38. More on Stationary Sets

Stationary sets play a central role in several areas of set theory. In this final chapter we address some of the issues dealing with stationary sets.

The Nonstationary Ideal on $\aleph_1$

The question of considerable interest is whether the ideal I_{NS} on $\aleph_1$ can be $\aleph_2$-saturated. By Theorem 37.16, the saturation of I_{NS} follows from MM, and thus is consistent relative to a supercompact cardinal. This can be improved:

Theorem 38.1 (Shelah). *If there exists a Woodin cardinal then there is a generic model in which the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated.*

Proof (sketch). The model is constructed by an RCS iteration (up to a Woodin cardinal), as in the proof of Theorem 37.9, iterating the forcings described in (37.5), for those maximal antichains for which the forcing (37.5) is semiproper. An argument similar to the one used in the proof of Theorem 34.8 shows that in the resulting model, I_{NS} is saturated. $\square$

Combining this result with Steel's Theorem 35.21, it follows that the consistency strength of the saturation of I_{NS} is approximately that of the existence of a Woodin cardinal.

In contrast to that, the consistency strength of the precipitousness of I_{NS} is only that of the existence of a measurable cardinal (Theorems 22.33 and 23.10).

A σ-complete ideal I on ω_1 is ω_1-*dense* if the Boolean algebra $P(\omega_1)/I$ has a dense subset of size $\aleph_1$. Clearly, every (nontrivial) ω_1-dense ideal is $\aleph_2$-saturated. The following result (that we state without proof) shows that the consistency strength of "I_{NS} is ω_1-dense" is exactly the existence of infinitely many Woodin cardinals.

Theorem 38.2 (Woodin). *The following are equiconsistent*:

(i) I_{NS} *is ω_1-dense.*
(ii) AD *holds in $L(\boldsymbol{R})$.* $\square$

The saturation of the nonstationary ideal implies (almost) that the Continuum Hypothesis fails:

Theorem 38.3 (Woodin). *If I_{NS} is $\aleph_2$-saturated and if there exists a measurable cardinal, then $\boldsymbol{\delta}^1_2 = \aleph_2$ (and hence $2^{\aleph_0} \geq \aleph_2$).* □

Note that the construction in the proof of Theorem 34.8 yields a model in which for some stationary set A, the ideal $I_{\mathrm{NS}} \restriction A$ is $\aleph_2$-saturated and the Continuum Hypothesis holds.

Saturation and Precipitousness

By Theorem 23.17, the nonstationary ideal on κ is not κ^+-saturated, for any $\kappa \geq \aleph_2$. The proof of Theorem 23.17 yields a somewhat stronger result: If κ and λ are regular cardinals such that $\lambda^+ < \kappa$, then $I_{\mathrm{NS}} \restriction E^\kappa_\lambda$ is not κ^+-saturated. Theorem 38.4 below shows that the saturation of $I_{\mathrm{NS}} \restriction \mathrm{Reg}$ is consistent (and not particularly strong). It remains open whether for a regular uncountable cardinal κ, $I_{\mathrm{NS}} \restriction E^{\kappa^+}_\kappa$ can be κ^{++}-saturated.

Let κ be a regular cardinal and let $\alpha < \kappa^+$. The cardinal κ is *α-Mahlo* if the order of κ (as defined in Chapter 8) is at least $\kappa + \alpha$. (Thus 0-Mahlo means weakly inaccessible, 1-Mahlo means weakly Mahlo, etc.)

Theorem 38.4.

(i) *Let κ be an α-Mahlo cardinal, with $0 < \alpha < \kappa^+$. If $I_{\mathrm{NS}} \restriction \mathrm{Reg}$ is κ^+-saturated then κ is a measurable cardinal of Mitchell order at least α in the model K^m.*

(ii) *Let κ be a measurable cardinal of Mitchell order α, with $0 < \alpha < \kappa^+$. There is a generic model in which κ is α-Mahlo and $I_{\mathrm{NS}} \restriction \mathrm{Reg}$ is κ^+-saturated.*

Proof. Cf. Jech and Woodin [1985]. For (i), see Exercise 38.1. □

By Theorem 23.10, the existence of a measurable cardinal is sufficient for the construction of a generic model in which the ideal I_{NS} on ω_1 is precipitous. The construction generalizes to obtain the precipitousness of $I_{\mathrm{NS}} \restriction E^{\kappa^+}_\kappa$, for every regular cardinal κ. For the precipitousness of the entire ideal I_{NS} on $\kappa \geq \aleph_2$, more than measurability is needed. For instance:

Theorem 38.5. *The following are equiconsistent:*

(i) *I_{NS} on $\aleph_2$ is precipitous.*

(ii) *There exists a measurable cardinal of Mitchell order 2.*

Proof. Cf. Gitik [1984]. For the lower bound, see Exercise 38.2. □

The consistency strength of the precipitousness of I_{NS} on $\kappa \geq \aleph_3$ is more than $o(\kappa) = \kappa^+$. In [1997], Gitik calculated the exact strength for successors or regulars, and nearly optimal lower and upper bounds for inaccessible κ (in both cases, it is the Mitchell order between κ^+ and κ^{++}). For successors of singulars the consistency strength is in the region of Woodin cardinals.

