective stationary. By SRP there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $M_\alpha \in S$ for all α. Let $M = \bigcup_{\alpha<\omega_1} M_\alpha$; we shall finish the proof by showing that $W \subset M$.

Let $A \in W$ and assume that $A \notin M$. Let N be the Skolem hull of $M \cup \{A\}$ and for each α, let N_α be the Skolem hull of $M_\alpha \cup \{A\}$. Let C be the closed unbounded set of all $\alpha < \omega_1$ such that $M_\alpha \cap \omega_1 = N_\alpha \cap \omega_1 = \alpha$, and let $\alpha \in C \cap A$. Since $M_\alpha \in S$, there exists some $B \in W \cap M_\alpha$ such that $\alpha \in B$. As $A \cap B$ is nonstationary and $A, B \in N_\alpha$, there exists a closed unbounded set $D \in N_\alpha$ such that $A \cap B \cap D = \emptyset$. This is a contradiction, since $\alpha = N_\alpha \cap \omega_1 \in D$, and also $\alpha \in A$ and $\alpha \in B$. □

Theorem 37.23. *For every regular* $\lambda \ge \omega_2$, SRP(λ) *implies* RP(λ).

Proof. Assuming SRP(λ) we prove a stronger version of RP(λ):

(37.6) If S is a stationary set in $[H_\lambda]^\omega$ then there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $\{\alpha : M_\alpha \in S\}$ is stationary.

Let $S \subset [H_\lambda]^\omega$ be stationary. By Exercise 37.19 (since I_{NS} is $\aleph_2$-saturated by SRP(ω_2)), there exists a stationary $A \subset \omega_1$ such that for every stationary $B \subset A$, the set $\{M \in S : M \cap \omega_1 \in B\}$ is stationary. Therefore the set $\{M : M \in S \text{ or } M \cap \omega_1 \notin A\}$ is projective stationary, and by SRP(λ) contains an elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$. It follows that $M_\alpha \in S$ for every $\alpha \in A$. □

We mention two other consequences of SRP: the Singular Cardinal Hypothesis (Todorčević, Exercise 37.20) and $2^{\aleph_0} = \aleph_2$ (Woodin [1999], Theorem 9.82, proves that SRP(ω_2) implies $\boldsymbol{\delta}^1_2 = \aleph_2$).

Forcing Axioms

Martin's Maximum (as well as MA and PFA) are principles that postulate the existence of sufficiently generic filter on every forcing notion of a given kind. In general, let $\mathcal{P}$ be a class of forcing notions.

Definition 37.24 (MA($\mathcal{P}$)). If P is a forcing notion in $\mathcal{P}$ and if $\{D_\alpha : \alpha < \omega_1\}$ are dense (or predense) subsets of P, then there exists a filter G on P that meets all the D_α.

Thus MA(c.c.c.) is $\mathrm{MA}_{\aleph_1}$, MA(proper) is PFA and MA(semiproper) = MA(stationary set preserving) is SPFA = MM. A useful strengthening of a given forcing axiom is the following:

Definition 37.25 (MA$^+$($\mathcal{P}$)). If P is a forcing notion in $\mathcal{P}$, if $\{D_\alpha : \alpha < \omega_1\}$ are dense (or predense) subsets of P and if $\dot{S}$ is a name of a subset of ω_1 such that $\Vdash \dot{S}$ is stationary, then there exists a filter G on P that meets all the D_α, and $\dot{S}^G = \{\alpha : \exists p \in G\ p \Vdash \alpha \in \dot{S}\}$ is stationary.

MM$^+$ is stronger than MM; its consistency follows by a modification of the proof of Theorem 37.9. While $\mathrm{MA}^+_{\aleph_1}$ is equivalent to $\mathrm{MA}_{\aleph_1}$ (Exercise 37.22), MA$^+$($\mathcal{P}$) is generally stronger than MA($\mathcal{P}$). A useful special case is MA$^+$(ω-closed). Among others, it implies the Reflection Principle RP and is therefore a large cardinal axiom. Moreover, it implies (37.6) (Exercise 37.23) and hence SCH. The following theorem shows that MA$^+$(ω-closed) follows from MM:

Theorem 37.26 (Shelah). MM *implies* MA$^+$(ω-closed).

