SAMPLE PROBLEMS FOR SET THEORY

Problem **I**. (a) Prove the following version of the Dushnik-Miller theorem:

$$\aleph_2 \to (\aleph_2, \aleph_0)^2$$

(b) Prove: if $\prec$ is an arbitrary well-order of the ordinal ω_2 then there is a subset X of ω_2 of cardinality $\aleph_2$ such that $\alpha \prec \beta$ if and only if $\alpha \in \beta$ for all $\alpha, \beta \in X$.

Problem **II**. Give a self-contained proof of the inequality $\kappa < \operatorname{cf} 2^\kappa$, that is, do not rely on results like Kőnig's inequality.

Problem **III**. Prove, without using the Axiom of Choice, that there is a surjection from $\mathcal{P}(\omega)$ onto ω_1.

Problem **IV**. Prove that the set $V_{\omega+\omega}$ satisfies the axioms of ZF except the Axiom of Replacement and give a specific instance of this axiom that fails in $V_{\omega+\omega}$.

Problem **V**. Prove Hartogs' theorem: the Wellordering Theorem is equivalent to the statement that for any two sets A and B there is an injection of A into B or an injection of B into A. *Hint*: given a set X prove that there is an ordinal that does not admit an injective map into X.

Problem **VI**. Let κ be a successor cardinal. Prove that there is a family $\{S_\alpha : \alpha < \kappa\}$ of pairwise disjoint stationary subsets of κ.