Reflection

Let $\kappa \geq \aleph_2$ be a regular cardinal. A stationary set $S \subset \kappa$ reflects at $\alpha < \kappa$ if $S \cap \alpha$ is stationary in α (see Definition 23.5). We shall now discuss briefly to what extent can stationary sets reflect.

First we consider the property "every stationary set $S \subset \kappa$ reflects (at some $\alpha < \kappa$)." This implies that κ is either (weakly) inaccessible or the successor of a singular cardinal, because if $\kappa = \lambda^+$ with λ regular, the set $E_\lambda^{\lambda^+}$ does not reflect (see Exercise 23.4). Let κ be an inaccessible cardinal. If κ is weakly compact then every stationary $S \subset \kappa$ reflects (Corollary 17.20). If $V = L$ then the converse is true as well: If every stationary set reflects then κ is weakly compact (Jensen [1972], Theorem 6.1).

Following Mekler and Shelah [1989], let us call κ a *reflecting cardinal* if there exists a normal ideal I on κ such that for every $X \in I^+$, $\{\alpha \in \kappa : X \text{ reflects at } \alpha\} \in I^+$. Every weakly compact cardinal is reflecting, and since being a reflecting cardinal is a Π_1^1 property (see Exercise 38.3), every weakly compact cardinal is a limit of reflecting cardinals.

Theorem 38.6. *The following are equiconsistent*:

(i) *There exists a cardinal κ such that every stationary $S \subset \kappa$ reflects.*
(ii) *There exists a reflecting cardinal.*

Proof. Mekler and Shelah [1989]. □

A cardinal κ is *greatly Mahlo* if κ is α-Mahlo for every $\alpha < \kappa^+$. If $V = L$ then every reflecting cardinal is greatly Mahlo and a limit of greatly Mahlo cardinals (Mekler and Shelah [1989]). Thus the consistency strength of "every stationary set reflects" is strictly between "greatly Mahlo" and "weakly compact."

Now let κ be a successor of a singular cardinal λ. The property "every stationary set $S \subset \lambda^+$ reflects" is a very large cardinal property. On the one hand there is this consistency result:

Theorem 38.7. *If there exist infinitely many supercompact cardinals, then there is a generic model in which every stationary set $S \subset \aleph_{\omega+1}$ reflects.*

Proof. Magidor [1982]. □

On the other hand, $\Box_\lambda$ implies that there exists a stationary subset of λ^+ that does not reflect (Exercise 38.5). As $\Box_\lambda$ holds in the core model K^{strong} then if $(\lambda^+)^{K^{\text{strong}}} = \lambda^+$, $\Box_\lambda$ holds in V as well (with the same square sequence) and one concludes (by Theorem 35.19) that if λ is a strong limit singular cardinal and every $S \subset \lambda$ reflects then there exists an inner model for a strong cardinal. This has been extended by Schimmerling and others to show that the consistency strength of this reflection property is more than the existence of a Woodin cardinal.

Now consider the question of what is the largest possible extent of reflection. Let us recall (Definition 8.18) that $S < T$ means that S reflects at almost all $\alpha \in T$. If $S < T$ then $o(S) < o(T)$ and one may ask whether it is possible that $S < T$ holds whenever $o(S) < o(T)$. This is possible for $\kappa = \aleph_2$: By Magidor's Theorem 23.23 it is consistent that every stationary $S \subset E^{\omega_2}_\omega$ reflects at almost all α of cofinality ω_1.

For $\kappa > \aleph_2$ it is impossible that $S < T$ whenever $o(S) < o(T)$. If $\mu < \lambda$ are regular cardinals such that $\lambda^+ < \kappa$ then there exist stationary sets $S \subset E^\kappa_\mu$ and $A \subset E^\kappa_\lambda$ such that S does not reflect at any $\alpha \in A$ (Exercise 38.7). Thus let us restrict ourselves to reflection at regular cardinals.

Definition 38.8. A weakly inaccessible cardinal κ satisfies *full reflection* if for every stationary set $S \subset \kappa$ and every stationary set $T \subset \kappa$ of regular cardinals, $o(S) < o(T)$ implies $S < T$.