Proof. Assume MM and let P be ω-closed, $\mathcal{D}$ a family of $\aleph_1$ dense subsets of P and $\dot{S}$ a P-name for a stationary set. Let $\{A_i : i \in W\}$ be a maximal antichain of those stationary sets for which $\Vdash_P A_i \cap \dot{S}$ is nonstationary. By MM, $|W| \le \aleph_1$. Let $A = \sum_{i \in W} A_i$ be the diagonal union and let $T = \omega_1 - A$. We have $\Vdash_P \dot{S} - T$ is nonstationary (hence T is stationary) and for every stationary $X \subset T$ there exists some $p \in P$ such that $p \Vdash \dot{S} \cap X$ is stationary.

Let Q be the countable support product of ω_1 copies of P; let $Q_\alpha = P$ and $\dot{S}_\alpha = \dot{S}$. For every stationary $X \subset T$ and every $q \in Q$ there exist some $q' \le q$ and $\alpha < \omega_1$ such that $q' \Vdash \dot{S}_\alpha \cap X$ is stationary. It follows that for every stationary $X \subset T$, $\Vdash_Q X \cap \sum_{\alpha < \omega_1} \dot{S}_\alpha$ is stationary. In V^Q, let $\dot{R}$ be the forcing notion that shoots a closed unbounded set $\dot{C}$ through $A \cup \sum_{\alpha < \omega_1} \dot{S}_\alpha$ (with countable conditions). It follows that $Q * \dot{R}$ preserves stationary sets.

By MM there exists a filter $G \times H$ on $Q * \dot{R}$ such that each $G_\alpha = G{\restriction}Q_\alpha$ is $\mathcal{D}$-generic, that for all α and β, G meets $\{q \in Q : q \text{ decides } \alpha \in \dot{S}_\beta\}$, and that for each α, $G \times H$ meets $\{(q, r) : \max(r) \ge \alpha\}$. Then $C = \dot{C}^{G \times H}$ is a closed unbounded set and $A \cup \sum_{\alpha < \omega_1} \dot{S}_\alpha^G \supset C$. Therefore there exists some α such that $\dot{S}_\alpha^{G_\alpha}$ is stationary, and MA$^+$(ω-closed) follows. ◻

While PFA is a large cardinal axiom and implies that $2^{\aleph_0} = \aleph_2$ there are weaker versions that do not need large cardinals, and are consistent with $\mathfrak{c} > \aleph_2$: For instance, there is a class $\mathcal{P}$ of proper forcings that includes, among others, the forcings for adding Cohen reals, Sacks reals, Mathias reals and Laver reals and MA($\mathcal{P}$) is consistent (relative to ZFC) with $2^{\aleph_0} > \aleph_2$ (Groszek and Jech [1991]).

Finally, forcing axioms can be further modified by restricting the size of predense sets that the filter should meet. If only D_α of size $\le \aleph_1$ are involved, these are known as *bounded* forcing axioms:

Definition 37.27 (Bounded MA($\mathcal{P}$)). If P is a forcing notion in $\mathcal{P}$ and if $\{D_\alpha : \alpha < \omega_1\}$ are predense subsets of P such that $|D_\alpha| \le \aleph_1$ for all α, then there exists a filter G on P that meets all the D_α.

The consistency strength of Bounded PFA is below a Mahlo cardinal (Goldstern and Shelah [1995]). An interesting equivalence for Bounded MM was proved by Bagaria:

Theorem 37.28 (Bagaria [2000]). *Bounded* MM *holds if and only if for every stationary set preserving forcing notion P,*

$$(H_{\omega_2}, \in) \prec_{\Sigma_1} (H_{\omega_2}, \in)^{V^P}. \qquad \square$$

Exercises

37.1. Let P be a notion of forcing such that for some stationary $S \subset \omega_1$, $\Vdash_P S$ is nonstationary. Then there exist $\aleph_1$ dense sets such that no filter G on P meets them all.