Obviously, the property is meaningful only if κ is at least a (weakly) Mahlo cardinal. The consistency strength of full reflection for cardinals in the Mahlo hierarchy has been established by Jech and Shelah. For instance:

Theorem 38.9. *The following are equiconsistent, for every* $n \geq 1$:

(i) *There exists an n-Mahlo cardinal that satisfies full reflection.*
(ii) *There exists a* Π^1_n*-indescribable cardinal.*

Proof. Jech and Shelah [1993]. See also Exercises 38.8 and 38.9. □

If κ is a large cardinal such as measurable, strong, or supercompact then there is a generic extension in which κ remains measurable (strong, supercompact) and in addition satisfies full reflection (Gitik and Witzany [1996]).

Stationary Sets in $P_\kappa(\lambda)$

By Theorem 8.28, the closed unbounded filter on $[\lambda]^\omega$ is generated by the sets $C_F = \{x \in [\lambda]^\omega : x \text{ is closed under } F\}$ where $F : [\lambda]^{<\omega} \to \lambda$. Thus in many applications one considers the space $[H_\lambda]^\omega$ and stationary sets are those $S \subset [H_\lambda]^\omega$ such that for every model $(H_\lambda, \in, \ldots)$ there exists an $M \in S$ with $M \prec (H_\lambda, \in, \ldots)$.

When $\kappa > \aleph_1$, the sets C_F do not generate the closed unbounded filter on $P_\kappa(\lambda)$ as the set $\{x \in P_\kappa(\lambda) : |x| \geq \aleph_1\}$ is closed unbounded and does not include any C_F (which contains a countable set). A generalization of Theorem 8.28 yields the following description of the closed unbounded filter: it is the filter generated by the sets C_F and the set $\{x \in P_\kappa(\lambda) : x \cap \kappa \in \kappa\}$ (Exercise 38.10; see also Exercises 8.18, 8.19 and 36.17). For more on this subject, see Exercises 38.11 and 38.12.

By Lemma 31.3, ω-closed forcing preserves stationary sets in $[\lambda]^\omega$. This does not generalize to $P_\kappa(\lambda)$ for $\kappa > \aleph_1$, as $<\kappa$-closed forcing may destroy stationary sets in $P_\kappa(\lambda)$. The following concept is relevant to this problem and has other applications:

A model $M \prec H_\lambda$ is *internally approachable* if there exists an elementary chain $\langle M_\alpha : \alpha < \gamma\rangle$ with $M = \bigcup_{\alpha<\gamma} M_\alpha$ such that for every $\beta < \gamma$, $\langle M_\alpha : \alpha < \beta\rangle \in M$. In $P_\kappa(H_\lambda)$, let IA denote the set of all internally approachable M. The set IA is stationary, and if $\kappa = \aleph_1$ then IA contains a closed unbounded set (since every countable M is internally approachable). A stationary set S is preserved by $<\kappa$-closed forcing if and only if $S \cap IA$ is stationary (Exercises 38.13 and 38.14).

By Theorem 8.10, every stationary subset of κ can be partitioned into κ disjoint stationary sets. The situation is more complicated for $P_\kappa(\lambda)$. Since $|P_\kappa(\lambda)| = \lambda^{<\kappa}$ we may ask whether stationary sets in $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets. This is generally not provable, for two reasons. One is cardinal arithmetic and the other are large cardinals.

The cardinal arithmetic reason is that there may exist stationary, or even closed unbounded, sets of size λ while $\lambda^{<\kappa} > \lambda$. For instance, there exists a stationary set $S \subset [\omega_2]^\omega$ of size $\aleph_2$ (Exercise 38.15), or a closed unbounded set $C \subset [\omega_4]^{\omega_2}$ of cardinality $\aleph_4^{\aleph_1}$ (Exercise 38.16).

A generalization of Solovay's proof of Theorem 8.10 gives that every stationary set in $P_\kappa(\lambda)$ can be partitioned into κ disjoint stationary sets (Exercise 38.18). This is best possible as Gitik [1985] constructs a model, using a supercompact cardinal, in which there is a stationary set $S \subset P_\kappa(\kappa^+)$ that cannot be partitioned into κ^+ disjoint stationary sets.

In view of this discussion, the following is best possible:

Theorem 38.10. *Let κ be regular uncountable and $\lambda \geq \kappa$.*

(i) *$P_\kappa(\lambda)$ can be partitioned into λ disjoint stationary sets.*
(ii) *If κ is a successor cardinal then every stationary subset of $P_\kappa(\lambda)$ can be partitioned into λ disjoint stationary sets.*
(iii) *If $0^\sharp$ does not exist then every stationary subset of $P_\kappa(\lambda)$ can be partitioned into λ disjoint stationary sets.*
(iv) *If* GCH *holds then $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ stationary sets, and if moreover $0^\sharp$ does not exist then every stationary subset of $P_\kappa(\lambda)$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

Proof. Let us consider the following set

$$E = \{x \in P_\kappa(\lambda) : |x \cap \kappa| = |x|\}. \tag{38.1}$$

It is easy to see that E is stationary and that if κ is a successor cardinal then E contains a closed unbounded subset (Exercise 38.19). The question whether, for an inaccessible κ, the complement of E is stationary, involves large cardinals; cf. the following lemma and Exercise 38.20.