[Let $\dot{C}$ be a closed unbounded set in V^P such that $\Vdash_P S \cap \dot{C} = \emptyset$. For each $\alpha < \omega_1$, let $D_\alpha = \{p : (\exists \beta \ge \alpha)\ p \Vdash \beta \in \dot{C}\}$ and $E_\alpha = \{p :$ either $p \Vdash \alpha \in \dot{C}$ or $\exists \gamma < \alpha$ such that $p \Vdash \xi \notin \dot{C}$ for all ξ between γ and $\alpha\}$. If G meets all the D_α and E_α, let $C = \{\alpha : \exists p \in G\ p \Vdash \alpha \in \dot{C}\}$. Show that C is closed unbounded and so $S \cap C \neq \emptyset$; a contradiction.]

37.2. Let $f_\eta : \omega_1 \to \omega_1$, $\eta < \omega_2$, be the canonical ordinal functions, and let $g : \omega_1 \to \omega_1$ be such that $g > f_\eta \bmod I_{\mathrm{NS}}$ for all η. A forcing condition in P_g is $(h, c, \{c_\eta : \eta \in A\})$ where $h : \alpha + 1 \to \omega_1$ for some $\alpha < \omega_1$, c and c_η are closed subsets of $\alpha + 1$, $A \subset \omega_2$ is countable, and $h < g$ on c, $h > f_\eta$ on c_η. The c's in a stronger condition are end-extensions. Show that P_g is stationary preserving.

[Shelah [1982], p. 255.]

Let $|A| \ge \aleph_2$. A set $C \subset [A]^\omega$ is *locally closed unbounded* if for closed unbounded many $X \in [A]^{\aleph_1}$, $C \cap [X]^\omega$ contains a closed unbounded set in $[X]^\omega$.

37.3. The filter of locally closed unbounded sets is a normal filter and extends the closed unbounded filter on $[A]^\omega$.

37.4. A notion of forcing P is stationary set preserving if and only if for every sufficiently large λ there is a locally closed unbounded set in $[H_\lambda]^\omega$ of countable elementary submodels such that $\forall p \in M\ \exists q \le p\ q$ is (M, P)-semigeneric. (Compare with Definition 34.3).

[Feng and Jech [1989], Theorem 2.1.]

37.5. P is stationary set preserving if and only if for every $p \in P$ and every set X of names for countable ordinals such that $|X| = \aleph_1$, player II has a winning strategy in the following game: I plays $\dot\alpha_n \in X$, II plays β_n, and II wins if $\exists q \le p\ q \Vdash \forall n\ \exists k$ $(\dot\alpha_n = \beta_k)$. (Compare with Theorem 31.9 and Exercise 34.4.)

[Feng and Jech [1989], Theorem 2.1.]

37.6. P is semiproper if and only if for every $p \in P$, every sufficiently large λ and every countable $M \prec (H_\lambda, \in, <)$ containing P and p, there exists a $q \le p$ that is (M, P)-semigeneric.

[As in Lemma 31.16.]

37.7. Show that the Prikry forcing is semiproper.

[Use Exercise 34.4.]

37.8. If P is semiproper and $\Vdash_P \dot Q$ is semiproper then $P * \dot Q$ is semiproper.

[As in Lemma 31.18, or use the semiproper game from Exercise 34.4.]

37.9. Let P_α be an RCS iteration, let $\gamma < \alpha$, and let $\dot P_\alpha^{(\gamma)}$ be an RCS iteration, in V^{P_γ}, of $\{\dot Q_\beta : \gamma \le \beta < \alpha\}$. Then $V^{P_\alpha} = V^{P_\gamma * \dot P_\alpha^{(\gamma)}}$.

37.10. For every stationary $S \subset [H_\lambda]^\omega$ and every $\gamma < \omega_1$ there exists an elementary chain $\langle M_\alpha : \alpha \le \gamma\rangle$ such that $M_\alpha \in S$ for all $\alpha \le \gamma$.