Lemma 38.11. *If $\{x \in P_\kappa(\lambda) : |x \cap \kappa| < |x|\}$ is stationary then $0^\sharp$ exists.*

Proof. By the assumption there exists a model $M \in P_\kappa(L_\lambda)$ such that $\kappa \in M \prec L_\lambda$, $\kappa_M = M \cap \kappa \in \kappa$, and $\kappa_M < |M|$. Let L_α be the transitive collapse of M. Thus there is an elementary embedding $j : L_\alpha \to L_\lambda$ with critical point κ_M, and since $\kappa_M < |\alpha|$, $0^\sharp$ exists. □

Consequently, the following two lemmas will complete the proof of Theorem 38.10:

Lemma 38.12. *Every stationary subset of the set E can be partitioned into λ disjoint stationary sets.*

Lemma 38.13. *If* GCH *holds and if* $\operatorname{cf}\lambda < \kappa$ *then every stationary set in $P_\kappa(\lambda)$ can be partitioned into λ^+ disjoint stationary sets.*

Proof of Lemma 38.12. Assume that $\lambda > \kappa$ (if $\lambda = \kappa$ we have Theorem 8.10). Let S be a stationary set in $P_\kappa(\lambda)$ such that $|x \cap \kappa| = |x|$ for every $x \in S$. For each $x \in S$, let $f_x : x \to x \cap \kappa$ be one-to-one. For each $\alpha < \lambda$, let $g_\alpha(x) = f_x(\alpha)$, for all $x \in S$ with $\alpha \in x$. There exists a stationary set S_α such that g_α is constant on S_α, with value $\gamma_\alpha < \kappa$.

Now if μ is any regular cardinal with $\kappa < \mu \le \lambda$, there exists a $\gamma < \kappa$ such that $\gamma_\alpha = \gamma$ for μ many α's. The corresponding sets S_α are pairwise disjoint stationary subsets of S. Thus for every regular cardinal $\mu \le \lambda$, every stationary subset of E has μ pairwise disjoint stationary subsets. It follows easily that every $S \subset E$ can be partitioned into λ disjoint stationary sets. □

Proof of Lemma 38.13. Assume GCH and let $\lambda > \kappa$ be such that $\operatorname{cf}\lambda < \kappa$. First we note that $|P_\kappa(\lambda)| = \lambda^+$, and that every unbounded (and therefore every stationary) subset of $|P_\kappa(\lambda)|$ has size λ^+: If Y is unbounded then $P_\kappa(\lambda) = \bigcup_{x \in Y} P(x)$ and the assertion follows.

Let $\langle f_\alpha : \alpha < \lambda^+ \rangle$ enumerate the set of all functions $f_\alpha : [\lambda]^{<\omega} \to P_\kappa(\lambda)$ such that each function appears cofinally often. By Lemma 8.26, for every closed unbounded set C and every $\gamma < \lambda^+$ there exists an $\alpha > \gamma$ such that $C \supset C(f_\alpha) = \{x : f(e) \subset x \text{ whenever } e \subset x\}$.

Now let S be a stationary set in $P_\kappa(\lambda)$. By induction on $\alpha < \lambda^+$ we construct one-to-one sequences $\langle x^\alpha_\xi : \xi < \alpha \rangle$ such that $\{x^\alpha_\xi : \xi < \alpha\} \subset S \cap C(f_\alpha)$, and that $\{x^\alpha_\xi : \xi < \alpha\}$ and $\{x^\beta_\xi : \xi < \beta\}$ are disjoint whenever $\alpha \ne \beta$. If we let, for each $\xi < \lambda^+$, $S_\xi = \{x^\alpha_\xi : \xi < \alpha < \lambda^+\}$, the sets S_ξ, $\xi < \lambda^+$, are pairwise disjoint, and we complete the proof by showing that each S_ξ is stationary.

If C is closed unbounded, then $C \supset C(f_\alpha)$ for some $\alpha > \xi$, and since $x^\alpha_\xi \in S_\xi \cap C(f_\alpha)$, we have $S_\xi \cap C$ nonempty. □

By Theorem 23.17 the nonstationary ideal on κ is not κ^+-saturated, for any $\kappa \ge \aleph_2$. A similar result is true for the nonstationary ideal on $P_\kappa(\lambda)$:

Theorem 38.14. *If κ is a regular uncountable cardinal and $\lambda > \kappa$ then the nonstationary ideal on $P_\kappa(\lambda)$ is not λ^+-saturated.* □

The result follows easily from Theorem 23.17 when λ is regular: Let $\kappa < \lambda$ be regular uncountable. The proof of Theorem 23.17 shows that there are almost disjoint stationary sets $A_\xi \subset \lambda$, $\xi < \lambda^+$, such that $\operatorname{cf}\alpha < \kappa$ for all $\alpha \in A_\xi$ and all $\xi < \lambda^+$. For each ξ let $S_\xi = \{x \in P_\kappa(\lambda) : \sup x \in A_\xi\}$. Then S_ξ, $\xi < \lambda^+$, are stationary sets with $S_\xi \cap S_\eta$ nonstationary if $\xi \neq \eta$ (Exercise 38.21).

When λ is singular, the result is a combination of several cases, depending on κ and $\operatorname{cf}\lambda$. The nonsaturation of I_{NS} on $[\lambda]^\omega$ for singular λ is an application of the concept of mutually stationary sets that we shall briefly describe in the next section (see Corollary 38.17).

Mutually Stationary Sets

The following definition, due to Foreman and Magidor, exploits the fact that if κ is a regular cardinal and $\lambda > \kappa$ then a set $S \subset \kappa$ is stationary if and only if for every model $\mathfrak{A} = \langle H_\lambda, \in, \ldots\rangle$ there exists some $M \prec \mathfrak{A}$ such that $\sup(M \cap \kappa) \in S$; i.e., if and only if the set $\{M \in P(H_\lambda) : \sup(M \cap \kappa) \in S\}$ is stationary in $P(H_\lambda)$.

Definition 38.15. Let A be a set of regular cardinals and let $\lambda = \sup A$. The sets S_κ, $\kappa \in A$, where $S_\kappa \subset \kappa$ for each $\kappa \in A$, are *mutually stationary* if the set $\{M : \sup(M \cap \kappa) \in S_\kappa$ for every $\kappa \in M\}$ is stationary in $P(H_\lambda)$.

Not much is known about mutual stationarity beyond the following theorem:

Theorem 38.16 (Foreman-Magidor). *Let A be a set of regular cardinals with $\lambda = \sup A$. If for each κ, S_κ is a stationary subset of κ such that* $\operatorname{cf}\alpha = \omega$ *for every $\alpha \in S_\kappa$, then the S_κ are mutually stationary. For every $\mathfrak{A} = \langle H_\lambda, \in, \ldots\rangle$ there exists a countable $M \prec \mathfrak{A}$ such that* $\sup(M \cap x) \in S_\kappa$ *for every $\kappa \in M$.*

Proof. Foreman and Magidor [2001]. □

One consequence of this result is that the nonstationary ideal on $[\lambda]^\omega$ is not λ^+-saturated when λ is singular:

Corollary 38.17. *If λ is a limit cardinal then there exist stationary stationary sets S_ξ, $\xi < \lambda^{\operatorname{cf}\lambda}$, in $[\lambda]^\omega$ such that $S_\xi \cap S_\eta$ is nonstationary whenever $\xi \neq \eta$.*

Proof. Let $\mu = \operatorname{cf}\lambda$ and let $A = \{\kappa_\alpha : \alpha < \mu\}$ be a set of regular cardinals with limit λ. For each $\alpha < \mu$, let $\{S^\alpha_\beta : \beta < \kappa_\alpha\}$ be a partition of $E^{\kappa_\alpha}_\omega$ into κ_α disjoint stationary sets. For each function $f \in \prod_{\alpha<\mu} \kappa_\alpha$, let $S_f = \{M \in [\lambda]^\omega : \sup(M \cap \kappa_\alpha) \in S^\alpha_{f(\alpha)}$ for all $\alpha \in M\}$. The sets S_f are stationary in $[\lambda]^\omega$ and if $f \neq g$ then for any α with $f(\alpha) \neq g(\alpha)$, the closed unbounded set $\{M : \alpha \in M\}$ is disjoint from $S_f \cap S_g$. □

Weak Squares

The theory of inner models shows that in the absence of very large cardinals, Jensen's principle $\square_\kappa$ holds whenever κ is a singular cardinal. In this last section we take a look at some weaker versions of Square.

Definition 38.18. Let κ be an uncountable cardinal, and let ν be a cardinal, $1 \le \nu \le \kappa$; $\square_{\kappa,\nu}$ is as follows:

(38.2) ($\square_{\kappa,\nu}$) There exists a sequence $\langle \mathcal{C}_\alpha : \alpha \in \text{Lim}(\kappa^+)\rangle$ such that
- (i) $\mathcal{C}_\alpha$ is nonempty and $|\mathcal{C}_\alpha| \le \nu$, and each $C \in \mathcal{C}_\alpha$ is a closed unbounded subset of α;
- (ii) if $C \in \mathcal{C}_\alpha$ and $\beta \in \text{Lim}(C)$ then $C \cap \beta \in \mathcal{C}_\beta$;
- (iii) if $\text{cf}\,\alpha < \kappa$ then $|C| < \kappa$ for every $C \in \mathcal{C}_\alpha$.