[It suffices to show that such a chain exists in some V^P where P collapses H_λ with countable conditions. In V^P, consider an elementary chain with limit H_λ and apply Exercise 8.5.]

37.11. Let $S \subset \omega_1$ and $T \subset E^\kappa_\omega$ be stationary and let λ be sufficiently large. Then there exists a countable $M \prec H_\lambda$ such that $M \cap \omega_1 \in S$ and $\sup(M \cap \kappa) \in T$.

[There exists $N \prec H_\lambda$ of size $\aleph_1$ such that $\omega_1 \subset N$ and $\eta = \sup(N \cap \kappa) \in T$ (because T is stationary). Then (because S is stationary) there exists a countable $M \prec N$ with $\sup(M \cap \kappa) = \eta$ and $M \cap \omega_1 \in S$.]

37.12. MM implies that for every regular $\kappa \ge \omega_2$, every stationary $A \subset E^\kappa_\omega$ contains a closed set of order-type ω_1. (Compare with Exercise 8.5.)

[Let P be the set of all continuous increasing $\langle p(\alpha) : \alpha \le \gamma\rangle$, $\gamma < \omega_1$, in A.]

37.13. RP implies that every stationary set preserving P is semiproper.

[Foreman, Magidor and Shelah [1988], Proposition 14.]

37.14. RP(λ) implies that for every stationary $S \subset [\lambda]^\omega$, the set $\{Y \subset \lambda : |Y| = \aleph_1$ and $S \cap [Y]^\omega$ is stationary$\}$ is stationary in $[\lambda]^{\aleph_1}$.

[Feng and Jech [1989], Theorem 3.1, (3) implies (2).]

37.15. RP(κ) implies that every stationary $A \subset E^\kappa_\omega$ reflects at some γ of cofinality ω_1. (Compare with Exercise 31.9.)

37.16. Let $\aleph_1 < \kappa < \lambda$.

(i) If $S \subset [\lambda]^\omega$ is projective stationary then $S{\restriction}\kappa$ is projective stationary.

(ii) If $S \subset [\kappa]^\omega$ is projective stationary then the lifting of S to λ is projective stationary.

37.17. Let $\kappa < \lambda$, $Y \subset [H_\kappa]^\omega$, let X be the lifting of Y to H_λ. Show that $X \cup X^\perp$ is projective stationary.

[Feng and Jech [1998], Claim 1.2. (Or, modify the proof of Claim 37.12.)]

37.18. SRP implies that for every regular $\kappa \geq \omega_2$, every stationary $A \subset E^\kappa_\omega$ contains a closed set of order-type ω_1.

[Apply SRP(κ) to the set $\{M : A \in M \text{ and } \sup(M \cap \kappa) \in M\}$.]

37.19. If I_{NS} is $\aleph_2$-saturated then for every stationary $S \subset [\lambda]^\omega$ there exists a stationary $A \subset \omega_1$ such that for every stationary $B \subset A$, the set $\{X \in S : X \cap \omega_1 \in B\}$ is stationary.

[For every stationary $A \subset \omega_1$, let $S_A = \{X \in S : X \cap \omega_1 \in A\}$, and let $W = \{A_\xi : \xi < \vartheta\}$, $\vartheta \leq \omega_1$, be a maximal antichain of stationary sets A such that S_A is not stationary. For each ξ let C_ξ be closed unbounded in $[\lambda]^\omega$ such that $S_{A_\xi} \cap C_\xi = \emptyset$. Let $A = \triangle_\xi(\omega_1 - A_\xi)$ and $C = \triangle_\xi C_\xi = \{X : (\forall \alpha \in X \cap \vartheta)\ X \in C_\alpha\}$. Since $C \cap S$ is stationary, A is stationary. A is as desired.]