The principle $\square_{\kappa,<\nu}$ is defined similarly, replacing $|\mathcal{C}_\alpha| \le \nu$ by $|\mathcal{C}_\alpha| < \nu$.

The weakest principle of these, $\square_{\kappa,\kappa}$, is also denoted by $\square^*_\kappa$ and is called *Weak Square*. By Jensen [1972], $\square^*_\kappa$ is equivalent to the existence of a special Aronszajn κ^+-tree, and therefore, if κ is regular, $\square^*_\kappa$ follows from $2^\kappa = \kappa^+$. The main interest in the principles $\square_{\kappa,\nu}$ and $\square_{\kappa,<\nu}$ is in the case when κ is a singular cardinal. The failure of $\square^*_\kappa$ for κ singular (which, as mentioned below, entails a Woodin cardinal) is consistent: In [1979] Shelah proved the consistency, relative to a supercompact cardinal, of the negation of $\square^*_{\aleph_\omega}$.

The failure of Weak Square for singular κ has the consistency strength of (roughly) at least one Woodin cardinal: If there is a measurable cardinal and there is no inner model for the Woodin cardinal, then $\square_{\kappa,\text{cf}\,\kappa}$ holds for every strong limit singular cardinal. This follows from results of Mitchell, Schimmerling and Steel; cf. Schimmerling [1995].

Exercise 38.5 shows that if $\square_\kappa$ holds then κ^+ has a nonreflecting stationary set. The proof is easily modified to show that $\square_{\kappa,<\omega}$ suffices, see Exercise 38.23. (In contrast, $\square_{\aleph_\omega,\omega}$ is consistent with "every stationary subset of $\aleph_{\omega+1}$ reflects;" cf. Cummings, Foreman and Magidor [2001].)

The proof of Theorem 31.28 can be modified to show that PFA implies the negation of $\square_{\kappa,\omega_1}$ for every $\kappa \ge \omega_1$; see Exercise 38.24. This, and the afore mentioned results on $\square_{\kappa,\text{cf}\,\kappa}$ and Woodin cardinals yields Schimmerling's Theorem 31.30.

As a final application of weak squares we mention the following; for simplicity, let $\kappa = \aleph_\omega$. By the pcf theory there exists a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n\in A} \aleph_n$ (mod finite) for some $A \subset \omega$. If $\square_{\aleph_\omega,\aleph_m}$ holds for some m, then there exists a scale $\langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ with this property:

(38.3) For every $\alpha < \aleph_{\omega+1}$ such that $\text{cf}\,\alpha > \omega$ there exists a closed unbounded set $C \subset \alpha$ and some k such that for all $\beta < \gamma$ in C and all $n \ge k$ in A, $g_\beta(n) < g_\gamma(n)$.

(See Exercise 38.25.) Such scales were used in Cummings, Foreman and Magidor [2001] to prove (negative) results on stationary reflection.

Exercises

38.1. If κ is α-Mahlo and $I_{\text{NS}}\restriction\text{Reg}$ is κ^+-saturated, then κ has Mitchell order α in K^m.

[Use Theorem 35.16 and generic ultrapowers. Find an almost disjoint collection W of stationary subsets of Reg such that the length of $<$ restricted to W is at least α, and that the dual of $I_{\text{NS}}\restriction S$ restricted to K^m is a normal measure $U_S \in K^m$. Then show that $S < T$ implies $K \vDash U_S < U_T$. For details, see Jech [1984]. (The symbol $<$ is used both for the hierarchy of stationary sets and for the Mitchell ordering.)]

38.2. If I_{NS} on $\aleph_2$ is precipitous then $\aleph_2$ is a measurable cardinal of Mitchell order 2 in K^m.

[Use Theorem 35.16 and generic ultrapowers, or see Gitik [1984].]

38.3. Let I_0 be the normal ideal generated by the sets that do not reflect, let $I_{\alpha+1}$ be the smallest normal ideal extending I_α that contains every X such that $\{\beta : X \text{ does not reflect at } \beta\} \in I_\alpha$, and let $I_\alpha = \bigcup_{\beta<\alpha} I_\alpha$ if α is limit. Then κ is a reflecting cardinal if and only if $\kappa \notin \bigcup_{\alpha<\kappa^+} I_\alpha$.

38.4. If κ is a reflecting cardinal then κ is a reflecting cardinal in L.

38.5. If $\Box_\lambda$ holds then for every stationary set $S \subset \lambda^+$ there exists a stationary $T \subset S$ that does not reflect.

[Let $\langle C_\alpha : \alpha < \lambda^+\rangle$ be a square sequence, and let $f(\alpha)$ be the order-type of C_α. There is a stationary set $T \subset S$ on which f is constant. Show that T does not reflect (as in Lemma 23.6).]

38.6. If κ is supercompact and $\nu < \kappa < \lambda$ (regular cardinals) then every stationary $S \subset E^\lambda_\nu$ reflects.

[Compare with Exercise 27.3.]

38.7. Let $\mu < \lambda < \kappa$ be regular with $\lambda^+ < \kappa$. There exist stationary set $S \subset E^\kappa_\mu$ and $A \subset E^\kappa_\lambda$ such that S does not reflect at any $\alpha \in A$.

[As in Exercise 23.12.]

The Π^1_n *filter* on κ is the filter F^1_n generated by the sets $\{\alpha < \kappa : V_\alpha \vDash \varphi\}$ where φ is a Π^1_n formula true in V_κ; the Π^1_n ideal I^1_n is the dual ideal. κ is Π^1_n-indescribable if and only if the Π^1_n filter is a filter, i.e., $\kappa \notin I^1_n$.

38.8. Let κ be a Mahlo cardinal, let E_0 be the set of all inaccessible non Mahlo cardinals and assume that every stationary set $S \subset \kappa$ of singular cardinals reflects at almost all $\alpha \in E_0$. If $A \in L$ is a subset of κ such that $A \in I^1_1$ in L, then $A \cap E_0$ is nonstationary.

[Jeh and Shelah [1993], Lemma 2.1.]

As a consequence, if κ is Mahlo and satisfies full reflection, then κ is Π^1_1-indescribable in L. The following generalization implies that if κ is n-Mahlo and satisfies full reflection, then κ is Π^1_n-indescribable in L:

38.9. Let κ be an n-Mahlo cardinal that satisfies full reflection and let E_{n-1} be the set of all $\alpha < \kappa$ that are $(n-1)$-Mahlo but not n-Mahlo. If $A \in L$ is a subset of κ such that $A \in I^1_n$ in L, then $A \cap E_{n-1}$ is nonstationary.

38.10. For every closed unbounded set C in $P_\kappa(\lambda)$ there exists a function $F : [\lambda]^{<\omega} \to \lambda$ such that $C \supset \{x : x \cap \kappa \in \kappa \text{ and } F``[x]^{<\omega} \subset x\}$.

Let $[\lambda]^\nu = \{x \in P_{\nu^+}(\lambda) : |x| = \nu\}$. A set $C \subset [\lambda]^\nu$ is *strongly closed unbounded* if $C = C_F \cap [\lambda]^\nu$ for some $F : [\lambda]^{<\omega} \to \lambda$.

38.11. If the set $\{x \in [\lambda]^\nu : x \supset \nu\}$ contains a strongly closed unbounded set then every closed unbounded set $C \subset [\lambda]^\nu$ contains a strongly closed unbounded set.

38.12. The following are equivalent:

(i) The closed unbounded filter on $[\omega_2]^{\aleph_1}$ is generated by strongly closed unbounded sets.
(ii) Chang's Conjecture.

38.13. If $S \subset IA$ is stationary and if P is $<\kappa$-closed, then S is stationary in V^P.

38.14. Let P be the forcing that collapses $|H_\lambda|$ to κ (with conditions of size $< \kappa$). In V^P, the set $(IA)^V$ contains a closed unbounded set.

38.15. There exists a stationary set $S \subset [\omega_2]^\omega$ of size $\aleph_2$.

[For each $\alpha < \omega_2$, let $f : \alpha \to \omega_1$ be one-to-one. If $\alpha < \omega_2$ and $\xi < \omega_1$, let $X_{\alpha,\xi} = \{\beta < \alpha : f_\alpha(\beta) < \xi\}$. Let $S = \{X_{\alpha,\xi} : \alpha < \omega_2,\ \xi < \omega_1\}$.]

38.16. There exists a closed unbounded set $C \subset [\omega_4]^{\omega_2}$ of size $\aleph_4^{\aleph_1}$.

[Baumgartner [1991], Corollary 3.5.]

38.17. If X_α, $\alpha < \lambda$, are stationary sets in $P_\kappa(\lambda)$ such that $X_\alpha \cap X_\beta$ is nonstationary for all $\alpha \neq \beta$, then there exist pairwise disjoint stationary sets Y_α with $Y_\alpha \subset X_\alpha$ for all $\alpha < \lambda$.

[$Y_\alpha = X_\alpha \cap \{x : \alpha \in x \text{ and } \forall \beta \in x \text{ if } \beta \neq \alpha \text{ then } x \notin X_\beta\}$.]

38.18. For every stationary set $S \subset P_\kappa(\lambda)$ the ideal $I_{\text{NS}} \restriction S$ is not κ-saturated.

[Gitik [1985], p. 893.]