37.20. SRP(κ) implies that $\kappa^{\aleph_1} = \kappa$.

[Let A_α ($\alpha < \kappa$), $f : \omega_1 \to \kappa$ and S_α ($\alpha < \omega_1$) be as in the proof of Theorem 37.13, and prove (37.4). The set $\{M : (\forall \alpha \in M \cap \omega_1)$ if $M \cap \omega_1 \in S_\alpha$ then $\sup(M \cap \kappa) \in A_{f(\alpha)}\}$ is projective stationary.]

37.21. SRP holds if and only if for all $\kappa < \lambda$ regular uncountable, if $S \subset [\kappa]^\omega$ is stationary then there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ such that $M_\alpha \cap \kappa \in S$ for every α for which there exists a countable M such that $M_\alpha \subset M \prec H_\lambda$, $M \cap \omega_1 = M_\alpha \cap \omega_1$, and $M \cap \kappa \in S$. (In other words, if T is the lifting of S to H_λ then $T \cup T^\perp$ contains an elementary chain. This reflection principle is due to Todorčević, see Bekkali [1991], p. 57.)

[Feng and Jech [1998], Theorem 1.2.]

37.22. $\text{MA}_{\aleph_1}$ implies $\text{MA}^+_{\aleph_1}$.

[Let P be c.c.c., $|\mathcal{D}| = \aleph_1$ (dense sets), and $\dot{S}$ a P-name for a stationary set. Let Q be the (finite support) product of ω copies of P; let $Q_n = P$ and $\dot{S}_n = \dot{S}$. Let $T = \{\alpha : \exists p \in P\ p \Vdash \alpha \in \dot{S}\}$ and show that $\Vdash_Q T = \bigcup_{n<\omega} \dot{S}_n$. Apply $\text{MA}_{\aleph_1}$ to Q which is c.c.c. Let G be a filter on Q such that each $G_n = G{\restriction}Q_n$ is $\mathcal{D}$-generic, and that for every α and every n, G meets $\{q : q \text{ decides } \alpha \in \dot{S}_n\}$. Then $T = \bigcup_{n<\omega} \dot{S}_n^G$ and therefore there exist some n such that $\dot{S}_n^{G_n}$ is stationary.]

37.23. MA^+(ω-closed) implies that for every stationary $S \subset [H_\lambda]^\omega$ there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ such that $\{\alpha : M_\alpha \in S\}$ is stationary.

[Apply MA^+(ω-closed) to the ω-closed forcing that produces a generic chain $\langle M_\alpha^G : \alpha < \omega_1\rangle$ such that $\bigcup_{\alpha<\omega_1} M_\alpha^G = (H_\lambda)^V$ (the conditions being countable chains) and the canonical name for the stationary set $S_G = \{\alpha : M_\alpha^G \in S\}$.]

Historical Notes

Martin's Maximum was formulated by Foreman, Magidor and Shelah [1988]. The consistency proof (Theorem 37.9) as well as the major applications (Theorem 37.13, Theorem 37.16 and Definition 37.17) are in that paper. The method of RCS iteration and Theorem 37.4 are due to Shelah and appear in [1982]. The (simplified) definition presented here follows Fuchs [1992] and Schlindwein [1993].

Theorem 37.10 was proved by Shelah in [1987].

Todorčević proved (in unpublished notes) that RP implies $2^{\aleph_0} \le \aleph_2$; he also formulated a strong reflection principle (see Exercise 37.21) and used it to prove Theorems 37.21, 37.22 and 37.23, as well as SCH. The present version of SRP (Definition 37.20) is due to Feng and Jech [1998]; so is the equivalence in Exercise 37.21.

$\mathrm{MA}^+(\omega\text{-closed})$ and MM^+ are discussed in Foreman, Magidor and Shelah [1988]. Theorem 37.26 was proved by Shelah in [1987].

Exercises 37.1, 37.12 and 37.23: Foreman, Magidor and Shelah [1988].

Exercise 37.3: Feng and Jech [1989].

Exercises 37.10 and 37.19: Feng and Jech [1998].

Exercises 37.18 and 37.20: Todorčević.

Exercises 37.22: Baumgartner.