38.19. Let $E = \{x \in P_\kappa(\lambda) : |x \cap \kappa| = |x|\}$.

(i) E is stationary.
(ii) If κ is a successor cardinal then E contains a closed unbounded subset.

38.20. If κ is supercompact and $\lambda > \kappa$ then the set $\{x \in P_\kappa(\lambda) : |x \cap \kappa| < |x|\}$ is stationary.

38.21. Let $\kappa < \lambda$ be regular uncountable and let $A \subset \lambda$ be such that $\operatorname{cf} \alpha < \kappa$ for all $\alpha \in A$. A is stationary if and only if $\{x \in P_\kappa(\lambda) : \sup x \in A\}$ is stationary in $P_\kappa(\lambda)$.

38.22. If κ is supercompact then for all $\lambda \geq \kappa$, $\square_{\lambda,<\kappa}$ fails.

38.23. If $\square_{\kappa,<\omega}$ holds then for every stationary $S \subset \kappa^+$ there exists a stationary $T \subset S$ that does not reflect.

[Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_{\kappa,<\omega}$ sequence and let $f(\alpha) = \{\text{o.t.}(C) : C \in \mathcal{C}_\alpha\}$. Then proceed as in Exercise 38.5.]

38.24. PFA implies that $\square_{\kappa,\omega_1}$ fails for every uncountable cardinal κ.

[Let $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ be a $\square_{\kappa,\omega_1}$ sequence, and let T be the tree of all (α, A) with $A \in \mathcal{C}_\alpha$, ordered by $(\alpha, A) \prec (\beta, B)$ if and only if $\alpha \in \operatorname{Lim}(B)$ and $A = B \cap \alpha$. Let $P = \{p \subset \kappa^+ : p \text{ is closed and countable}\}$, ordered by end-extension. In V^P, $T \restriction \bigcup \dot{G}$ has no ω_1-branch. Let $\dot{Q}$ be the c.c.c. forcing that specializes $T \restriction \bigcup \dot{G}$. Applying PFA to $P * \dot{Q}$ leads to a contradiction as in Theorem 31.28. For details, see Schimmerling [1995].]

38.25. Prove (38.3) using $\square_{\aleph_\omega,\aleph_m}$.

[Let $\langle \mathcal{C}_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be a $\square_{\aleph_\omega,\aleph_m}$ sequence, and assume that $m < n$ for all $n \in A$. For a limit γ, let g_γ be such that $g_\gamma > g_\alpha$ for all $\alpha < \gamma$, $g_\gamma(n) > f_\gamma(n)$ for all $n \in A$, and such that for all $n \in A$, $g_\gamma(n) > \sup\{\sup_{\beta\in C} g_\beta(n) : C \in \mathcal{C}_\gamma, |C| < \aleph_n\}$.]

Historical Notes

The first consistency proof for the saturation of I_{NS} was obtained by Steel and van Wesep [1982], forcing over a model of ZF + $\text{AD}_{\boldsymbol{R}}$ + "Θ is regular" ($\text{AD}_{\boldsymbol{R}}$ is the determinacy of games where moves are real numbers). Following the proof of Theorem 37.16 (by Foreman, Magidor and Shelah), Shelah obtained the consistency from the existence of a Woodin cardinal (Theorem 38.1).

Theorem 38.2 will appear in the forthcoming book on AD, cf. Woodin et al. [∞]. Theorem 38.3 is proved in Woodin [1999].

Theorem 38.4: Jech and Woodin [1985].

Theorem 38.5: Gitik [1984].

Theorem 38.6: Mekler and Shelah [1989].

Theorem 38.9: Jech and Shelah [1993].

Theorem 38.10 is a combination of several results, including Jech [1972/73], Matsubara [1987], [1988] and [1990], Di Prisco and Baumgartner.

Lemma 38.11: Baumgartner.

Lemma 38.12: Di Prisco.

Lemma 38.13: Matsubara.

Theorem 38.14 is a combination of several results, including Gitik and Shelah [1997], Baumgartner and Taylor [1982], Donder and Matet [1993], Burke and Matsubara [1999] and Foreman and Magidor [2001].

Mutually stationary sets are investigated in Foreman and Magidor [2001]. For weak squares, see Schimmerling [1995] and Cummings, Foreman and Magidor [2001].

Exercises 38.3 and 38.4: Mekler and Shelah [1989].

Exercises 38.5: Jensen.

Exercises 38.6 and 38.22: Solovay.

Exercises 38.7: Shelah.

Exercises 38.8 and 38.9: Jech and Shelah [1993].

Exercises 38.10: Kueker.

Exercises 38.11, 38.12, 38.13 and 38.14: Foreman, Magidor and Shelah [1988].

Exercises 38.15, 38.16, 38.19 and 38.20: Baumgartner.

Exercises 38.24: Magidor.

Exercises 38.25: Cummings, Foreman and Magidor [2001].